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PERTURBATION THEORY, BOREL SUMMABILITY, AND ASYMPTOTICS

© E. Caliceti, V. Grecchi, M. Maioli

Abstract. We discuss some results in stability, perturbation theory, and summation methods applied to anharmonic oscillators and Stark problems. In particular, using also other technique, we show how to get exact results or asymptotic behavior and to relate different asymptotics in a rigorous way.

§1. Perturbation theory and summability

As is well known, in the nonregular case, the perturbation theory is an asymptotic method for the study of the spectrum of an operator. Moreover, perturbation theory allows one to use regular methods of summation of the divergent series and to pass from asymptotic results to exact ones. Also, in the regular case, if the unperturbed problem is parameter dependent, the perturbation theory could be one of the techniques used in order to get precise asymptotic estimates.

From the very beginning of the new quantum theory, the Rayleigh–Schrödinger Perturbation Theory (RSPT) has played a central role in the understanding of physical problems (see [Si] for a review). For instance the weak field energy levels of the hydrogen Stark effect resonances were very well approximated by perturbation series.

From a rigorous point of view, we should note that in this important case the perturbation theory is not regular and the perturbation series is not convergent. Moreover, we expect complex energy states (resonances) with exponentially small imaginary part. Since the perturbation series is real, it is clear that the imaginary part is not directly given by the perturbation series.

Such difficulties can be overcome by an appropriate summation method of the series. Following G.'t Hooft ['t] , we define (and give a criterion for) a distributional Borel (DB) sum of a series which extends the original Borel one to critical cases [CaGrMa1]. We note that in such cases the "uniqueness" of the DB sum is not on the same footing as the usual Borel one. Indeed, we use the principle of "maximal continuity" of a Borel transform as a distribution on the positive half-axis in place of the usual continuity. An equivalent principle is the one of the real sum of a real series, or the choice of the principal part of a singular integral (this rule was the one used by 't Hooft). Actually, our the summability criterion defines directly a pair of complex conjugate sums (called upper and lower Borel sums (US, LS) : $\sum^+$, $\sum^-$) whose difference is called the discontinuity of the Borel sum (DOS) : $\Delta\sum = \sum^+ - \sum^-$, and whose mean is the DB sum itself. We note that the DB sum is the only reasonable candidate for "the" sum even if it is not possible to disregard the existence of the parents US and LS giving a measure of the imaginary "error" (DOS). The usefulness of the Borel sum can be tested on the Airy function expansion in different Stokes sectors. It is known that the Stokes paradox of different asymptotic expansions in the same sector cannot be solved without the use of the Borel sum [Vo]. The problem of the expansion on one Stokes direction between two sectors was left open. We have shown that this case also is solved by the use of DB sums. This solution makes rigorous the usual association of the Bi function on the positive half-axis with its asymptotic power series expansion at $+\infty$ [CaGrMa1, AbSt]. Of course, the DB sums form a linear space, but they do not form not an algebra with respect to multiplication and composition as US's (LS's) do, so that in complex WKB [Vo] calculations we should use US (LS) sums in intermediate steps and should take the DB sum (the mean of (US, LS)) only as the last operation.

One should be warned about the use of mixed asymptotic expansions of power series and exponentials of the inverse variable: such series could be ambiguous and meaningless (as in the Stokes paradox) if the sum of the series of the leading term is not given. Before the introduction of the DB sum, we consider the limit of the usual Borel sum in the critical direction. We prove that in this case the limit from above (below) coincides with the US (LS sum) given by the criterion of summability proposed. It is clear that the proof of the DB summability is a stronger result than the proof of the simple existence of upper and lower limits of Borel sums. In particular, this allows one to connect directly the asymptotics of the perturbation series with the asymptotics of the imaginary part and the nature of the first singularities of the Borel transform on the positive half-axis.

§2. Stability criteria

In order to prove the analyticity and strong asymptotics of eigenvalues [Si] in a rigorous way, we use, with some adaptation, the stability theory of Hunziker–Vock [HuVo]. This theory plays a fundamental role in the analysis of many problems of interest in physics, in the framework of nonanalytic perturbation theory. In fact it allows us to treat the cases where norm resolvent convergence is not available, but only the strong convergence in the generalized sense, as in the models illustrated below. The operator class explicitely provided for applications in [HuVo] is a family of Schrödinger operators $H_g = p^2 + V_g(x)$ in $L^2(\mathbb{R}^\nu)$ with $C_0^\infty(\mathbb{R}^\nu)$ as a core. The perturbation parameter g varies in a sector S of the complex plane with vertex $g = 0$, and the potential $V_g(x)$ is in $L^2_{\text{loc}}(\mathbb{R}^\nu)$. Moreover, the following conditions must be satisfied:

(a)

$$\lim_{g\to 0, g\in S} \|(V_g - V_0)u\| = 0, u \in C_0^\infty(\mathbb{R}^\nu).$$

This assumption garantees the strong convergence of the resolvents.

(b) There exist positive constants ξ, η, ϵ such that

$$\langle u, p^2 u\rangle \leq \xi[\Re(e^{-i\gamma_g}\langle u, H_g u\rangle) + \eta\langle u, u\rangle],$$

$u \in C_0^\infty(\mathbb{R}^\nu), |\gamma_g| \leq \pi/2 - \epsilon$. In particular, it follows that the numerical range of H_g lies in a half-plane S_g contained in $Q \equiv \{ z \in \mathbb{C}J \mid -\pi + \epsilon \leq \arg(z - \eta/\epsilon) \leq \pi - \epsilon \}$.

(c) The range $(z - \dot{H}_g)$ is dense, for some $z \notin Q$, where $\dot{H}_g$ is the restriction of H_g to $C_0^\infty(\mathbb{R}^\nu)$. Consequently, the set of uniform boundedness of the resolvents $\Delta = \{ z \in \mathbb{C} \mid z \notin \sigma(H_g)$ and $(z - H_g)^{-1}$ is uniformly bounded as $g \to 0 \}$ is nonempty, in fact $\mathbb{C} \setminus Q \subset \Delta$.

The fundamental inequality in this method, namely,

$$\|(1 + p^2)^{1/2} u\| \leq a(\|u\| + \|H_g u\|), \quad u \in C_0^\infty(\mathbb{R}^\nu), \tag{1}$$

for some constant $a > 0$ independent of g and u, follows from (b) and is used in order to prove that if we are given two sequences:

$$\begin{aligned} &g_n \to 0, \ u_n \in D(H_{g_n}) \\ &\qquad \text{such that } \|u_n\| = 1, \ u_n \to 0 \text{ weakly}, \ \|(\lambda - H_{g_n})u_n\| \to 0, \end{aligned} \tag{2}$$

then the functions u_n can be chosen so as to be supported at infinity (more precisely, $u_n(x) = 0$ for $|x| \leq n$). Under the hypotheses (a), (b), (c) and the further conditions

(i) $\operatorname{dist}(\lambda, \sigma_{\text{ess}}(H_g)) \geq \delta > 0$, as $g \to 0$;

(ii) $d_n(\lambda, g) \geq \delta > 0$, for $n > n_0$ and $g \to 0$, where

$$d_n(\lambda, g) = \inf\{ \|(\lambda - H_g)u\| : u \in D(H_g), \|u\| = 1, u(x) = 0 \text{ for } |x| \leq n \},$$

the following alternative holds:

1) if $\lambda \notin \sigma_d(H_0)$, then $\lambda \in \Delta$;
2) if $\lambda \in \sigma_d(H_0)$, then λ is stable (in the sense of Kato) with respect to the family H_g.

We note, first of all, that condition (ii) is usually easy to verify, since it can be reduced to a uniform bound from below of the distance from λ to the "asymptotic" (i.e. restricted to functions supported at infinity) numerical range of H_g. Such a bound can be further improved, by decomposing the "asymptotic" numerical range into a finite number of parts, each satisfying a similar estimate. This condition allows us to obtain easily the first alternative, since any $\lambda \notin \sigma_d(H_0) \cup \Delta$ generates a sequence as in (2) (called a Weyl-type sequence), contradicting in fact (ii).

This stability result is complete, in the sense that the first alternative implies that there are no "dying" eigenvalues: if $\lambda_g \to \lambda_0$ as $g \to 0$ and λ_g is a discrete eigenvalue of H_g, then λ_0 is an eigenvalue of H_0.

Finally, we recall the definition of stability. An isolated eigenvalue of finite multiplicity λ of H_0 is called stable in the sense of Kato with respect to the family H_g if

(I) for any sufficiently small $r > 0$, $\Gamma_r = \{ z \in \mathbb{C} : |z - \lambda| = r \}$ is contained in the resolvent set of H_g, as $g \to 0$;

(II) $\lim_{g \to 0} \|P_g - P_0\|$, where $P_g = (2\pi i)^{-1} \oint_{\Gamma_r} (z - H_g)^{-1} dz$ is the spectral projection of H_g relative to the part of its spectrum encircled by Γ_r.

In particular, this implies that $\dim(P_g) = \dim(P_0)$ as $g \to 0$; thus, if λ is stable and simple, for any small g there exists a unique simple eigenvalue λ_g of H_g such that $\lambda_g \to \lambda$ as $g \to 0$.

Since the examples described below are rather singular and do not immediately fall in the class suggested in [HuVo], we now proceed to illustrating how we extend this class, analyzing the difficulties presented by the different models under examination. To this end it is useful to provide some further details about the method of Hunziker and Vock in order to show the variants that we suggest in different cases.

Let $\chi \in C_0^\infty(\mathbb{R})$be such that $\chi(x) = 1$ if $|x| \leq 1$, $0 \leq \chi(x) \leq 1$ for all $x \in \mathbb{R}$, and let $M_n(x) = 1 - \chi_n(x), n \in \mathbb{N}$. In order to construct Weyl-type sequences supported at infinity, the basic estimate (1) is used in [HuVo] to prove the following fundamental results:

(A) $\|[H_g, \chi_n]u\| \leq cn^{-1}(\|H_g u\| J + J\|u\|), \ \forall u, \ g \to 0;$

(B) Given two sequences g_m, u_m such that $g_m \to 0$, $u_m \to 0$ weakly, and $\|u_m\| = 1$, $\|H_{g_m} u_m\| \leq c$, we have

$$\lim_{m \to \infty} \|\chi_n u_m\| = 0 \quad \text{for all } n.$$

Now we are ready to examine some examples, which will be analyzed in further detail below.

Anharmonic oscillator and the Stark effect. To prove stability for the resonance operators

$$H_\rho = \alpha J[p f(r)^2 p + 4^{-1}(f(r)^2)^{(2)} + k/\xi(r)^2 + \rho^2 \xi(r)^4] + \alpha^{-1}\xi(r)^2$$

described below, as $g \to 0$ in the Nevanlinna disk $-\Re g^{-2} > R^{-1}$, we cannot proceed directly as in [HuVo], since neither the basic estimate (1) is available, nor condition (b) which implies (1), but we can obtain only the weaker estimates (3) and (4) for the quadratic form h_ρ associated with H_ρ (see below). These estimates are still enough to ensure the condition

(A_1) $\|[H_\rho, \chi_n]u\| \leq cn^{-1/4}(\|H_\rho u\|J + J\|u\|)$.

As for (B), instead, the situation is more delicate, since in [HuVo] (B) is deduced immediately from (1) by using the compactness of the operator $\chi_n(1+p^2)^{-1/2}$ and the boundedness of the sequence $(1+p^2)^{1/2}u_m$.

A completely new method is used in this case in [CaGrMa3]; this method employs more directly the compactness of the resolvent of the limiting operator H_0 and the estimates of the difference $H_\rho - H_0$ in the disk, which can be reduced to the difference of the corresponding potentials on the interval $[0, 2n]$.

It is worth noting that this method can be viewed as a generalization of [HuVo] to more singular cases where (1) does not hold, but only a weaker condition like (3), provided that H_0 has compact resolvents.

Symmetric double well. In the case of $H_0(g) = p^2 + x^2(1-gx)^2$ we need, first of all, to revise the very notion of stability, since, because of the two potential wells, we expect, as in the selfadjoint case (g real), a couple of distinct eigenvalues converging to the same limiting eigenvalue of the harmonic oscilator as $g \to 0$ in the Nevanlinna disk.

This difficulty had already been overcome in [CaGrMa2], by splitting the operator $H_0(g)$ into two components $H_0^+(g)$ and $H_0^-(g)$, obtained by projecting, respectively, onto the (orthogonal) subspaces of the even and odd functions with respect to the barrier point $x = (2\rho)^{-1}$; this proves the stability of the eigenvalues of $H_0(0)$ separately with respect to the two families $H_0^+(g)$ and $H_0^-(g)$ as $g \to 0$ in any fixed sector $-\pi/4 + \epsilon \leq \arg g \leq \pi/4 - \epsilon$, $\epsilon > 0$. Thus, the existence of an entire disk of analyticity $\Re g^{-2} < -R^{-1}$ was not yet garanteed. At this point there are difficulties similar to those encountered in the previous example, and they are treated with similar techniques in [CaGrMa4], with one further question: the condition $d_n(\lambda, g) \geq \delta > 0$ cannot be satisfied in this case unless $d_n(\lambda, g)$ is redefined so as to take into account also the second well. For

$$d_n'(\lambda, g) = \inf\{\, \|(\lambda - H_g)M_n^\rho u\| : u \in D(H_0(g)), \|M_n^\rho u\| = 1\,\}$$

condition (ii) still holds, with $M_n^\rho = 1 - \chi_n^\rho$, $\chi_n^\rho = \chi_n(x) + \chi_n(\rho^{-1} - x)$. Indeed, if we do not want the distance from λ to the "asymptotic" numerical range to go to zero, we must cut the potential both around the origin (the first well) and around ρ^{-1} (the second well).

Asymmetric double well. The case of $H_j(g) = p^2 + x^2(1 - gx)^2 - j(gx - 1/2)$, with $j/2 \in \mathbb{R} \setminus \mathbb{Z}$, presents a different anomaly compared to the previous examples, since alternative 1) of the theorem of Hunziker and Vock is now false. In fact, if we consider $H_j(0) = p^2 + x^2 + j/2$ as the natural limiting operator, then, on one hand, 2) holds, i.e., the eigenvalues are stable with respect to $H_j(g)$, but on the other hand, the operator $H_j(g)$ possesses "dying" eigenvalues. More precisely , for every $n, \lambda_n = 2n + 1 + j/2$ is stable with respect to $H_j(g)$ as $g \to 0(g > 0)$; moreover, there exists a unique eigenvalue $\lambda'_n(g)$ of $H_j(g)$ such that $\lambda'_n(g) \to 2n+1-j/2$, as $g \to 0^+$. This result was proved in [CaGrMa5] and it is due, also in this case, to the presence of a second well which gives rise to a second group of eigenvalues associated with another "limiting operator" $K_j(0) = p^2 + x^2 - j/2$. Indeed, it is proved that the eigenvalues of $K_j(0)$ are stable with respect to the following family of translated operators:

$$K_j(g) = T_{1/g} H_j(g) (T_{1/g})^{-1} = p^2 + x^2(1 + gx)^2 - j(gx + 1/2),$$

where $(T_{1/g}u)(x) = u(x + 1/g), u \in L^2(\mathbb{R}), g > 0$. Since $K_j(g)$ and $H_j(g)$ are unitarily equivalent, they have the same eigenvalues: the ones converging to $2n + 1 - j/2$, regarded as eigenvalues of $H_j(g)$, appear as "dying", because they do not converge to an eigenvalue of the limiting operator $H_j(0)$. Thus, working in parall with the two families $H_j(g)$ and $K_j(g)$ we can suitably modify once again the stability theory of Hunziker and Vock so as to obtain the above illustrated result, and to extend it further (in [CaGrMa4]), with techniques and estimates similar to those already seen in the preceding examples, thus obtaining again the analyticity of the eigenvalues in a Nevanlinna disk.

§3. The models and the results

In the Stark effect we cannot expect that the resonances are given by the DB sum of the perturbation series. Indeed, the resonances are complex and, in principle, any unperturbed eigenvalue is associated with a pair of complex conjugate resonances. For this reason, we can expect that a resonance is determined by LS (or US, depending on the convention on the physical sheet), the position of the resonance by the DB sum, and the physical uncertainty on the position by $|DOS|$. We have proved that this is the case (see [CaGrMa3]).

Actually, we have first proved the DB summability of the unstable anharmonic oscillator (the "volcano") perturbation series of the eigenvalues. As in the Stark case, a "resonance" of the "volcano" is given by the LS (US) of its perturbation series .

In order to prove this, it suffices to prove that the eigenvalues are analytic in a Nevanlinna disk of the coupling constant complex plane. More precisely, let

$$H(g) = p^2 + x^2 + g^2 x^4$$

for $-\frac{\pi}{4} < \theta + \frac{\pi}{2} < \frac{\pi}{4}$, where $g = \rho \exp(i\theta)$, defined by analytic dilation: $x \to x \exp(-i\frac{\theta}{3})$. Each eigenvalue $E_n^j(g)$ of $H(g)$ is analytic in a Nevanlinna disk $-\Re g^{-2} > R^{-1}$. The proof is based on the stability of the eigenvalue $E_n^j(0)$ for the radial dilated operator

$$A(\rho) = \alpha\Big(p^2 + \frac{(j^2-1)}{4r^2} + \rho^2 r^4\Big) + \frac{1}{\alpha} r^2$$

in $L^2(\mathbb{R}^+)$, where $p = -i\frac{d}{dr}$, $j = d + 2(l-1)$, $j \geq 0$ or $j = -1$, $\alpha = i^{-1}\exp(i2\epsilon/3)$, $\epsilon = \rho^2/2R$, with the following boundary condition at the origin: $\psi(r) \sim r^{(j+1)/2}$. In order to prove the stability as $\rho \to 0$, we use a further distortion transformation: $U\psi(r) = \xi'(r)^{\frac{1}{2}}\psi(\xi(r))$, where $\xi(r)$ is in $C^\infty(\mathbb{R}^+)$ and satisfies

$$\xi(r) - r = 0$$

for $r - r_0 > \eta > 0$, $r_0 = 1/\sqrt{2}\rho$, and

$$\xi(r) - r = -2i\eta(1 - (1 + r^3)^{-1/6})J$$

for $0 < r \leq r_0$. Th transformed operator is as follows:

$$H_\rho = \alpha J[pf(r)^2 p + 4^{-1}(f(r)^2)^{(2)} + k/\xi(r)^2 + \rho^2\xi(r)^4] + \alpha^{-1}\xi(r)^2,$$

where $f(r) = 1/\xi'(r)$ and $k = (j^2-1)/4$. We claim that $H_\rho \to H_0$ in the strong resolvent sense, and the eigenvalue $E_n^j(0)$ of H_0 is stable with respect to the family H_ρ. We prove this by a position-dependent energy bound. Let $h_\rho[.]$ be the quadratic form associated with H_ρ; for ρ smaller than ρ_0 we have

$$\Re h_\rho[u] \geq a\eta \int_1^{r_0} r^2(1 + r^3)^{-7/6}|pu|^2 dr - c\|u\|^2, \tag{3}$$

and if supp (u) $\subset (n, \infty)$, then

$$\Re h_\rho J[u] \geq (aR^{-1} - c)\|u\|^2 \tag{4}$$

for n large. These estimates are much weaker than the ones required in [HuVo], but they are sufficient for obtaining the stability result, as remarked in the previous section. The latter inequality is optimal because of the existence of a barrier of height of the order of ρ^{-2} multiplied by a coefficient of the order of ϵ. This means that we expect top resonances of the barrier to interact with our eigenvalue and to give rise to Bender–Wu singularities near the disk. Only in the limit case of zero energy we can expect that the radius R of the disk goes to ∞. Now, let $\chi \in C_0^\infty(\mathbb{R}^+)$ and $\chi_n(r) = \chi(r/n)$; then condition (A_1) holds for ρ small or equal to zero. If we set $M_n = 1 - \chi_n$ and

$$d_n(E,\rho) = \inf\{ J\|(E - H_\rho)M_n u\| : \|M_n u\| = 1, u \in D(H_\rho) \},$$

then for every $E \in \mathbb{C}$ there exist $R, n_0, \rho_0, \delta > 0$ such that $d_n(E,\rho) > \delta$ for any $\rho < \rho_0, n > n_0$ in the Nevanlinna disk $\Re g^{-2} < -1/R$. Moreover, if $u_m \to 0$ weakly and $\rho_m \to 0$, with $\|u_m\| = 1$, and $H_{\rho_m} u_m$ bounded, for R small we have $\|M_n u_{m(n)}\| \to 1$, and $\|H_{\rho_{m(n)}} M_n u_{m(n)}\|$ is bounded for some $m(n) \to \infty$ as $n \to \infty$. In this way we prove that $E_n^j(g)$ is analytic in the Nevanlinna disk $\Re g^{-2} < -1/R$.

We recall the relationship between Hydrogen Stark effect resonances and anharmonic oscillator eigenvalues. Because of the separability of the Stark operator

$$K(F) = -(1/2)\Delta - 1/|x| + F x_1$$

in $L^2(\mathbb{R}^3)$, in parabolic coordinates, the resonances $E^m_{n_1,n_2}(F)$ are given implicitly by the following equation in E, F:

$$f^m_{n_1,n_2}(u) = e^{-i\pi/3}\zeta,$$

where, for $m = 0, 1, 2, \ldots$,

$$f^m_{n_1,n_2}(u) = [E^{2m}_{n_1}(u^{3/2}/i) + E^{2m}_{n_2}(u^{3/2}/i^2)]/u = [E^{2m}_{n_1}(u^{3/2}/i) - E^{2m}_{n_2}(u^{3/2}i)]/u,$$

$u = (i^2 F)^{1/3}(-2E)^{-1/2}$, and $\zeta = 4(F)^{-1/3}$. We have used the complex scaling relation $E_n^j(g/i^2) = -E_n^j(ig)$, valid for any complex $g \neq 0$. In [CaGrMa3] it was proved that each resonance $E^m_{n,n}(F)$ is analytic in a Nevanlinna disk $B((R,0),R)$.

The UB sum can be written as follows:

$$\Phi(z) = 1/z \int_0^\infty B(t + i0)\rho(t/z)dt,$$

where $\rho_\alpha(t) = e^{-t}t^\alpha$, and $\Phi(z) \sim \sum c_n z^n$, $\Im c_n = 0$, $\Im\Phi(z) = g(z)$ for positive z. If $B(t)$ is analytic in the disk $|t|J < 1$ and has $t = 1$ as the only singularity on the circle $|t| = 1$, then the following properties are equivalent:

i) $\Im B(t + i0) = \pi\delta(t-1) + b_1\pi\theta(t-1) + O((t-1)\theta(t-1))$ as $t \to 1$,
ii) $c_n = \Gamma(n + \alpha + 1)(1 + b_1/n + O(1/n^2))$ as $n \to \infty$,
iii) $g(z) = (\pi/z)\rho_\alpha(1/z)(1 + b_1 z + O(z^2))$ as $z \to 0^+$.

Similar relations apply to the Stark effect resonances, with ii) replaced by

ii′) $C_{2n} = 2\Gamma(2n+\alpha+1)[1+b_1/2n+O(1/n^2)]$ as $n \to \infty$, $C_{2n+1} = 0$.

In particular (see [BeGr]), $E_0^0(g/i) = \gamma\phi(\sigma g^2)$, where $\gamma = -\sigma 8/\pi$, $\sigma = 3/2$, and $c_n \sim n!(1-4.41\overline{6}\sigma/n)$. Thus,

$$\Im E_0^0(g/i) \sim -\gamma\pi/(\sigma g^2)\exp(-1/\sigma g^2)(1-4.41\overline{6}/g^2).$$

The general behavior is given by

$$\Im E_n^{2m}(g/i) = 2(n!(n+m)!)^{-1}(4/g^2)^{2n+m+1}\exp(-2/3g^2)(1+O(g^2)),$$

for g small positive. The coefficients of the expansion of

$$\phi(F) \equiv E_{0,0}^0(F/\sigma)/\delta,$$

where $\delta = -2\sigma/\pi$, have the asymptotics $C_{2n} = 2(2n)!(1+\sigma B_1/2n+O(1/n^2))$, where $B_1 = -8.91\overline{6}$, so that $g(F) = (\pi/F)\exp(-1/F)(1+B_1F+O(F^2))$, and

$$\Im E_{0,0}^0(F) = \delta(\pi/\sigma F)\exp(-1/\sigma F)(1+F\sigma B_1+O(F^2))$$

for small positive F. Thus, we generalize the well-known Herbst–Simon relations on asymptotics. A surprising result of the small coupling analysis of the anharmonic oscillators and of the implicit formula for the Stark resonances given above, was the infinite field behavior of such resonances [BGHS, BeGr]. Indeed,

$$f_{k,k}^m(u) = (1/u)\Delta\sum c_n(k,m)u^{3n} = (1/u)2i\Im E_k^{2m}(u^{3/2}/i)$$

for positive u. Thus, $f_{k,k}^m(u)$ is analytic and decays exponentially as u vanishes about the positive direction, so that for any k and m we have a family of resonances defined by a phase index q. More precisely, for any k, m, q we have a resonance trajectory $u(\zeta)$ decaying as $O(\ln(\zeta))^{-1/3}$ for small positive ζ [BeGr]:

$$\begin{aligned} u(\zeta)^{-3} &= (3/2)\ln(1/\zeta) + (3/2)(2k+m+4/3)\ln(\ln(1/\zeta)) \\ &\quad + i\pi(3q-1/4) - (3/2)\ln[k!(k+m)!(2/3)^{1/3}(1/6)^{2k+m+2}] \\ &\quad + O(\ln(\ln(1/\zeta)))/\ln(1/\zeta)). \end{aligned}$$

This means that for large F we have an infinite number of resonances diverging as $(1/2)((1/2)F\ln F)^{2/3}$ in the asymptotic direction $\arg E = -\pi/3$. Other rigorous results give the absence of any resonance in complex plane domains [GrGr1]. Moreover, using complex scaling and regular perturbation theory for the operator

$$H(a,1) = p^2 + ax^2 + x^4,$$

we have proved that all resonances are in the region $|E| > c(F)^{2/3}$ for a suitable $c > 0$, and, for F large, in the sector $-2/3\pi \leq \arg E \leq 0$. Thus, we conjecture that all the resonances coming from the hydrogen bound states at zero field go to infinity with the asymptotics given above. In this case, any resonance trajectory with $n_1 - n_2 = p$ should cross at least $|p|$ Bender–Wu cuts [BeWu] of the anharmonic eigenvalues. Numerical investigation of three resonances $(n_1, n_2, m) = (0,0,0), (1,0,0), (0,1,0)$ [BeGr] suggest a one to one relationship between large field resonances and the zero field levels. Simple relations of the new indices with the original parabolic ones, namely, $k = \min(n_1, n_2)$ and $q = n_1 - n_2$, and minimal crossings of $|q|$ Bender–Wu cuts are suggested by numerical results.

Now we consider the most singular perturbation problem: the double well,

$$H_j(g) = p^2 + x^2(1 - gx)^2 - j(gx - 1/2)$$

for g positive and j real, in $L^2(\mathbb{R})$. In order to make the difficulties of this problem apparent, we recall the particular case considered earlier by Herbst–Simon [HeSi]. The formal operator $H_{-2}(ig)$ can be naturally defined as a closed operator $H^+_{-2}(ig)$ having the eigenvalue 0 for any positive g and eigenvector $\psi_0^+(ig, x) = \exp(-x^2/2 + ix^3 g/3)$. The operator $H^+_{-2}(ig)$ (and, similarly, $H^-_{-2}(-ig)$) can be continued to positive g, as a "resonance" operator. Perturbation theory does not distinguish between the two operators having different conditions at ∞, and the perturbation series for the eigenvalue 0 at $g = 0$ vanish identically. But the original operator $H_{-2}(g)$ is positive semidefinite, formally vanishing only on the diverging vector $\psi_0^+(g, x)$, so that it is positive for $g > 0$. Of course, the regular sum of the zero series is zero, so that in this case the matter does not reduce to summability. Indeed, for g real, the perturbation series, if suitably summed, gives the "resonance", which is the bound state eigenvalue for imaginary g. In order to get the eigenvalues of the original operator we should change the notion of perturbation series for this class of problems. In this case we consider the original perturbation fraction $E_n^j(g) = N_n^j(g)/D_n^j(g)$, well defined for a suitable choice of the unperturbed vector, and define the perturbation series for the numerator and denominator separately. We use a method of comparison based on the two operators defining the complex conjugate resonances. In this way we obtain a complex series whose real part coincides with the original one and should be summed by the DB method, and whose imaginary part contributes through the exponentially small DOS. Thus, we show explicitly the difference between the possibility of the usual perturbation series and the actual energy levels. We first fix the energy parameter z and use the following relation between Green functions:

$$G(x, y) = [G^+(x, y) + G^-(x, y) + ih(G^+(x, y) - G^-(x, y))]/2,$$

where $G^{\pm}(x,y)$ are the Green functions of the "resonance" operators $H_j^{\pm}(g)$, $h = h(g,z) = -i(1-k)/(1+k)$ is real for real parameters,

$$k = k(z) = W_{0,1}W_{3,-1}/W_{0,-1}W_{3,1},$$

and $W_{m,n}$ is the Wronskian of two fundamental solutions that vanish in the complex directions such that $[3\arg(x)/\pi]$ is equal, modulo 3, to m and n respectively. We note that $k(z)$ can be approximated by the complex WKB method and goes to $k_0(z+j)$ as $g \to 0$, where $k_0 = W_{0,1}W_{2,-1}/W_{0,-1}W_{2,1} = \exp(iz\pi)$. If we restrict ourselves to dilation analytic vectors, we can relate the mean values of resolvents corresponding to the above Green functions:

$$\langle R\rangle_\psi = [\langle R^+\rangle_\psi + \langle R^-\rangle_\psi + ih(\langle R^+\rangle_\psi - \langle R^-\rangle_\psi)]/2.$$

We fix the unperturbed eigenvalue $E_0^j(0) = E_0^j = 1 + j/2$ (i.e., the first eigenvalue of the first unperturbed well; the corresponding eigenvalue of the second unperturbed well is E_0^{-j}), for non integer j, with eigenfunction ψ. Of course, the perturbation series for the numerator (denominator), comes from the formula

$$N = -(2\pi i)^{-1} \int z^n \langle R_z\rangle_\psi dz,$$

where $n = 1$ ($n = 0$ for the denominator D), and from the expansion of the resonance resolvents as above. We consider $h = h(g,z)$ in accordance with the complex WKB method. Each term of the form

$$A^{\pm}(n,m) = -(2\pi i)^{-1} \int z^n h^m \langle R_z^{\pm}\rangle_\psi dz$$

is the upper or lower sum of a perturbation series obtained by expanding the resolvent in powers of the parameter. The series for $m = 0$ gives the usual one for the numerator (or the denominator), the other one ($m = 1$) is new, not explicitely known, and parameter dependent. Putting all together, we get $N(g) = \sum a_n g^n + i\Delta \sum b_n(g) g^n/2$, where the first series is the usual one summed by the DB method and the $b_n(g)$ are regarded as fixed in the summation of the series. As for the $b_n(g)$'s, we should recall that they depend on $k(g,z)$ which can be computed with the help of the complex WKB method and DB sums, or can be approximated uniformly on the integration path by $\exp i(z - E_0^{-j})\pi$ for g small. This simple approximation shows that $b_n \sim \cot((z - E_0^{-j})\pi/2)a_n$ and for $z = E_0^j$, and

$$N(g) \sim \sum a_n g^n + i\cot(j\pi/2)\Delta \sum a_n g^n/2,$$

where the second term is the leading one of order of $O(\exp -2S)$, and S is the absolute value of the classical action on the barrier. We consider the case where j is an integer, and, in particular, the symmetric one for $j = 0$. In this case we should use the symmetric and antisymmetric, parameter dependent, test vectors defined as $\psi^{\pm}(x) = [\psi(x) \pm \psi(1/g - x)]/\sqrt{2}$. Thus, the metod is not changed, but now the a_n's are also parameter dependent. The approximated $h(g,z) \sim \cot((z - E_0^{-j})\pi/2)$ cannot be extracted from the integral that determines the perturbation series. The splitting term, of the ordrer of $O(\exp -S)$, clearly comes from the g-dependence of the test function and the a_n's.

As an example of nonzero integral j, we return to the first problem we have mentioned above, namely, $j = -2$. In this case the first eigenvalue is stable, and we have $a_n = 0, n = 0, 1, \ldots$. As test vectors we use a sequence of dilation analytic vectors like $\psi_n(g,x) = \exp(-x^2/2 + gx^3/(3 + x^2/n)), n = 1, 2, \ldots$, with $|\arg x^3 g| = \pi/2$, having a limit $\psi(g,x)$ in L^2. Therefore, we can justify the use of the resonance eigenvector as a test vector, and employ the formal equation $R^{\pm}(g,z)\psi(g,x) = -\psi(g,z)/z$. Thus, we obtain

$$A^{\pm}(m,n) \sim (\delta_1^m \delta_1^n C_1(g) + \delta_0^n(\delta_0^m + \delta_1^m C_2(g)))G^{\pm}(g^2),$$

where $C_1(g), C_2(g)$ are the limits of $zh(g,z)$ and $h(g,z) - C_1(g)/z$ as z tends to zero, and the $G^{\pm}(g^2)$ are the "norms" of the test vector for two suitable complex dilations:

$$G^{\pm}(g^2) = g^2 \exp i\alpha^{\pm} \int \psi(g, x \exp i\alpha^{\pm})^2 dx,$$

where $\alpha^{\pm} = \pm\pi/6 - \arg(g)/3$. The function $G^{\pm}(z)$ is analytic in a Nevanlinna disk, the corresponding asymptotic series $\sum a_n z^n$ is DB summable for small positive z to the real part of $G^{\pm}(z)$, and $\Delta \sum a_n z^n = \pm 2i\Im G^{\pm}(z)$ for real z. Since in this case $h(g,z) \to \cot(\pi z/2)$, we have

$$E_0^{-2}(g) \sim (2/\pi)\Im G^{+}(g^2)/ReG^{+}(g^2) = (2/\pi)Ai((2g)^{-4/3})/Bi((2g)^{-4/3})$$

as $g \to 0$. Actually, in the original paper [CaGrMa2] the LS (and US) sums for $j > -2$ are substituted by limits of odinary Borel sums. These results are now improved by the results obtained for the unstable anharmonic oscillator "resonances" discussed above and by their connections with the double well "resonances" [BuGr]. A direct rigorous study on the analyticity of the double well eigenvalues will follow [CaGrMa4]. The double well appears as the most complicated perturbation problem, but also in this case the perturbation theory, adapted and assisted by summation methods and the complex WKB, can give the best asymptotic results.

It is appropriate for this exposition to add some comments on the paper [BuGr] quoted above. In this paper there is the solution, in terms of the identity of the spectrum of suitable operators, of a problem started as the strange identity of asymptotic perturbation series [SeZi, An] and developed in terms of Borel summability [GrGr]. The proof is based not on perturbation theory, but on controls of the asymptotic behavior of the eigenfunctions. In particular, the operator of the unstable anharmonic oscillator (the "volcano"), suitably defined by complex translation on the radial variable,

$$B_j(ig) = (1/2)(-d^2/dx^2 + (j^2 - 1)/4x^2 + x^2) - g^2x^4$$

in $L^2(\mathbb{R} - i\epsilon)$, for any $\epsilon > 0$ has the same spectrum as the double well operator $H_j(g)$ defined above. On the other hand, the stable anharmonic oscillator

$$P_j(g) = (1/2)(-d^2/dr^2 + (j^2 - 1)/4r^2 + r^2) + g^2r^4$$

in $L^2(\mathbb{R}^+)$, with the boundary condition $\psi \sim r^{(j+1)/2}$ for $j \geq 0$ and analytically continued to $j > -2$, has the same spectrum as the double well resonance operator continued to imaginary parameter, i.e., the operator $H_j^+(ig)$ defined in $L^2(\mathbb{R})$ as above and satisfying $\log\psi(x) \sim -x^2/2 + igx^3/3$ at infinity.

For other examples of the use of perturbation theory for obtaining asymptotic results we quote two papers on delocalization instability: [GrGrJo] and [GrMar]. In the first case, a symmetric double well was considered, with a small fixed and localized perturbation in the semiclassic regime [GrGrJo]. In the second case, a mutidimensional double well nonlinear Stark effect was considered, as a model of molecular localization in a gas for strong pressure [GrMar]. The localization is proved by comparison with the linear Stark effect and by using perturbation theory. The degenerate localized level results from a bifurcation of the ground state level.

In the case of crystals with external electric and magnetic field it is possible to use peturbation theory taking the (field-dependent) one-band approximation as an unperturbed problem [GrMaSa1, GrMaSa2, GrSa]. In the magnetic case, a double well effect arises, with the splitting asymptotics rigorously given by perturbation theory and semiclassic approximation [GrSa]. In the electric case (known as Wannier–Stark problem), the second order approximation gives the resonance width asymptotics modulo a numerical factor ($\pi^2/9$) [GrMaSa1, GrMaSa2, BuDm]. Another challenge for perturbation theory and summability theory is the following problem:

$$H(g) = p^2 + x^2 + gx^{2n}\exp(ax),$$

where $n \in \mathbb{N}_0$ and a is positive. In this case the perturbation coefficients grow faster than any power of the factorial, $\log a_n \sim a^2n^2/4$ [DoPo], so that it is necessary to apply a well defined "logarithmic Borel" method of summation [GrMa], but the complete proof of summability is still missing.

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Dipartimento di Matematica,
Universitá di Bologna,
JI-40127 Bologna (Italy)

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Dipartimento di Matematica,
Universitá di Modena,
I-41100 Modena (Italy)

Dipartimento di Matematica,
Universitá di Bologna,
JI-40127 Bologna (Italy)