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QUASI-INVARIANT MEASURES FOR TOPOLOGICAL DYNAMICAL SYSTEMS

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Abstract. It is proved that for a topological dynamical system to admit an ergodic quasi-invariant measure of type III (a measure which is not equivalent to any σ-finite invariant measure) it is necessary and sufficient that this system have a recurrent point. For systems with a recurrent point, it is shown that there exist a nondenumerable number of pairwise singular ergodic quasi-invariant measures of type III.

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§1. Introduction

By a dynamical system is meant a triplet (X, μ, T), where X is a Lebesgue space with measure μ, and $T: X \to X$, together with its inverse, is one-to-one (mod 0) and measurable. The measure μ is assumed to be finite and quasi-invariant with respect to T. Moreover, in what follows we shall consider only ergodic measures.

The well-known classification of ergodic quasi-invariant measures is closely related to the classification of von Neumann factors (see [2]). This classification is given in the following table.

Type of Measure	Characteristic property of a measure of the given type.
I_n $(n = 1, 2, \ldots)$	μ is a discrete measure concentrated on a trajectory with period n.
I_∞	μ is a discrete measure concentrated on a nonperiodic trajectory.
II_1	μ is a continuous measure, and there exists a finite measure ν invariant with respect to T and equivalent to μ.
II_∞	μ is a continuous measure, and there exists an infinite but σ-finite measure ν invariant with respect to T and equivalent to μ.
III.	There does not exist any (finite or infinite) σ-finite measure ν invariant with respect to T and equivalent to μ.

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Examples of measures of each type, except for type III, are not difficult to cite. The first example of a measure of type III was constructed by Ornstein [1]. If X is a compact metric space, and if T is a homeomorphism on it, then we speak of the *topological* dynamical system (X, T). In this case the measure μ is not assumed to be given, but on the contrary one raises the question of the existence of quasi-invariant Borel measures of the different types for the system (X, T).

In [3] it is shown that when X is the circle $|z| = 1$ and T rotates the circle through a π-irrational angle α: $Tz = z\exp(i\alpha)$, there exists a continuum of nonequivalent continuous quasi-invariant ergodic measures for (X, T).

Krieger [4] proved that an uncountable number of such measures exist for every strongly ergodic system.

Katznelson and Weiss [5] proved this under more general hypotheses: for every topological dynamical system with a recurrent point.

We note that in [3], [4], and [5] it is not set forth to which types the constructed measures belong. In particular, the possibility is not excluded that these measures are of type II_∞.(1) Therefore [3], [4], and [5] leave open the question: For which systems (X, T) does there exist a quasi-invariant measure of type III? The answer to this question is given in Theorem 1 below.

We now proceed to the necessary definitions and formulate the results obtained. A measure μ is called quasi-invariant with respect to a transformation T if for each measurable set $A \subseteq X$ the equalities $\mu(A) = 0$ and $\mu(TA) = 0$ either hold or do not hold simultaneously.

A measure μ (not necessarily quasi-invariant) is called ergodic with respect to T if $\mu(A)\mu(X\setminus A) = 0$ for each T-invariant set A.

A measure μ is called continuous if $\mu(A) = 0$ for each singleton A.

A measure μ is called σ-finite if the space X can be represented as the union of a countable number of sets with finite measure.

Measures μ and ν are called equivalent if for each set A the equalities $\mu(A) = 0$ and $\nu(A) = 0$ either hold or do not hold simultaneously.

Measures μ and ν are called singular if there exist sets A and B such that $A \cup B = X$ and $\mu(A) = \nu(B) = 0$.

A point $x \in X$ is called recurrent with respect to T (here (X, T) is a topological dynamical system) if in each of its neighborhoods there exist infinitely many distinct points of the form $T^k x$ for $k > 0$.

Theorem 1. *In order for the system (X, T) to have an ergodic quasi-invariant measure of type* III, *it is necessary and sufficient that there exist a recurrent point $x \in X$.*

The necessity of the condition was in fact proved in [5] (it is necessary only to note that a measure of type III is required to be continuous). The fundamental difficul-

(1) We note that measures of type II_∞ (as well as all other types) are realizable in topological systems.

ty consists in proving the sufficiency. This part of Theorem 1 is obtained as a consequence of the following theorem.

Theorem 2. *Let X be a compact metric space, and let $T: X \to X$ be a homeomorphism having a continuous ergodic quasi-invariant measure. Then the system (X, T) has an uncountable number of pairwise singular ergodic quasi-invariant measures of type* III.

We note that the theorem of Katznelson and Weiss [5] is a consequence of Theorem 2.

This article contains three sections. In §2 two lemmas are proven which are needed for the proof of Theorem 2. Lemma 1 treats not the topological but the purely metric situation and gives in approximate terms some sufficient conditions under which a measure would be of type III. Essentially, these conditions are obtained from the formal properties of the construction which was proposed by Ornstein [1] for constructing the first example of a measure of type III. Lemma 2 gives the possibility of "embedding" the corresponding metric construction in an arbitrary topological dynamical system with a recurrent point.

Theorem 2 is proved in §3.

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§2. Two lemmas

Lemma 1. *Let (X, μ, T) be a dynamical system (the measure μ is quasi-invariant with respect to $T: X \to X$). Assume that for each natural number N there are given a natural number k_N, a positive number ϵ_N and a collection of measurable sets $G_1^N, \ldots, G_{k_N}^N$, satisfying the following conditions:*

1) $k_N > k_{N-1}$ $(N = 2, 3, \ldots)$,
2) $\epsilon_N \to 0$,
3) $G_i^N = T^{i-1}G_1^N$ $(i = 2, 3, \ldots, k_N)$,
4) $G_i^N \cap G_j^N = \emptyset$ *for* $i \neq j$, $1 \leq i, j \leq k_N$,
5) $\sum_{i=1}^{k_N}\mu(G_i^N) = 1 - \epsilon_N$,
6) $\sum_{i=1}^{k_{N-1}}\mu(G_i^N) = \frac{1}{2}$,
7) $\mu(G_i^N) \leq 1/100(N-1)k_{N-1}$ $(N = 2, 3, \ldots; k_{N-1} < i \leq k_N)$,
8) *if* $A \subseteq G_i^N$ *is measurable* $(1 \leq i < k_N)$, *then*

$$\frac{\mu(TA)}{\mu(A)} = \frac{\mu(G_{i+1}^N)}{\mu(G_i^N)}$$

(the expression 0/0 is regarded as equal to any number).

Then there does not exist a σ-finite measure which is equivalent to μ and invariant with respect to T.

Proof. Following Ornstein [1], we will say that a set B is a copy of a set A if it is possible to represent A and B in the following form:

$$A = \bigcup_{s=1}^{\infty} A_s, \quad A_p \cap A_q = \varnothing \text{ for } 1 \leq p \neq q < \infty,$$

$$B = \bigcup_{s=1}^{\infty} B_s, \quad B_p \cap B_q = \varnothing \text{ for } 1 \leq p \neq q < \infty,$$

in which $B_s = T^{n_s} A_s$, where n_s is some integer. We show that for any measurable set S with $\mu(S) \geq 9/10$, and for any natural number N, we can find a measurable set $M \subseteq S$, $\mu(M) \geq 1/8$, such that S contains at least N nonintersecting copies of M. The assertion of the lemma follows from this, as shown in [1].

And so, let a set S, $\mu(S) \geq 9/10$, and a natural number N be given.

We consider the set G_1^{N+1} and assign to each $x \in G_1^{N+1}$ a corresponding sequence $\{\xi_i\}_{i=1}^{k_{N+1}}$ of k_{N+1} zeros and ones according to the rule:

$$\xi_i = \begin{cases} 0, & \text{if } T^{i-1}x \in S, \\ 1, & \text{if } T^{i-1}x \notin S. \end{cases}$$

By the same token we have defined a transformation

$$\overline{R}: G_1^{N+1} \to (\mathbf{Z}_2)^{k_{N+1}}.$$

We now consider the set

$$G^{N+1} = \bigcup_{i=1}^{k_{N+1}} G_i^{N+1}$$

and extend the transformation $\overline{R}$ to the transformation

$$R: G^{N+1} \to (\mathbf{Z}_2)^{k_{N+1}}$$

in the following manner: if $x \in G_i^{N+1}$ $(1 \leq i \leq k_{N+1})$, then set $R(x) = \overline{R}(T^{1-i}x)$. For $\xi \in (\mathbf{Z}_2)^{k_{N+1}}$ we denote by $G_i^{N+1}(\xi)$ the set $G_i^{N+1} \cap R^{-1}(\xi)$.

By virtue of condition 8) we have

$$\frac{\mu(G_1^{N+1}(\xi))}{\mu(G_1^{N+1})} = \frac{\mu(G_2^{N+1}(\xi))}{\mu(G_2^{N+1})} = \ldots = u_\xi,$$

where u_ξ is a number which does not depend upon i.

A point $\xi \in (\mathbf{Z}_2)^{k_{N+1}}$ is called good if among its coordinates ξ_i with $k_N < i \leq k_{N+1}$ it has no fewer than Nk_N zeros; in the contrary case we call the point ξ bad. We will divide the sets $B_\xi = R^{-1}(\xi)$ into good and bad, depending upon whether the corresponding ξ is good or bad. We shall show that the sum of the measures of the good B_ξ is greater than ½. From conditions 5) and 6) it follows that

$$\sum_{i=k_N+1}^{k_{N+1}} \mu(G_i^{N+1}(\xi)) = \left(\frac{1}{2} - \varepsilon_{N+1}\right) u_\xi. \tag{2.1}$$

If a point ξ is bad, then from condition 7) we obtain

$$\sum \mu (G_i^{N+1}) \leqslant \frac{1}{100},$$

where the summation is over those i for which $\xi_i = 0$, $k_N < i \leq k_{N+1}$.

Taking condition 8) into account, we find

$$\sum \mu (G_i^{N+1}(\xi)) \leqslant \frac{1}{100} u_\xi \tag{2.2}$$

(the summation is over those same i).

Subtracting (2.2) from (2.1), we obtain

$$\sum (\mu (G_i^{N+1}(\xi)) > \left(\frac{1}{2} - \varepsilon_{N+1} - \frac{1}{100}\right) u_\xi,$$

where the summation is over i such that $\xi_i = 1$, $k_N < i \leq k_{N+1}$. But if $\xi_i = 1$, then $G_i^{N+1}(\xi) \subseteq (X \setminus S) \cap B_\xi$. Since for different i the sets $G_i^{N+1}(\xi)$ do not intersect, we have

$$\mu ((X \setminus S) \cap B_\xi) > \left(\frac{1}{2} - \varepsilon_{N+1} - \frac{1}{100}\right) u_\xi.$$

It is clear that we may suppose that N is so large that, by virtue of condition 2),

$$\mu ((X \setminus S) \cap B_\xi) > \frac{2}{5} u_\xi.$$

Combining such inequalities for all bad ξ, we obtain on the left-hand side a number no larger than $\mu(X \setminus S) \leq 1/10$. Therefore

$$\frac{1}{10} > \frac{2}{5} \sum_{\xi \text{ bad}} u_\xi.$$

This means

$$\sum_{\xi \text{ bad}} u_\xi \leqslant \frac{1}{4}, \qquad \sum_{\xi \text{ good}} u_\xi > \frac{3}{4}.$$

But

$$u_\xi = \frac{\mu (B_\xi)}{\mu (G^{N+1})} = \frac{\mu (B_\xi)}{1 - \varepsilon_{N+1}}.$$

Therefore if N is so large that $\epsilon_{N+1} < 1/3$, then

$$\sum_{\xi \text{ good}} \mu (B_\xi) \geqslant \frac{3}{4} (1 - \varepsilon_{N+1}) > \frac{1}{2}.$$

We may now define the set M, about which we were speaking in the beginning of the proof:

$$M = S \cap \left(\bigcup_{\xi \text{ good}} B_\xi\right) \cap \left(\bigcup_{i=1}^{k_N} G_i^{N+1}\right).$$

We have

$$\mu(M) \geqslant \mu\left[\left(\bigcup_{\xi \text{ good}} B_\xi\right) \cap \left(\bigcup_{i=1}^{k_N} G_i^{N+1}\right)\right] - \mu(X \setminus S) \geqslant \frac{1}{2} \sum_{\xi \text{ good}} \mu(B_\xi) - \mu(X \setminus S).$$

The last inequality follows from condition 6). Consequently $\mu(M) \geq 1/4 - 1/10 > 1/8$. Moreover, $M = \bigcup_{\xi \text{ good}} M_\xi$, where $M_\xi = S \cap B_\xi \cap (\bigcup_{i=1}^{k_N} G_i^{N+1})$, and the set M_ξ, by virtue of the definition of good ξ, has at least N disjoint images lying in S. Lemma 1 is proved.

Before formulating Lemma 2, we make a few remarks which may facilitate the following reading. In Lemma 2 it will be shown that for every dynamical system with a recurrent point and for every N it is possible to find sets G_i^N which behave like the sets in Lemma 1 having the same name. But we do not now have the measure μ, and therefore, together with the sets G_i^N, we construct numbers μ_i^N—the values of a future measure μ for the sets G_i^N.

The measure μ for the proof of Theorem 2 will be constructed as the limit of some discrete measures μ_m, concentrated on finite parts of the trajectory of a recurrent point x. In this connection we naturally wish to guarantee the possibility of passing to the limit

$$\lim_{m \to \infty} \mu_m(G_i^N) = \mu(G_i^N).$$

For this we have to ensure that the measures μ_m do not accumulate in neighborhoods of the boundaries of the sets G_i^N. This explains the appearance of the sets ${}^N\Gamma_p^q$, constructed in Lemma 2: ${}^N\Gamma_p^q$ is a small neighborhood of the boundary of G_p^q.

We further remark that Lemma 2 will be proved by induction on N, and the origin of some points in its formulation (for example, 12, 13, 15, 16, and 17) is purely "inductive". The assertions contained in it are not all needed by us for the proof of the lemma, and we will use them only as supplementary inductive hypotheses (this phenomenon is common in inductive arguments). Such "inductive" points, it is understood, arose not prior to, but in the process of the proof; therefore, to try to understand them probably makes sense only after having run across them in context.

Lemma 2. *Let X be a compact metric space, let $T: X \to X$ be a homeomorphism, let $x \in X$ be a recurrent point, let $\nu_1, \nu_2, \cdots$ be a sequence of Borel probability measures on X, and let $j_1, j_2, \ldots$ be a sequence of natural numbers such that each natural number appears in it an infinite number of times. Then for each natural number N it is possible to define*:

a) *a natural number k_N*;

b) *a positive number ϵ_N*;

c) *a collection of open sets $G_1^N, \ldots, G_{k_N}^N$ (these sets will be referred to below as sets of rank N)*;

d) *a collection of open sets $\{{}^N\Gamma_p^q\}$ $(1 \leq q \leq N;\ 1 \leq p \leq k_q)$; and*

e) *a collection of positive numbers $\mu_1^N, \ldots, \mu_{k_N}^N$*

in such a way that the following conditions are satisfied:

1) $k_N > k_{N-1}$ $(N = 2, 3, \ldots)$.

2) $0 < \epsilon_N \leq 1/2^N$.

3) G_i^N *is a neighborhood of the point* $x_i = T^{i-1}x$ *with diameter* $\leq 1/N$, *and* $G_i^N = T^{i-1}G_1^N$ $(i = 2, 3, \ldots, k_N)$.

4) $\overline{G}_i^N \cap \overline{G}_j^N = \varnothing$ *for* $1 \leq i \neq j \leq k_N$.

5) $\Sigma\mu_r^s = 1 - \epsilon_N$, *where the summation is over pairs* (r, s) *such that* α) $s \leq N$ *and* β) *there does not exist a set* $G_{r'}^{s'}$ *of rank* $s' < s$ *such that* $x_r \in G_{r'}^{s'}$.

6) $\Sigma_{i=1}^{k_{N-1}} \mu_i^N = ½$ $(N = 2, 3, \ldots)$.

7) $\mu_i^N \leq 1/100Nk_{N-1}$ $(N = 2, 3, \ldots; k_{N-1} < i \leq k_N)$.

8) *If* $G_p^q \subseteq G_r^s$ $(q, s = 1, 2, \ldots; 1 \leq p < k_q; 1 \leq r < k_s)$, *then*

$$\frac{\mu_{p+1}^q}{\mu_p^q} = \frac{\mu_{r+1}^s}{\mu_r^s}.$$

9) *If* G_i^N *is any set of rank* N, *and if* q *is a natural number*, $q > N$, *then*

$$\left(1 - \frac{1}{2^q}\right)\mu_i^N \leqslant \Sigma\mu_r^s < \mu_i^N, \tag{2.3}$$

where the summation is taken over pairs (r, s) *such that* α) $N < s \leq q$, β) $x_r \in G_i^N$, *and* γ) *there does not exist a set* $G_{r'}^{s'}$ $(N < s' < s)$ *such that* $G_r^s \subseteq G_{r'}^{s'} \subseteq G_i^N$; *moreover, the number* $\alpha(i, N, q) = \Sigma\mu_r^s/\mu_i^N$ *is independent of* i *(the summation is over the same pairs* (r, s)*).*

10) *If* $\overline{G}_p^q \cap \overline{G}_r^s \neq \varnothing$ $(q = 1, 2, \ldots; s = q+1, q+2, \ldots; 1 \leq p \leq k_q; 1 \leq r \leq k_s)$, *then* $\overline{G}_r^s \subseteq G_r^q$.

11) ${}^N\Gamma_p^q$ *is a neighborhood of the sets*

$$\Gamma_p^q = \overline{G}_p^q \diagdown G_p^q \ (1 \leqslant q \leqslant N;\ 1 \leqslant p \leqslant k_q);$$

moreover,

$${}^N\Gamma_p^q \subseteq {}^q\Gamma_p^q, \quad {}^N\Gamma_p^q = T^{p-1}({}^N\Gamma_1^q).$$

12) *For each* $m < N$ *the set*

$$U_m^N = \left[T\left(X \diagdown \bigcup_{\substack{1 \leqslant q \leqslant N \\ 1 \leqslant p \leqslant k_q}} {}^q\overline{G}_p^q\right)\right] \cap \left[G_1^m \diagdown \bigcup_{\substack{m < q \leqslant N \\ 1 \leqslant p \leqslant k_q}} {}^q\overline{G}_p^q\right]$$

$$\cap \left[T^{-k_m-1}\left(X \diagdown \bigcup_{\substack{1 \leqslant q \leqslant N \\ 1 \leqslant p \leqslant k_q}} {}^q\overline{G}_p^q\right)\right] \cap [\{x_k\}_{k=1}^{\infty}],$$

where ${}^qG_p^q = G_p^q \cup {}^q\Gamma_p^q$, *is nonempty.*

13) $X \diagdown \bigcup_{i=1}^{k_N} {}^N\overline{G}_i^N \supseteq \{T^{-1}x;\ T^{k_N+1}x\}$.

14) *If* G_p^q *is any set of rank* $q \leq N$, *then* $\Sigma\mu_r^s < 1/N$, *where the summation is over pairs* (r, s) *such that* α) $q < s \leq N$, β) $\overline{G}_r^s \subseteq {}^N\Gamma_p^q$, *and* γ) *there does not exist a set*

$G_{r'}^{s'}$ $(q < s' \le N)$ *such that* $G_r^s \subseteq G_{r'}^{s'} \subseteq {}^N\Gamma_p^q$.

15) *If* G_p^q *is any set of rank* $q < N$, *and if* $x_i \in {}^l\Gamma_p^q$ *for some* i, $1 \le i \le k_N$, *and some* l, $q \le l < N$, *then* $\bar{G}_i^N \subseteq {}^l\Gamma_p^q$.

16) *If* G_p^q *is any set of rank* $q < N$ *and if* $x_i \notin {}^l\Gamma_p^q$ *for some* i, $1 \le i \le k_N$, *and some* l, $q \le l < N$, *then*

$$\bar{G}_i^N \cap {}^l\bar{\Gamma}_p^q = \varnothing.$$

17) *The boundaries of the sets* G_i^N *and* ${}^l\Gamma_p^q$ *do not intersect the trajectory of the point* x.

18) $\nu_{i_N}(\bigcup_{i=1}^{k_N} G_i^N) \le 1/2^N$.

The proof will proceed by induction on N. For $N = 1$ we obtain $k_1 = 2$, $\epsilon_1 = 1/3$ and $\mu_1^1 = \mu_2^1 = 1/3$. As G_1^1 and G_2^1 we take neighborhoods of x_1 and x_2, respectively, of diameter < 1 such that

α) $G_2^1 = TG_1^1$;

β) $\bar{G}_1^1 \cap \bar{G}_2^1 = \varnothing$;

γ) $X \setminus (\bar{G}_1^1 \cap \bar{G}_2^1) \supseteq \{T^{-1}x,\ T^2x\}$;

δ) the boundaries of G_1^1 and G_2^1 do not intersect the trajectory of x.

As ${}^1\Gamma_1^1$ and ${}^1\Gamma_2^1$ we take neighborhoods of the sets $\Gamma_1^1 = \bar{G}_1^1 \setminus G_1^1$ and $\Gamma_2^1 = \bar{G}_2^1 \setminus G_2^1$, respectively, such that

α) ${}^1\bar{G}_1^1 \cap {}^1\bar{G}_2^1 = \varnothing$, where ${}^1G_p^1 = G_p^1 \cup {}^1\Gamma_p^1$ $(p = 1, 2)$;

β) $X \setminus ({}^1\bar{G}_1^1 \cup {}^1\bar{G}_2^1) \supseteq \{T^{-1}x,\ T^2x\}$;

γ) the boundaries of ${}^1\Gamma_1^1$ and ${}^1\Gamma_2^1$ do not intersect the trajectory of x.

For $N = 1, \ldots, n$ (and also for $1 \le p, q, r, s, r', s' \le n$), let there be given all that is required in a)–d) and for which the inductive hypotheses 1)–18) are satisfied. Now let $N = n + 1$.

We begin with the definition of the numbers μ_i^{n+1} $(1 \le i \le k_n)$. We assume that

$$\mu_i^{n+1} = \frac{1}{2(1-\varepsilon_n)} \mu_i^n \quad (1 \leqslant i \leqslant k_n).$$

Condition 6) is satisfied for these numbers, since

$$\sum_{i=1}^{k_n} \mu_i^{n+1} = \frac{1}{2(1-\varepsilon_n)} \sum_{i=1}^{k_n} \mu_i^n = \frac{1}{2(1-\varepsilon_n)} \cdot (1-\varepsilon_n) = \frac{1}{2}.$$

For convenience of exposition, let us agree now to some terminology. Let G_i^N be an arbitrary set of some rank N, and let q be a natural number with $q > N$. If for G_i^N and q the left inequality in (2.3) in the induction hypothesis 9) is violated, then G_i^N will be called unsaturated (with respect to rank q). If the right inequality is violated, we call G_i^N supersaturated. If both inequalities are satisfied, we call G_i^N saturated. Moreover, we call the number $\Sigma\mu_r^s/\mu_i^N$ the level of saturation of the set G_i^N (with respect to rank q). Here the summation extends over those pairs (r, s) which appear in condition 9), exception being made for the following circumstance: for some pairs (r, s) which, according to 9), enter into the summation, the numbers μ_r^s cannot yet be determined; in such a case, these pairs (r, s) are excluded from the summation. Of course, the level of saturation, as well as the property of a set to be saturated, unsaturated, or supersaturated, depends upon which of the μ_r^s are already defined, and which are not.

Turning now to the proof of the lemma, we can easily verify that all sets of rank n are not supersaturated with respect to rank $n+1$, and the level of saturation is the same for them (in fact, if $n > 1$, then they are necesarily unsaturated, but we do not need this).

We will define the numbers μ_i^{n+1} for $i > k_n$ in such a way that sets of all ranks $\leq n$ turn out to be saturated. We first show how to make a set of rank n saturated (if it is not already saturated).

Let

$$^1k^n = \min\left\{k : x_k \in G_1^n ; T^{-1}x_k \overline{\in} \bigcup_{i=1}^{k_n} {}^n\overline{G}_i^n; \right.$$

$$\left. T^{k_n+1}x_k \notin \bigcup_{i=1}^{k_n} {}^n\overline{G}_i^n;\ k > k_n\right\}, \quad ^1\overline{k}^n = {}^1k^n + k_n - 1.$$

From the induction hypotheses it follows that the number $^1k^n$ exists. In fact $T^{-1}x \notin \bigcup_{i=1}^{k_n} {}^n\overline{G}_i^n$, and the point $T^{-1}x$ is recurrent (all points in the trajectory of a recurrent point are recurrent). Therefore each of its neighborhoods contains infinitely many points x_k for $k > k_n$. If we now consider the neighborhood

$$G(T^{-1}x) = T^{-1}(G_1^n) \cap T^{-k_n-1}\left(X \setminus \bigcup_{i=1}^{k_n} {}^n\overline{G}_i^n\right) \cap \left(X \setminus \bigcup_{i=1}^{k_n} {}^n\overline{G}_i^n\right)$$

(this set is nonempty), we will have

$$^1k^n = \min\{k : x_{k-1} \in G(T^{-1}x);\ k > k_n\}.$$

We define the number μ_p^{n+1} for values of p belonging to the segment $[^1k^n, {}^1\overline{k}^n]$. We note that if some value $\mu_{^1k^n}^{n+1} = \lambda$ has been chosen, then the remaining μ_p^{n+1} for this segment are uniquely defined by condition 8). We begin to increase continuously the value of λ, starting from zero; at the same time the values of all $\mu_p^{n+1}(\lambda) = (\mu_p^n/\mu_1^n)\lambda$ for $p \in [^1k^n, {}^1\overline{k}^n]$ are continuously increasing.

It is here possible to distinguish two cases:

a) For that value $\lambda = \overline{\lambda}$ for which

$$\max_{p \in [^1k^n, {}^1\overline{k}^n]} \mu_p^{n+1}(\lambda) = \frac{1}{100(n+1)k_n},$$

sets of rank n are still not unsaturated relative to rank $n+1$. Then from continuity considerations it is clear that for some $\widetilde{\lambda} \in (0, \lambda]$ the first saturations arise. In this case we set $\lambda = \widetilde{\lambda}$ and thereby define the numbers μ_p^{n+1} for $p \in [^1k^n, {}^1\overline{k}^n]$. We then pass to the maximum of the ranks not yet saturated.

β) Case α) does not obtain. Then we put $\lambda = \bar{\lambda}$. This thus defines μ_p^{n+1} for $p \in [{}^1k^n, {}^1\bar{k}^n]$. But we cannot yet pass to the following rank, since the rank n is not yet saturated (saturation of all sets of rank n necessarily happens simultaneously, and we therefore speak of a saturated rank). We take

$$ {}^2k^n = \min\{k : x_{k-1} \in G(T^{-1}x);\ k > {}^1\bar{k}^n\}, \qquad {}^2\bar{k}^n = {}^2k^n + k_n - 1, $$

and define the numbers μ_p^{n+1} for $p \in [{}^2k^n, {}^2\bar{k}^n]$ by the same method as we previously defined them in the interval $[{}^1k^n, {}^1\bar{k}^n]$. In this connection we may also come upon one of two cases: α) or β). In case α) the process of saturation of the nth rank is completed; in case β) it is necessary to give one more step—passage to the interval $[{}^3k^n, {}^3\bar{k}^n]$, which is defined analogously to the previous ones. Since at each step the level of saturation of the nth rank increases by a constant, after a finite number of steps case α) will necessarily be encountered, and rank n will be saturated. We note that in the construction, the level of saturation of each of the ranks $\neq n$ is not altered. We proceed now to the maximal of the ranks which are not yet saturated. We shall, for definiteness, assume that this is rank $n-1$.

Let

$$ {}^1k^{n-1} = \min\left\{k : k > k^n_{\max};\ x_k \in \left[T\left(X \setminus \bigcup_{\substack{1\leqslant q\leqslant n\\ 1\leqslant p\leqslant k_q}} {}^q\bar{G}^q_p\right)\right] \cap \left[G_1^{n-1} \setminus \bigcup_{k=1}^{k_n} {}^n\bar{G}^n_i\right] \cap \left[T^{-k_{n-1}}\left(X \setminus \bigcup_{\substack{1\leqslant q\leqslant n\\ 1\leqslant p\leqslant k_q}} {}^q\bar{G}^q_p\right)\right]\right\}, $$

$$ {}^1\bar{k}^{n-1} = {}^1k^{n-1} + k_{n-1} - 1. $$

Here $k^n_{\max}$ is the maximal number k for which the value of μ_k^{n+1} is determined for the saturation of rank n.

From the inductive hypotheses, it follows that the number ${}^1k^{n-1}$ exists. By the same method as used for saturation of rank n, we define values μ_p^{n+1} on the interval $[{}^1k^{n-1}, {}^1\bar{k}^{n-1}]$; if necessary, then on the interval $[{}^2k^{n-1}, {}^2\bar{k}^{n-1}]$ and so on, until the rank $n-1$ is saturated. Here in this construction, the level of saturation of each rank $\neq n-1$ is not changed. Carrying the process further, we obtain that all sets of rank $\leq n$ turn out to be saturated with respect to rank $n+1$, and the level of saturation will be the same for all sets of a given rank $\leq n$. We now need to take care of fulfilling condition 2) for $N = n+1$. We assume that $\epsilon_n > 2^{-n-1}$. Let

$$ a = \left[\frac{\varepsilon_n - 2^{-n-1}}{100\,(n+1)\,k_n}\right] + 1. $$

From the inductive hypotheses it follows that it is possible to find a sequence of natural numbers $j_1 < j_2 < \cdots < j_a$ satisfying the following conditions:

α) $k_{\max} < j_1$, where $k_{\max}$ is the largest number k for which μ_k^{n+1} has till now been defined.

β) $x_{j_t} \in X \setminus \bigcup_{i=1}^{k_n} {}^n\bar{G}_i^n$ $(t = 1, 2, \ldots, a)$.

We assume that

$$\mu_{j_t} = \frac{\varepsilon_n - 2^{-n-1}}{a} \quad (t = 1, 2, \ldots, a).$$

If $\epsilon_n \leq 2^{-n-1}$, it is not necessary to define the numbers j_t and, naturally, $\mu_{j_t}^{n+1}$.

In any case, we have

$$\sum \mu_r^s \geqslant 1 - \frac{1}{2^{n+1}},$$

where the summation is over those (r, s) for which μ_r^s is already defined, and with respect to which the summation in condition 5) is carried out for $N = n + 1$.

We may now define the number k_{n+1}. Namely, we put

$$k_{n+1} = \min\left\{k : k > j_a;\ x_{k+1} \in X \setminus \bigcup_{i=1}^{k_n} {}^n\bar{G}_i^n\right\}.$$

We shall concern ourselves presently with the definition of the numbers μ_i^{n+1} $(1 \leq i \leq k_{n-1})$ for those i for which this has not yet been done. For this let us agree to call the index i $(1 \leq i \leq k_{n+1})$ bound if from condition 8) with the value μ_i^{n+1} either the value of μ_{i+1}^{n+1} or the value of μ_{i-1}^{n+1} is uniquely defined. In the contrary case, the index i will be called free. We divide the indices i for which we have yet to define μ_i^{n+1} into segments.

A segment is a set of indices i, $1 \leq i \leq k_{n+1}$, satisfying the following conditions:

α) Indices appearing in this set fill some interval $[c; d]$ of natural numbers (possibly $c = d$).

β) For indices appearing in this set, μ_i^{n+1} is not yet defined.

γ) Either all indices appearing in the set are free, or all are bound.

δ) This set cannot be enlarged while retaining properties α), β), and γ).

We will consider segments successively and define the numbers μ_i^{n+1} for each of these. If some segment A is free (consists of free indices), then all μ_i^{n+1} for this segment take the same value, which is sufficiently small so that the following conditions are satisfied:

α) $\mu_i^{n+1} \leq 1/100(n + 1)k_n$ for $i \in A$.

β) $\Sigma \mu_r^s < 1$, where the summation is over the same pairs (r, s) as in condition 5) with $N = n + 1$ (with regard to the fact that μ_r^s must already be defined).

γ) None of the sets G_p^q $(1 \leq q \leq n;\ 1 \leq p \leq k_q)$ is supersaturated with respect to rank $n + 1$.

δ) The conditions in 14) for each of these sets already constructed are not violated.

If some segment is bound (consists of bound indices), then μ_i^{n+1} for it is defined analogously, with the sole difference that in the beginning we take a sufficiently small value for the first μ_i^{n+1} of this segment; subsequently we uniquely define values using condition 8) so long as this is possible (the entire segment is not required to be eliminated here). At the price of the smallness of the first μ_i^{n+1} in the segment, we have satisfied conditions α), β), γ) and δ). If a segment is not exhausted, we take a sufficiently small value for the first of the remaining μ_i^{n+1}, and so on, until the segment is exhausted.

It is necessary to remark here that the level of saturation of the sets G_p^q with respect to rank $n+1$, although it increases, remains equal for all sets of a fixed rank.

And so, with the help of the process described, we define all μ_i^{n+1} $(1 \le i \le k_{n+1})$.

We now construct the sets G_i^{n+1} $(1 \le i \le k_{n+1})$. For this it is sufficient to note that if a sufficiently small neighborhood of the point x was taken as G_1^{n+1}, so that the sets

$$T^i\Gamma_1^{n+1} = T^i(\bar{G}_1^{n+1} \setminus G_1^{n+1}) \quad (i = 0, 1, \ldots, k_{n+1} - 1)$$

are disjoint from the trajectory of the point x, and if we assume henceforth that

$$G_i^{n+1} = T^{i-1}G_1^{n+1} \quad (i = 2, 3, \ldots, k_{n+1}),$$

then all assertions concerning G_i^{n+1} contained in conditions 1)–18) will be fulfilled.

We assume $\epsilon_{n+1} = 1 - \Sigma\mu_r^s$, where the summation is over the same pairs (r, s) used in condition 5) with $N = n+1$. It is easy to see that $0 < \epsilon_{n+1} \le 1/2^{n+1}$.

In order to conclude the proof of the lemma, it remains to construct the sets ${}^{n+1}\Gamma_p^q$ $(1 \le q \le n+1;\ 1 \le p \le k_q)$. Using the fact that the sets Γ_p^q do not intersect the trajectory of x, we may take as ${}^{n+1}\Gamma_p^q$ a sufficiently small neighborhood, not intersecting the set $\{x_i\}_{i=1}^{k_{n+1}}$. At the price of the smallness of the neighborhood, it is possible to satisfy all assertions relating to ${}^{n+1}\Gamma_p^q$. The lemma is proved.

§3. Proof of Theorem 2

The proof will be conducted with the help of transfinite induction. We suppose for all transfinite numbers η smaller than some countable transfinite ξ we have constructed a Borel probability measure μ_η of type III, quasi-invariant with respect to T, for which, if $\eta_1 \ne \eta_2$ and $\eta_1, \eta_2 < \xi$, then the measures μ_{η_1} and μ_{η_2} are singular. We construct the measure μ_ξ. For this all measures $\{\mu_\eta\}$ with $\eta < \xi$ are written in the form of a sequence $\nu_1, \nu_2, \ldots$. It is well known (see [6]) that if a homeomorphism admits a continuous ergodic measure, then it has a recurrent point. Let $x \in X$ be recurrent. Taking any sequence of natural numbers $\{j_n\}$ which contains each natural number an infinite number of times, we may apply Lemma 2.

In what follows we keep the notation given in the proof of Lemma 2. We define on the space X a sequence of Borel measures $\mu_1, \mu_2, \ldots$ in the following manner. If $E \subseteq X$ is a Borel set, let

$$\mu_n(E) = \sum_{x_i \in E} \mu_i^n \quad (n = 1, 2, \ldots).$$

Moreover, if E is one of the sets X, G_p^q, or ${}^l\Gamma_p^q$, and if n is a natural number, we assume that $\lambda_n(E) = \Sigma\mu_r^s$, where the summation ranges over pairs (r, s) such that α) $s \leq n$, β) $G_r^s \subseteq E$, and γ) there do not exist sets $G_{r'}^{s'}$ such that $s' < s$ and $G_r^s \subseteq G_{r'}^{s'} \subseteq E$.

We break the rest of the argument into several subsidiary assertions.

Assertion 1. *If $m > n$, then $\mu_m(G_i^n) < \mu_i^n$ $(1 \leq i \leq k_n)$.*

The proof proceeds by induction on $d = m - n$. If $m - n = 1$, the assertion follows from Lemma 2. Assume it is proved for pairs (m, n) such that $d < d_0$, and let $m - n = d_0$. From Lemma 2 it follows that

$$\lambda_m (G_i^n) < \mu_i^n. \tag{3.1}$$

But $\lambda_m(G_i^n) = \Sigma\mu_r^s$ (we will not reiterate which pairs the summation is over). We shall take any set G_r^s, where (r, s) is one of those pairs. Since $s > n$, it follows that then $m - s < d_0$, and by the induction hypothesis $\mu_m(G_r^s) < \mu_r^s$.

Summing anew over the same pairs (r, s) and taking (3.1) into account, we obtain the desired inequality.

Assertion 2. *For each $\epsilon > 0$ there is a natural number n such that if $s \geq m \geq n$, then*

$$\mu_s(G_i^m) \geqslant (1-\varepsilon)\mu_i^m \quad (1 \leqslant i \leqslant k_m).$$

Proof. We take n such that $2^{-n+1} \leq \epsilon$. Now let G_i^m be any set of rank $m \geq n$. From Lemma 2 it follows that

$$\mu_i^m - \mu_{m+1}(G_i^m) \leqslant \frac{\mu_i^m}{2^m} \leqslant \frac{\mu_i^m}{2^n}.$$

If j is such that $G_j^{m+1} \subseteq G_i^m$, then, as in Lemma 2, we obtain

$$\mu_j^{m+1} - \mu_{m+2}(G_j^{m+1}) \leqslant \frac{\mu_j^{m+1}}{2^{m+1}} \leqslant \frac{\mu_j^{m+1}}{2^{n+1}}.$$

Summing these inequalities for all such j, we obtain

$$\mu_{m+1}(G_i^m) - \sum_j \mu_{m+2}(G_j^{m+1}) \leqslant \frac{\mu_i^m}{2^{n+1}}.$$

Analogously, considering sets G_h^{m+2} lying in some G_j^{m+1}, we find

$$\mu_h^{m+2} - \mu_{m+3}(G_h^{m+2}) \leqslant \frac{\mu_h^{m+2}}{2^{n+2}}.$$

We sum now, first on h, then on j:

$$\sum_j \mu_{m+2}(G_j^{m+1}) - \sum_j \sum_h \mu_{m+3}(G_h^{m+2}) \leqslant \frac{\mu_i^m}{2^{n+2}}$$

and so on for higher ranks. Now it is easy to deduce the required inequality:

$$\mu_i^m - \mu_s(G_i^m) \leqslant [\mu_i^m - \mu_{m+1}(G_i^m)] +$$

$$+ \left[\mu_{m+1}(G_i^m) - \sum_j \mu_{m+2}(G_j^{m+1})\right] + \left[\sum_j \mu_{m+2}(G_j^{m+1}) - \sum_j \sum_h \mu_{m+3}(G_h^{m+2})\right] + \dots,$$

where there are $s-m$ square brackets on the right-hand side. This means

$$\mu_i^m - \mu_s(G_i^m) \leqslant \left(\frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots + \frac{1}{2^{s-1}}\right)\mu_i^m < \frac{\mu_i^m}{2^{n-1}} \leqslant \varepsilon\mu_i^m.$$

Assertion 3. $\mu_m(G_i^n) \to \mu_i^n$ *as* $m \to \infty$ $(n = 1, 2, \dots;\ 1 \le i \le k_N)$.

Proof. From Assertion 1 it follows that it is sufficient to prove only the inequality

$$\varliminf_{m\to\infty} \mu_m(G_i^n) \geqslant \mu_i^n.$$

We take $\epsilon > 0$. Using Assertion 2, we find an N such that if $s \ge m \ge N$, then

$$\mu_s(G_j^m) \geqslant \left(1 - \frac{\varepsilon}{2}\right)\mu_j^m \quad (1 \leqslant j \leqslant k_m).$$

Now by Lemma 2 we find a $q > n$ such that

$$\lambda_q(G_i^n) \geqslant \left(1 - \frac{\varepsilon}{4}\right)\mu_i^n.$$

We note, as above, that $\lambda_q(G_i^n) = \Sigma\mu_r^s$. We consider all possible sets G_r^s, where (r, s) is a pair used in the summation. Their ranks are $\ge n+1$. For each G_r^s using Lemma 2, we find $t > s$ such that

$$\lambda_t(G_r^s) \geqslant \left(1 - \frac{\varepsilon}{8}\right)\mu_i^s.$$

As above, we may consider sets corresponding to all possible summands entering in $\lambda_t(G_r^s)$. Varying the pairs (r, s), we obtain a collection of nonintersecting sets lying in G_i^n, the ranks of which are $\ge n+2$. Continuing this process, we obtain, after a finite number of steps, a collection of sets whose ranks are $\ge N$. Let this collection be $\{G_j^m\}$. We take s larger than the maximal of the ranks of G_j^m. As was proved earlier,

$$\mu_s(G_j^m) \geqslant \left(1 - \frac{\varepsilon}{2}\right)\mu_j^m.$$

Summing these inequalities over all pairs (j, m) which appear in our collection, we obtain

$$\mu_s(G_i^n) \geqslant \sum_{(j,m)} \mu_s(G_j^m) \geqslant (1 - \varepsilon)\,\mu_i^n.$$

Assertion 4. $\mu_s(X) \to 1$ *as* $s \to \infty$.

Proof. From Assertion 1 it follows that $\mu_s(X) \leq \lambda_s(X)$ $(s = 1, 2, \ldots)$. Moreover, by Lemma 2, $\lambda_s(X) \leq 1$. Therefore it suffices to prove that

$$\underline{\lim}\ \mu_s(X) \geqslant 1.$$

We take $\epsilon > 0$. By Lemma 2 we find an n such that $\lambda_n(X) \geq 1 - \epsilon/2$.

Furthermore, using Assertion 2, we find a q such that for any $m \geq q$ and any $t \geq m$

$$\mu_t(G_j^m) \geqslant \left(1 - \frac{\varepsilon}{2}\right)\mu_j^m \quad (j = 1, 2, \ldots, k_m).$$

We recall that $\lambda_n(X_n) = \Sigma \mu_r^s$. Among all sets G_r^s corresponding to pairs (r, s) which appear in the summation, we consider those whose rank is $< q$. For each of these we find, using Assertion 3, an integer $N = N(r, s)$ such that for $t \geq N(r, s)$ we will have

$$\mu_t(G_s^r) \geqslant \left(1 - \frac{\varepsilon}{2}\right)\mu_r^s.$$

If $s \geq q$ is chosen so that $s \geq \max\{N(r, s)\}$, then we obtain $\mu_s(X) \geq 1 - \epsilon$. The assertion is proved.

Now it is clear that the sequence of measures $\{\mu_n\}$ is weakly compact and if μ is any of its weak limit points, then μ is a Borel probability measure. In order not to introduce an additional index to denote a subsequence, we will assume that $\mu = \lim_{n\to\infty}\mu_n$.

It remains for us to show that we have the right to suppose $\mu_\xi = \mu$.

Assertion 5. $\mu(\Gamma_p^q) = 0$ $(q = 1, 2, \ldots;\ 1 \leq p \leq k_q)$.

Proof. We take $\epsilon > 0$. Let $N \geq \max\{q, 1/\epsilon\}$. From Lemma 2 it follows that for $m \geq N$

$$\lambda_m({}^N\Gamma_p^q) \leqslant \frac{1}{N}.$$

Using Assertion 1, we obtain that $\mu_m({}^N\Gamma_p^q) \leq 1/N$. We can find an open set ${}^N\tilde{\Gamma}_p^q$ such that the μ-measure of its boundary is equal to zero, and $\Gamma_p^q \subseteq {}^N\tilde{\Gamma}_p^q \subseteq {}^N\Gamma_p^q$. It is clear that

$$\mu_m({}^N\tilde{\Gamma}_p^q) \leqslant \frac{1}{N}.$$

But by a theorem concerning the convergence of probability measures (see [6]) we have $\mu_m({}^N\tilde{\Gamma}_p^q) \to \mu({}^N\tilde{\Gamma}_p^q)$. This implies

$$\mu(\Gamma_p^q) \leqslant \mu({}^N\tilde{\Gamma}_p^q) \leqslant \frac{1}{N} \leqslant \varepsilon.$$

The assertion is proved. From it, with the help of the same theorem concerning the convergence of probability measures, we obtain

$$\mu(G_i^n) = \mu_i^n \quad (n = 1, 2, \ldots;\ 1 \leqslant i \leqslant k_n).$$

Assertion 6. *If A is a Borel set and $A \subseteq G_i^n$ ($n = 1, 2, \ldots$; $1 \leq i < k_n$), then*

$$\frac{\mu(TA)}{\mu(A)} = \frac{\mu(G_{i+1}^n)}{\mu(G_i^n)}.$$

Proof. To begin with we assume that A is a closed set whose boundary has μ-measure equal to zero. Then for $m \geq n$ we have

$$\frac{\mu_m(TA)}{\mu_m(A)} = \frac{\mu_m(G_{i+1}^n)}{\mu_m(G_i^n)}.$$

Passing to the limit as $m \to \infty$, we obtain the assertion for such A.

Since an arbitrary closed set can be represented in the form of an intersection of a decreasing sequence of open sets whose boundaries have μ-measure zero, the assertion is proved for all closed sets, and hence for all Borel sets.

Assertion 7. $\omega_m = \max_{1 \leq i \leq k_m} \{\mu_i^m\} \to 0$ *as* $m \to \infty$.

Proof. $\omega_m = \max\{\delta_m, \gamma_m\}$, where

$$\delta_m = \max_{1 \leqslant i \leqslant k_{m-1}} \{\mu_i^m\}, \quad \gamma_m = \max_{k_{m-1} < i \leqslant k_m} \{\mu_i^m\}.$$

But by Lemma 2, $\gamma_m \leq 1/m$. On the other hand,

$$\delta_m \leqslant \frac{1}{2(1-\varepsilon_{m-1})} \delta_{m-1} \leqslant \frac{1}{2^2(1-\varepsilon_{m-1})(1-\varepsilon_{m-2})} \delta_{m-2} \leqslant \ldots$$

$$\ldots \leqslant \frac{1}{2^{m-2}} \left[\prod_{n=2}^{m-1} (1-\varepsilon_n)\right]^{-1} \delta_2.$$

Since the series $\Sigma_1^\infty \epsilon_n$ converges, there exists a constant $C > 0$ such that

$$\prod_{n=2}^{\infty} (1-\varepsilon_n) \geqslant C.$$

Therefore $\delta_m \leq C \cdot \delta_2 / 2^{m-2}$.

Assertion 8. *The measure μ is quasi-invariant with respect to T.*

Proof. Let $A \subseteq X$ be a Borel set with $\mu(A) > 0$. Using Assertions 4 and 7, we can find numbers m and i such that $1 < i < k_m$ and $\mu(A \cap G_i^m) > 0$. From Assertion 6 it follows that

$$\mu(T^{-1}A \cap G_{i-1}^m) > 0, \quad \mu(TA \cap G_{i+1}^m) > 0.$$

Assertion 9. *The measure μ is ergodic with respect to T.*

Proof. Let the set E, $\mu(E) > 0$, be invariant with respect to T and let $\epsilon > 0$ be given. We first find a number m_0 such that for $m \geq m_0$

$$\mu\left(\bigcup_{i=1}^{k_m} G_i^m\right) \geqslant 1 - \frac{\varepsilon}{2}.$$

From Lemma 2 it follows that the collection of sets

$$\{G_i^m : m = m_0,\ m_0+1,\ \ldots;\ 1 \leqslant i \leqslant k_m\}$$

generates the entire σ-algebra of Borel sets of the space (X, μ). Therefore we can find numbers m and i such that $m \geq m_0$, $1 \leq i \leq k_m$, and

$$\mu(E \cap G_i^m) > \left(1 - \frac{\varepsilon}{2}\right)\mu(G_i^m).$$

But then for all j, $1 \leq j \leq k_m$, we will have

$$\mu(E \cap G_j^m) > \left(1 - \frac{\varepsilon}{2}\right)\mu(G_j^m),$$

whence it follows that $\mu(E) \geq 1 - \epsilon$.

Assertion 10. *For each $\eta < \xi$ the measures μ and μ_η are singular.*

Proof. Let, for example, $\mu_\eta = \nu_m$. We take an infinite increasing sequence $\{n_l\}_1^\infty$ such that $j_{n_l} = m$. Selecting, if need be, some subsequence (which we also designate $\{n_l\}$), we can ensure, by virtue of Assertion 4, the inequality

$$\mu\left(\bigcup_{i=1}^{k_{n_l}} G_i^{n_l}\right) > 1 - \frac{1}{2^l} \quad (l = 1, 2, \ldots).$$

Moreover, by Lemma 2

$$\mu_\eta\left(\bigcup_{i=1}^{k_{n_l}} G_i^{n_l}\right) \leqslant \frac{1}{2^l} \quad (l = 1, 2, \ldots).$$

Let

$$G_l = \bigcup_{i=1}^{k_{n_l}} G_i^{n_l}, \quad E_l = X \setminus G_l,$$

$$G = \underline{\lim}\, G_l = \bigcup_{s=1}^{\infty} \bigcap_{l=s}^{\infty} G_l, \quad E = \underline{\lim}\, E_l = \bigcup_{s=1}^{\infty} \bigcap_{l=s}^{\infty} E_l.$$

Then $E \cap G = \varnothing$ and $\mu(G) = \mu_\eta(E) = 1$. The assertion is proved.

From Lemma 2 it follows that μ is a measure of type III. Setting $\mu_\xi = \mu$, we conclude the transfinite induction step, and with it the proof of the theorem.

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