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CORRELATION DECAY AND GAP OF THE TRANSFER OPERATOR

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Abstract. The aim of this article is to present a new proof for a universal estimate of the gap between the two largest eigenvalues of the Kac operator in the case of a convex potential. This is done by means of a careful analysis of the technique of the transfer matrix in connection with the recent study of the decay of correlations associated to Laplace integrals by Helffer-Sjöstrand and Sjöstrand.

§1. Introduction

In collaboration with J. Sjöstrand, we gave in [18] a rather general approach to estimation of the decay of correlations attached to "Gaussian like" measures of type

$$\exp -\Phi(X)dX \tag{1.1}$$

on $\mathbb{R}^n$, with Φ convex. More general and refined estimates were given by J. Sjöstrand in [33] and were applied in [13] to estimation of the splitting between the two largest eigenvalues of the Kac operator. We give here a new proof of this estimate. Our version of the proof is based on the analysis of the transfer matrix. This method was suggested to us by M. Derrida and is very standard at least in the study of the Ising model. In order to present this new approach, we consider a particular case where $n = mp$ and then adopt a matrix notation for the variable X:

$$X = x_j^i,$$

where $i = 1, \dots, m$ corresponds to the rows and $j = 1, \dots, p$ corresponds to the columns. We use alternatively the notation $X = (x_1, \dots, x_p)$ with $x_j \in \mathbb{R}^m$, or $X = (x^1, \dots, x^m)$ with $x^j \in \mathbb{R}^p$. We consider the following particular potential Φ:

$$\Phi^{(p)}(X) \equiv \Phi(X) \equiv \sum_{j=1}^{p} V(x_j) + \frac{|x_j - x_{j+1}|^2}{4h^2}, \tag{1.2}$$

where we agree that $x_{p+1} = x_1$ and where h can serve as a semiclassical parameter; h is sometimes chosen equal to one if we are not interested in the "semiclassical" aspects. More generally, we could consider examples of the form

$$\Phi(X) = \sum_{j=1}^{p}(V(x_j) + I(x_j, x_{j+1})), \tag{1.3}$$

where I is a symmetric, "interaction" potential on $\mathbb{R}^m \times \mathbb{R}^m$ (since this will not be necessary for our purpose), we shall not advance in this direction). However, we mention that the example (1.2) appears naturally in quantum field theory when the so called "lattice approximation" is introduced. For this special class of potentials, we shall see that together with the technique mentioned before we can also use the "classical" technique of the transfer matrix. We shall demonstrate that the pieces of information given by these two points of view are complementary. Our main object in this article is to establish the "dictionary" between the properties of the measure $\exp -\Phi(X)dX$ and the spectral properties of the transfer matrix (also called the Kac operator in former papers)

$$K_V(x,y) = \exp -\frac{V(x)}{2} . \exp -\frac{|x-y|^2}{4h^2} . \exp -\frac{V(y)}{2} . \tag{1.4}$$

Employing this dictionary more intensively than in [13], we can improve the majoration of the quotient μ_2/μ_1 of the first two largest eigenvalues of the transfer matrix and, also, improve the control of the rate convergence of thermodynamic quantities. In particular, for fixed m and h, this rate of convergence can be exponentially rapid as $p \to \infty$. For proving this we do not need the convexity of the potential. The phenomenon is similar to what is observed in the one-dimensional Ising model. In particular, there is apparently no phase transition. But all these problems reappear in the semiclassical context $h \to 0$ or in the limit $m \to \infty$. Actually, this our initial motivation, following M. Kac, for the study of these questions.

§2. The dictionary

We first recall some properties of the Kac operator (or transfer matrix). This operator is associated with a potential V. We do not try to give the minimal assumptions which must be imposed on V; in order to simplify the discussion, we assume the following two conditions:

$$V(y) \geq (1/C)|y|^2 - C \tag{2.1}$$

for some strictly positive constant C, and

$$\forall \alpha \in \mathbb{Z}^m,\ \exists C_\alpha \quad \text{s.t. } |D_y^\alpha V(y)| \leq C_\alpha < y >^{(2-|\alpha|)_+} . \tag{2.2}$$

Under these assumptions, K_V is a compact strictly positive selfadjoint operator with strictly positive kernel. In particular, it satisfies the assumptions of the extended Perron-Frobenius Theorem (see, for example, Theorem 3.3.2 in [6]) also called Krein-Rutman Theorem; consequently, K_V admits a largest eigenvalue μ_1 equal to $||K_V||$ which is simple and corresponds to a unique strictly positive normalized eigenfunction u_1. Let μ_j be the sequence of eigenvalues ordered as a decreasing sequence tending to 0:

$$0 \leq \mu_{j+1} \leq \mu_j \leq \ldots \leq \mu_2 < \mu_1.$$

Thermodynamic limit. First, we look at the thermodynamic limit. We start with the decomposition

$$\exp -\Phi(X) = K_V(x_1, x_2).K_V(x_2, x_3)\ldots K_V(x_{p-1}, x_p).K_V(x_p, x_1) \tag{2.3}$$

and observe that

$$\int\limits_{\mathbb{R}^{mp}} \exp -\Phi(X)dX = \int\limits_{\mathbb{R}^m} K_V^{(p)}(y, y)dy, \tag{2.4}$$

where $K_V^{(p)}(x, y)$ is the distribution kernel of $(K_V)^p$. Our assumption on V implies that $(K_V)^p$ is in the trace class; so, we can rewrite (2.4) in the form

$$\int\limits_{\mathbb{R}^{mp}} \exp -\Phi(X)dX = \operatorname{tr}[(K_V)^p] = \sum_j \mu_j^p. \tag{2.5}$$

In particular, we get

$$\lim_{p \to \infty} \frac{\ln \int_{\mathbb{R}^{mp}} \exp -\Phi(X)dX}{p} = \ln \mu_1, \tag{2.6}$$

and the rate of convergence is easily estimated:

$$\left| \frac{\ln \int_{\mathbb{R}^{mp}} \exp -\Phi(X)dX}{p} - \ln \mu_1 \right| = \frac{k_2}{p} \left(\frac{\mu_2}{\mu_1}\right)^p [1 + \mathcal{O}(\exp -\delta_2 p)]. \tag{2.7}$$

Of course, the number $\delta_2 > 0$ and the multiplicity k_2 of μ_2 are usually not under control (and depend, in particular, on m) but we shall only use the trivial minoration for k_2 by one. We shall use (2.7) in the form

$$\begin{aligned} &- \ln \left| \frac{\ln \int_{\mathbb{R}^{mp}} \exp -\Phi(X)dX}{p} - \ln \mu_1 \right| \\ &\quad = -p \ln \left(\frac{\mu_2}{\mu_1}\right) - \ln k_2 + \ln p + \mathcal{O}(\exp -\delta_2 p). \end{aligned} \tag{2.8}$$

If $k_2 \geq 1$, the two dominant terms in the second line of (2.8) are independent of k_2, δ_2. Relation (2.8) can be used in two directions. Since we know that $\frac{\mu_2}{\mu_1} < 1$, we get immediately the exponential rate of convergence of the thermodynamic limit. The convergence is exponential and the rate is given by $-\ln(\frac{\mu_2}{\mu_1})$:

$$\lim_{p\to\infty} -\frac{1}{p}\ln\left|\frac{\ln\int_{\mathbb{R}^{mp}}\exp-\Phi(X)dX}{p} - \ln\mu_1\right| = \ln\left(\frac{\mu_2}{\mu_1}\right). \tag{2.9}$$

Conversely, if a majorant for the speed of convergence is known, we get as a consequence a majorant for $\frac{\mu_2}{\mu_1}$. We shall present here the following improvement of a result from [10].

Theorem 2.1. *If for some* $\sigma > 0$

$$\mathrm{Hess}\, V(x) \geq \sigma \tag{2.10}$$

for all $x \in \mathbb{R}^m$, *then*

$$\frac{\mu_2}{\mu_1} \leq \exp - \left[\cosh^{-1}[h^2\sigma + 1]\right]. \tag{2.11}$$

Remark 2.2. Here $\cosh^{-1} t$ is the positive solution of $\cosh u = t$ for $t \geq 1$, that is $\cosh^{-1} t = \ln(t + \sqrt{t^2 - 1})$. We can rewrite (2.11) as follows:

$$\frac{\mu_2}{\mu_1} \leq \frac{1}{1 + h^2\sigma + \sqrt{2\sigma h^2 + \sigma^2 h^4}}. \tag{2.12}$$

Remark 2.3. In [13], we got only the following not optimal result:

$$\frac{\mu_2}{\mu_1} \leq \exp - \left[\cosh^{-1}[h^2\sigma + 1]\right]/2.$$

Our new result is optimal in the sense that in the quadratic case where $V(y) = \sum_{j=1}^m \sigma_j y_j^2/2$ we have for $\sigma = \inf \sigma_j$ the equality

$$\frac{\mu_2}{\mu_1} = \exp - \left[\cosh^{-1}[h^2\sigma + 1]\right]. \tag{2.13}$$

The difficulty in [13] was that we have to control the 4-th order correlations. We think that it is possible to improve this technique using the ideas given by J. Sjöstrand in his proof of the corresponding result in the Schrödinger case (cf. [31]). Here we choose a different approach.

For reference, we recall the result of J. Sjöstrand [31], which looks very classical but was obtained only recently using the Maximum Principle (in the same spirit as in [28]):

Theorem2.4. *If, for some* $\sigma > 0$,

$$\text{Hess}\, V(x) \geq \sigma \tag{2.14}$$

for all $x \in \mathbb{R}^m$*, then the splitting between the first two eigenvalues* λ_2 *and* λ_1*of the Schrödinger operator* $-h^2\Delta + V$ *satisfies*

$$\lambda_2 - \lambda_1 \geq \sqrt{2\sigma} h. \tag{2.15}$$

Remark 2.5. In the semi-classical limit, the semiclassical technique allows us to prove that (see for example [2])

$$\frac{\mu_2(h)}{\mu_1(h)} = \exp(\lambda_1(h) - \lambda_2(h)) + \mathcal{O}(h^2).$$

An easy computation shows that the two results are coherent. We shall return to the link between the two statements in Section 4.

Mean value. Continuing the dictionary, now we consider the mean-value with respect to our measure of some cylindrical functions. More precisely, if f is a function of x_1 in $C_0^\infty(\mathbb{R}^m)$, we introduce

$$\langle f^{(1)} \rangle_p = \frac{\int f(x_1) \exp -\Phi(X) dX}{\int \exp -\Phi(X) dX}\,. \tag{2.16}$$

An immediate computation gives the formula

$$\langle f^{(1)} \rangle_p = \frac{\sum_j \mu_j^p (\int f(y) u_j(y)^2 dy)}{\sum_j \mu_j^p}\,. \tag{2.17}$$

In particular, we obtain

$$\lim_{p \to \infty} \langle f^{(1)} \rangle_p = \int f(y) u_1(y)^2 dy\,, \tag{2.18}$$

and, moreover, we have the following control of the above convergence:

$$\langle f^{(1)} \rangle_p - \int f(y) u_j(y)^2 dy = \mathcal{O}\Big(\Big(\frac{\mu_2}{\mu_1}\Big)^p\Big). \tag{2.19}$$

Pair correlation. We pass to the correlation functions. If f and g are two functions in $C_0^\infty(\mathbb{R}^m)$, we denote by $f^{(j)}$ the corresponding function on $\mathbb{R}^{mp}$:

$$X \to f(x_j)\ .$$

We shall now compute the pair correlation:

$$\mathrm{Cor}^{(p)}(f^{(i)}, g^{(j)}) = \langle (f^{(i)} - \langle f^{(i)}\rangle_p)(g^{(j)} - \langle g^{(j)}\rangle_p)\rangle_p\ .$$

Since everything is invariant under circular permutations, we can take $i = 1$. First, we have

$$\begin{aligned}\langle f^{(1)} g^{(j)}\rangle_p &= \frac{\int f(x_1)K^{(j)}(x_1, x_j)g(x_j)K^{(p-j)}(x_j, x_1)dx_1.dx_j}{\sum_i \mu_i^p}\\ &= \frac{\sum_{k,l} \mu_k^j \mu_l^{p-j}[\int f(y)u_k(y)u_l(y)dy][\int g(y)u_k(y)u_l(y)dy]}{\sum_i \mu_i^p}\ .\end{aligned}$$

In particular, if $f = g$ (and this will be the case of interest for us), we get

$$\langle f^{(1)} f^{(j)}\rangle_p = \frac{\sum_{k,l} \mu_k^j \mu_l^{p-j}[\int f(y)u_k(y)u_l(y)dy]^2}{\sum_i \mu_i^p}\ ,$$

and

$$\begin{aligned}&\Big(\sum_i \mu_i^p\Big)^2 . \mathrm{Cor}^{(p)}(f^{(1)}, f^{(j)})\\ &\quad= \Big[\sum_{k,l} \mu_k^j \mu_l^{p-j}\Big[\int f(y)u_k(y)u_l(y)dy\Big]^2\Big]\Big(\sum_i \mu_i^p\Big)\\ &\qquad- \Big[\sum_k \mu_k^p\Big(\int f(y)u_k^2(y)\,dy\Big)\Big]^2\\ &\quad= \Big[\sum_k \mu_k^p\Big[\int f(y)u_k(y)^2\,dy\Big]^2\Big]\Big(\sum_i \mu_i^p\Big)\\ &\qquad- \Big[\sum_k \mu_k^p\Big(\int f u_k^2\Big)\Big]^2\\ &\qquad+ \Big[\sum_{k\neq l} \mu_k^j \mu_l^{p-j}\Big[\int f(y)u_k(y)u_l(y)dy\Big]^2\Big]\Big(\sum_i \mu_i^p\Big).\end{aligned}$$

Using the Cauchy-Schwarz inequality, we see that this expression is minorized by

$$\Big[\sum_{k\neq l} \mu_k^j \mu_l^{p-j}\Big[\int f(y)u_k(y)u_l(y)dy\Big]^2\Big]\Big(\sum_i \mu_i^p\Big)\,.$$

Finally, we arrive at the following inequality:

$$\mathrm{Cor}^{(p)}(f^{(1)},f^{(j)}) \geq \frac{\sum_{k\neq l} \mu_k^j \mu_l^{p-j}[\int f(y)u_k(y)u_l(y)dy]^2}{\sum_i \mu_i^p}\,. \tag{2.20}$$

Assuming that $p \gg j$, we keep the term corresponding to $l = 1$, $k = 2$ in the preceding inequality:

$$\mathrm{Cor}^{(p)}(f^{(1)},f^{(j)}) \geq \frac{\mu_2^j \mu_1^{p-j}[\int f(y)u_2(y)u_1(y)dy]^2}{\sum_i \mu_i^p}\,. \tag{2.21}$$

The passage to the limit as $p \to \infty$ yields

$$\liminf_{p\to\infty}\Big[\mathrm{Cor}^{(p)}(f^{(1)},f^{(j)})\Big] \geq \Big(\frac{\mu_2}{\mu_1}\Big)^j \Big[\int f(y)u_2(y)u_1(y)dy\Big]^2\,. \tag{2.22}$$

If it is known how to control the decay with respect to j uniformly with respect to p ($p \gg j$), and if we can choose f such that $\int f(y)u_2(y)u_1(y)dy \neq 0$, then we find a way to get a new estimate for $\frac{\mu_2}{\mu_1}$. We recall that we can extend our estimates can be extended to the case where f is C^∞ but with bounded gradient (see [18]), and a possible choice for f is $f = u_2$. Here we observe that if μ_2 is not of multiplicity one, then u_2 is not uniquely determined, but coresponds to some arbitrary choice of a second eigenvector. Another natural choice is $f(y) = y^l$ for some l, but it is not always true that $\int y^l u_2(y)u_1(y)dy \neq 0$ for some l. We note here that this condition appears, e.g., in [27], and the above possibility to choose another f is quite useful. So, our idea is to use the results of our recent study [18] of the correlation functions (and some extensions based on an improvement of J. Sjöstrand (see [33]) in the same spirit as in [13] or [14]) in order to prove a universal estimate for μ_2/μ_1.

§3. Estimates of correlations with the help of the Maximum Principle:

In this section we describe briefly how we can control the pair correlation using the Maximum Principle. The basic idea of the proofs is essentially the same as in [18], but we need further improvements given by Sjöstrand in [33] and already used or presented in [12], [13] and [14]. First, we recall the following theorem.

Theorem 3.1. *Let* $\Phi \in C^\infty(\mathbb{R}^m;\mathbb{R})$ *satisfy*

$$\text{Hess}\,\Phi(x) \geq \sigma \quad \textit{for all} \quad x \in \mathbb{R}^m, \tag{3.1}$$

$$|\partial^\alpha \Phi(x)| \leq C \quad \textit{for all} \quad \alpha \quad \textit{such that} |\alpha| \geq 2. \tag{3.2}$$

Let $g \in C^1(\mathbb{R}^m;\mathbb{R})$ *such that* $x \to \nabla g(x)$ *is bounded. Then the equation*

$$g - \langle g\rangle = \nabla_x \Phi \nabla u - \Delta u \quad \text{in } \mathbb{R}^m \tag{3.3}$$

admits a unique temperate solution u *such that*

$$u \in L^2(\mathbb{R}^m, \exp -\Phi.dx) \tag{3.4}$$

and

$$\langle u\rangle = 0. \tag{3.5}$$

Moreover, the gradient of this solution is bounded, and

$$\sup_x ||\nabla u(x)|| \leq \frac{1}{\sigma} \sup_x ||\nabla g(x)||. \tag{3.6}$$

Sketch of the proof. As observed in [18], the operator A

$$u \to \nabla_x \Phi \nabla u - \Delta u \tag{3.7}$$

turns to be essentially selfadjoint (starting from C_0^∞) on the weighted space $L^2(\mathbb{R}^m, \exp -\Phi.dx)$. By using the conjugation $f \to f \exp -\Phi/2$, the problem of existence of solutions can be reduced to the study of a "pseudo" harmonic oscillator:

$$f \to -\Delta f + \frac{1}{4}|\nabla\Phi|^2 f - \Delta\Phi f\,. \tag{3.8}$$

Under the weaker condition that, for some $C > 0$,

$$\text{Hess}\,\Phi(x) \geq \sigma\,, \text{ for all } \quad x \in \mathbb{R}^m \setminus B(O,C)\,, \tag{3.9}$$

and condition (3.2), using standard technique we find that this new operator $\widetilde{A}$ admits a unique selfadjoint extension; this determines a positive operator whose domain is characterized as follows:

$$B^2(\mathbb{R}^m) = \{u \in L^2(\mathbb{R}^m) \mid x^\alpha \partial_x^\beta u \in L^2(\mathbb{R}^m) \quad \text{for all } (\alpha,\beta) \text{ such that } |\alpha| + |\beta| \leq 2\};$$

consequently, this operator has compact resolvent. The smallest eigenvalue is 0 and the first eigenfunction is the function $x \to \exp -\Phi(x)/2$. The first part of the statement can be viewed as a consequence of the Fredhom alternative. As for the other statements, we come back to A and apply a regularization argument in order to work with the Maximum Principle. This argument consists in working with a function Φ of the form $x^2/2 + \Psi$ with Ψ compactly supported, and in obtaining more precise information about the solution constructed in the first part of the proof. As proved in [31] (see also [18] and [33]), we observe that if in equation (3.3) g belongs to C_0^∞, then the solution u has the property that $\nabla u(x)$ tends to 0 as $|x|$ tends to ∞ and that the same property is true for the higher derivatives of u. One way to prove this property, which is also useful for application of the Maximum Principle, is to use the following observation.

Remark 3.2. If u is a solution of (3.3), then $v =: \nabla u$ is a solution of the "basic equation"

$$w = (-\Delta + \nabla\Phi.\nabla)v + (\text{Hess}\,\Phi)v \tag{3.10}$$

with

$$w = \nabla g. \tag{3.11}$$

Proof of (3.6), the Maximum Principle. In order to prove the last part of the theorem we apply the Maximum Principle to the solution of the basic equation. Let x_0 be a point where $||v(x)||$ is maximal. We take the scalar product with $v(x_0)$ in (3.10) and apply the Maximum Principle to the function $x \to \langle v(x)|v(x_0)\rangle$. Using the assumption (3.1) we obtain (3.6). Note that the argument is valid because $||v(x)||$ was assumed to tend to 0 at ∞.

As explained in [31, 18] or, more recently, in [33], a regularization argument is can then be applied in order to treat the general case.

The generalized Maximum Principle of Sjöstrand. As already mentioned, we need a more sophisticated version of the preceding theorem. Here we follow the presentation given in [33]. Let B be a finite $n-$dimensional real normed space and $B^\star$ the dual space. Let $A : B \to B$ be a linear map, and let $\delta \geq 0$. We say that A satisfies $MP(\delta)$ with respect to B (where "MP" is in reference to "Maximum Principle"), if we have the following property:

$$\text{if } t \in B, s \in B^\star \quad \text{and} \quad \langle t, s\rangle = ||t||_B||s||_{B^\star}, \tag{3.12}$$

$$\text{then } \langle At, s\rangle \geq \delta ||t||_B||s||_{B^\star}. \tag{3.13}$$

Basic example. Our application corresponds to $B = \oplus_{j=1}^{p} E_j$ with $E_j = \ell^2(\mathbb{Z}/m\mathbb{Z})$, where the norm on B is defined by the formula

$$||x||_B = \sup_j(\rho(j)||x_j||_{E_j}).$$

Our matrix A is Hess $\Phi^{(p)}$; the $\Phi^{(p)}$ was introduced in (1.2). Then, if Hess $V(x) \geq \sigma$ for all x , and if ρ is a weight on $\mathbb{Z}/p\mathbb{Z}$ such that

$$\exp -\kappa \leq \rho(j+1)/\rho(j) \leq \exp \kappa \tag{3.14}$$

for every κ satisfying

$$\cosh \kappa < 1 + \sigma, \tag{3.15}$$

then the matrix A satisfies $MP(\delta)$ with respect to B for

$$\delta = -\cosh \kappa + 1 + \sigma. \tag{3.16}$$

Indeed, we have $B^\star = \oplus_{j=1}^{p} E_j$, and the norm on $B^\star$ can be defined by

$$||x||_{B^\star} = \sum_j \frac{1}{\rho(j)} ||x_j||_{E_j}.$$

Now, if (3.12) is satisfied for some t in B and some s in $B^\star$, then we have necessarily

$$t_j = \lambda_j s_j, \quad \text{for } \lambda_j > 0$$

(we have used the Hilbertian character of E_j). Using this property, we easily complete the proof.

Application of the MP-property (cf. [33]). Returning to the situation analyzed above for the second basic equation, we assume that Hess $\Phi(x)$ satisfies $MP(\delta)$ for all $x \in \mathbb{R}^n$ for some fixed $\delta > 0$. We also assume that $v(x) = \nabla u(x)$ tends to 0 as $|x| \to \infty$. Let $x_0 \in \mathbb{R}^n$ be a point where $||v(x)||_B$ is maximal:

$$M_v = \sup_x ||v(x)||_B = ||v(x_0)||_B.$$

Let $s \in B^\star$ be a unit vector satisfying

$$\langle v(x)|s\rangle = M_v.$$

In particular, we have

$$\langle v(x)|s\rangle = ||v(x)||_B \,.\, ||s||_{B^\star}.$$

We now take the duality product with s in the basic equation. At the point x_0, we observe that

$$\langle -\Delta v(x_0)|s\rangle \geq 0$$

and

$$\langle \nabla\Phi.\nabla v(x_0)|s\rangle = 0.$$

Finally, using the $MP(\delta)$-property for Hess $\Phi(x_0)$, we get

$$\delta M_v \leq M_w,$$

or, more explicitely,

$$||\nabla u(x)||_B \leq \frac{1}{\delta}\sup_x ||\nabla g(x)||_B. \tag{3.17}$$

We observe that this estimate remains valid after the regularization procedure. Taking $B = \ell^2$, we recover the estimate (3.6).

Proof of Theorem 2.1. As mentioned after (2.22) ; we have to control the pair correlation $\mathrm{Cor}^{(p)}(f^{(i)}, f^{(j)})$. We employ here a trick used in [18], which is merely a suitable integration by parts formula. First, we solve, using Theorem 2.1, the equation:

$$f^{(j)} - \langle f^{(j)}\rangle_p = \nabla_X\Phi.\nabla_X u^{(j)} - \Delta_X u^{(j)}; \tag{3.18}$$

integrating by parts, we obtain immediately

$$\mathrm{Cor}^{(p)}(f^{(i)}, f^{(j)}) = \langle \nabla_X f^{(i)}.\nabla_X u^{(j)}\rangle_p. \tag{3.19}$$

Our estimate of the correlation will then be an immediate consequence of an estimate of $\nabla_X f^{(i)}.\nabla_X u^{(j)}$. Observe that the function depends only on the variable x_i; so, we must control the expression $(\nabla_y f)(x^{(i)}).\nabla_{x_i} u(j)$ with respect to i, j and uniformly with respect to p for p large. By Cauchy-Schwarz, this will result from an estimate of $||\nabla_{x_i} u^{(j)}(X)||_{\ell^2(\{1,\ldots,m\})}$ uniformly with respect to X in $\mathbb{R}^{mp}$. This estimate is an immediate consequence of the control of $||\nabla_X u(j)(X)||_B$ where B in the space introduced after (3.13). Our weight is $\rho_j(k) = \exp \kappa d_{\mathbb{Z}/p\mathbb{Z}}(k, j)$, where κ satisfies (3.15). Applying (3.17) with $g = f^{(j)}$ and $u = u^{(j)}$, we see that

$$||\nabla_X f^{(j)}(X)||_B \leq C(m),$$

where $C(m)$ is independent of i, j, p satisfying $1 \leq i \leq j \leq p/2$. This implies the same property of $||\nabla_X u^{(j)}(X)||_B$; consequently,

$$||\nabla_{x_i} u^{(j)}(X)||_{\ell^2(\{1,\dots,m\})} \leq C(m,\kappa) \exp -\kappa d_{\mathbb{Z}/p\mathbb{Z}}(i,j)$$

for any κ satisfying (3.15). Playing with κ, for $h = 1$ we obtain

$$\begin{aligned} &|\operatorname{Cor}^{(p)}(f^{(i)}, f^{(j)})| \\ &\quad \leq C(m,\epsilon) \exp -[\cosh^{-1}(1+\sigma) - \epsilon] d_{\mathbb{Z}/p\mathbb{Z}}(i,j), \quad \forall\ \epsilon > 0 \end{aligned}$$

for every $\varepsilon > 0$, and Theorem 2.1 follows.

§4. On the link between the Kac operator and the Schrödinger operator

Our main aim is to show how semiclassical analysis can be applied to recover the result of J. Sjöstrand [31] (stated above in this paper as Theorem 2.4) as a consequence of Theorem 2.1 on the Kac (or transfer) operator. Roughly speaking, the first theorem can be regarded as an infinitesimal version of the second one. If A is the "positive" Laplacian $-\Delta$ and B is the operator of multiplication by V, we just analyze the operator

$$K(t) = \exp -tB/2. \exp -tA. \exp -tB/2$$

by two different techniques, the semiclassical one and the universal one. We suspect that there exists a more abstract proof, but our present proof has a semiclassical aspect. Heuristically, we believe that the largest eigenvalues of $\exp -tB/2. \exp -tA. \exp -tB/2$ are the same as the eigenvalues of $\exp -t(A+B)$ modulo $\mathcal{O}(t^2)$, and this gives the expected link.

Lemma 4.1. *Let V be a C^∞ potential satisfying*

$$V(x) \geq C \tag{4.1}$$

and

$$|D_x^\alpha V(x)| \leq C_\alpha, \quad \textit{for all} \quad \alpha \textit{such that} |\alpha| \geq 2\,. \tag{4.2}$$

Let $\mu_1(t)$ and $\mu_2(t)$ denote the two largest eigenvalues of $K(t)$ for $t \in]0,1]$. Then

$$\ln \mu_1(t) - \ln \mu_2(t) \leq t(\lambda_2 - \lambda_1) + \mathcal{O}(t^{3/2}); \quad j = 1,2.$$

Assume this lemma, we show how to deduce Theorem 2.4 from Theorem 2.1 . From Theorem 2.1 with $t = h^2$ and $V(t) = tV$ it follows immediately that

$$\frac{\mu_2(t)}{\mu_1(t)} \leq \exp - \cosh^{-1}[t.(t\sigma) + 1].$$

Taking the logarithm and dividing by t, we arrive at Theorem 2.4 using Lemma 4.1.

Proof of the lemma. We now pass to the proof of the lemma. We apply the semi-classical analysis for the two h-pseudodifferential operators (near the maximum of their principal symbol) $K(t)$ and $\widetilde{K}(t) = \exp -t(-\Delta + V)$. Here our semiclassical parameter will be chosen equal to $\sqrt{t}$. We can follow the analysis presented in [25] (see also [15]); however, it should be mentioned that we do not work with completely classical pseudo-differential operators. For the convenience of the reader who is not familiar with the h-pseudodifferential calculus (for which we refer to [25]) we discuss this point more precisely. After some change of the potential not affecting the conclusion of the Theorem, we can assume (instead of (4.1) that

$$0 \leq V(y).$$

It is a classical result that (4.2) and the positivity of V imply that

$$|\nabla V(x)| \leq C V^{1/2}(x) \tag{4.3}$$

for all x and some constant C. Now we regard our operator $K(t)$ as a $\sqrt{t}$-pseudodifferential operator $k^w(x, \sqrt{t}D_x, t)$ associated via the Weyl quantification to the symbol $(x, \xi, t) \rightarrow k(x, \xi, t)$ by the formula

$$\begin{aligned} &k^w(x, \sqrt{t}D_x, t)u(x) \\ &\quad = (2\pi)^{-m} \iint \exp -i((x-y).\xi).k\Big(\Big(\frac{x+y}{2}\Big), \sqrt{t}\xi, t\Big)u(y)dyd\xi\ . \end{aligned}$$

Conversely, k is called the h-symbol of $K(t)$ (with $h = \sqrt{t}$). There the parameter $\sqrt{t}$ plays the role of the semiclassical parameter h. Of course, we must be careful, because the class of symbols we deal with is not completely standard. We introduce the class

$$\Sigma^s = \{p \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m \times]0, h_0])\ s.t.\ |\partial_x^\alpha \partial_\xi^\beta p(x, \xi, h)| \leq C_{\alpha\beta} h^s\} \tag{4.4}$$

(here C^∞ is meant with respect with the (x, ξ) variables. For $m = 0$, this class is stable under composition, and the usual law of h-composition for the Weyl symbols can be used. We recall that the law of composition $r = p\sharp_h q$ of two symbols p and q can be defined by the formula

$$p_h^w(x, hD_x, h) \; o \; q_h^w(x, hD_x, h) = r_h^w(x, hD_x, h) \,. \tag{4.5}$$

Of course, it is well known that if p and q are in Σ^0 then $p\sharp_h q - pq \in \Sigma^1$, but in our proof we shall need to do all computations modulo $\mathcal{O}(h^3)$; we shall use the first three terms of the law of computation. We recall the following formula (see [25] p. 80):

$$p\sharp_h q = \sum_{|\alpha|+|\beta|\leq 2} \frac{1}{\alpha!\beta!)}\Big(\frac{1}{2}\Big)^{|\alpha|}\Big(-\frac{1}{2}\Big)^{|\beta|}.h^{(|\alpha|+|\beta|)}(\partial_\xi^\alpha D_x^\beta p).(\partial_\xi^\beta D_x^\alpha q) + \mathcal{O}(h^3). \tag{4.6}$$

Now, we represent $K(t)$ as the composition of three h-pseudodifferential operators. The h-symbol of $\exp -tA$ is clearly $\exp -\xi^2$:

$$\sigma_t^w(\exp -tA) = \exp -\xi^2 \,, \tag{4.7}$$

and the h-symbol of $\exp -\frac{t}{2}B$ is as follows:

$$\sigma_h^w\Big(\exp -\frac{t}{2}B\Big) = \exp -tV(x) \,. \tag{4.8}$$

We establish the estimate for the symbol and prove that it belongs to Σ^0 (and, actually, to a better class). Inequality (4.2) implies that

$$\nabla_x\Big(\exp -\frac{t}{2}V(x)\Big) = -\frac{t}{2}(\nabla V)(x)\Big(\exp -\frac{t}{2}V(x)\Big),$$

i.e.,

$$\Big|\nabla_x(\exp -\frac{t}{2}V(x))\Big| \leq C_0 h. \tag{4.9}$$

We now look at the second derivative. This is a sum of terms to which the preceding argument applies, except for one term, namely,

$$t(\nabla^2 V)(x)\Big(\exp -\frac{t}{2}V(x)\Big),$$

which is clearly $\mathcal{O}(h^2)$:

$$\Big|\nabla_x^2\Big(\exp -\frac{t}{2}V(x)\Big)\Big| \leq C_2 h^2 \,.$$

These estimates show that the symbol behaves a little better, and we actually have

$$\nabla\Big(\exp -\frac{t}{2}V(x)\Big) \in \Sigma^1 \tag{4.10}$$

and

$$\nabla^2\Big(\exp -\frac{t}{2}V(x)\Big) \in \Sigma^2. \tag{4.11}$$

We now compute the Weyl symbol of $K(t)$ using the composition law for p.d.o.

Lemma 4.2.

$$\sigma_h^w(K(t)) - \big[\exp -[\xi^2 + tV(x)]\big] \in \Sigma^{3/2}.$$

First, we observe that

$$\begin{aligned}&\Big[\exp -\frac{t}{2}V(x)\Big]\sharp_h[\exp -\xi^2]\\ &\quad\sim \exp -\Big[\xi^2 + \frac{t}{2}V(x)\Big] + \Big(\frac{t}{4i}\Big)[\xi.\nabla V(x)]\exp -\Big[\xi^2 + \frac{t}{2}V(x)\Big].\end{aligned}$$

Here we have used the fact that the symbol has a more regular behavior for the two first derivatives; the symbol $\sim$ means modulo $\mathcal{O}(h^3)$. Composing again with $\exp -\frac{t}{2}V(x)$ on the right, we see that the second term in the expansion disappears and we get the lemma. The functional calculus of Helffer-Robert [25] also gives the following statement.

Lemma 4.3.

$$\sigma_h^w(\widetilde{K}(t)) - \big[\exp -[\xi^2 + tV(x)]\big] \in \Sigma^3.$$

If f is a good function and if $p^w(x, hD_x, h)$ belongs to a good family of h-pseudodifferential operators, then the functional calculus implies (see [25] p. 143) that $f(p^w(x, hD_x, h))$ is also a pseudo-differential operator and provides explicit formulas for the symbol. Here we take $f(s) = \exp -s$ and $p^w(x, hD_x, h) = -h^2\Delta + h^2V$; for $h = \sqrt{t}$ and $a_t(x,\xi) = \xi^2 + tV(x)$ we obtain

$$\sigma_t^w(\widetilde{K}(t)) \sim f(a_t(x,\xi)) + h^2\Big[\sum_{k=1}^{3}(-1)^k(k!)^{-1}d_{2,k}f^{(k)}(a_t(x,\xi))\Big].$$

But the explicit computation of the symbol $d_{2,k}$ as a polynomial function of the derivatives of a_t, combined with the improvements (4.10) and (4.11), shows that

$$\Big[\sum_{k=1}^{3}(-1)^k(k!)^{-1}d_{2,k}f^{(k)}(a_t(x,\xi))\Big] \in \Sigma^3$$

and the lemma is proved. Here we have used the Lemma III-7 from [25] and the fact that a_t satisfies the estimates

$$|\partial_x^\alpha \partial_\xi^\beta a_t(x,\xi)| \leq C_{(\alpha,\beta)}(a_t(x,\xi)+1),$$

where the constants $C_{(\alpha,\beta)}$ are independent of $t \in]0,1]$. From these two lemmas we deduce that

$$K(t) - \widetilde{K}(t) = O(t^{3/2}) \quad \text{in } \mathcal{L}(L^2) \tag{4.12}$$

(we have used the Calderon-Vaillancourt theorem on the L^2-continuity of the pseudo-differential operators in order to control the remainder terms).

End of the Proof of Lemma 4.1 Let u_1 be the first eigenfunction of the Schrödinger operator corresponding to λ_1. Using (4.12), we obtain the inequality

$$\mu_1(t) \geq \exp -t\lambda_1 + \mathcal{O}(t^{3/2})$$

With the help of the first eigenfunction of $K(t)$, in the same way we obtain

$$\mu_1(t) \leq \exp -t\lambda_1 + \mathcal{O}(t^{3/2}),$$

and, finally,

$$\mu_1(t) = \exp -t\lambda_1 + \mathcal{O}(t^{3/2}). \tag{4.13}$$

Using the second normalized eigenvector u_2 of the Schrödinger operator corresponding to λ_2 as a quasi-mode, we establish the existence of an eigenvalue of $K(t)$ near $\exp -t\lambda_2$ (which cannot be $\mu_1(t)$, in accordiance with (4.12) and the fact that λ_1 is simple); consequentl,

$$\mu_2(t) \geq \exp -t\lambda_2 + \mathcal{O}(t^{3/2}). \tag{4.14}$$

Using (4.13) and (4.14) and taking the logarithm, we arrive at Lemma 4.1. A more careful semiclassical analysis (in the spirit of [15]) could probably give a more precise result but this is not needed for our purpose.

§5. Complements

The models considered by M. Kac in [20] are not cyclic. But a modified dictionary can be established. In accordance with the philosophy of our proofs it can be useful to start with the study of the periodic case and then estimate the "boundary" effect. What we can do here is completely parallel to what is usually computed in the case of the one-dimensional Ising model. We just give one example of computation. We consider

$$\Psi^{(p)}(X) = \Psi(X) = \sum_{j=1}^{p} V(x_j) + \sum_{j=1}^{p-1} \frac{|x_j - x_{j+1}|^2}{4}. \tag{5.1}$$

This case can be treated along the same lines. We can write

$$\exp -\Psi(X) = \exp -\frac{V(x_1)}{2}.K_V(x_1, x_2).\cdots .K_V(x_{p-1}, x_p).\exp -\frac{V(x_p)}{2}$$

and get

$$\int \exp -\Psi(X)dX = \int_{\mathbb{R}^m} (K_V^{(p-1)}h)(y).h(y)dy \tag{5.2}$$

with

$$h(y) = \exp -V(y)/2\ .$$

Also, (5.2) can be rewritten in the form

$$\int \exp -\Psi(X)dX = \sum_j \mu_j^{p-1}\Big[\int_{\mathbb{R}^m} h(y)u_j(y)dy\Big]^2. \tag{5.3}$$

Similarly to (2.6), for the thermodynamic limit we obtain

$$\lim_{p\to\infty} \frac{\ln \int_{\mathbb{R}^{mp}} \exp -\Phi(X)dX}{p} = \ln \mu_1, \tag{5.4}$$

but the rate of convergence is no more exponential; this can be seen explicitely from the formula

$$\begin{aligned}\Big|\frac{\ln \int_{\mathbb{R}^{mp}} \exp -\Phi(X)dX}{p} - \ln \mu_1\Big| \\ = \frac{1}{p}\Big(2\ln\Big(\int h(y)u_1(y)\,dy\Big) - 1\Big) + \mathcal{O}\Big(\frac{\mu_2}{\mu_1}\Big)^p.\end{aligned} \tag{5.5}$$

Again, a good knowledge of μ_2/μ_1 will imply sharp estimates. We leave to the reader the computation of similar formulas for the pair correlations.

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