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Asymptotics of Solutions of Some Integral Equations Connected with Differential Systems with a Singularity

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Our studies concern some aspects of scattering theory of the singular differential systems $y' - x^{-1}Ay - q(x)y = \rho By$, $x > 0$ with $n \times n$ matrices $A, B, q(x)$, $x \in (0, \infty)$, where A, B are constant and ρ is a spectral parameter. We concentrate on investigation of certain Volterra integral equations with respect to tensor-valued functions. The solutions of these integral equations play a central role in construction of the so-called Weyl-type solutions for the original differential system. Actually, the integral equations provide a method for investigation of the analytical and asymptotical properties of the Weyl-type solutions while the classical methods fail because of the presence of the singularity. In the paper, we consider the important special case when q is smooth and $q(0) = 0$ and obtain the classical-type asymptotical expansions for the solutions of the considered integral equations as $\rho \rightarrow \infty$ with $o(\rho^{-1})$ rate remainder estimate. The result allows one to obtain analogous asymptotics for the Weyl-type solutions that play in turn an important role in the inverse scattering theory.

Keywords: differential systems, singularity, integral equations, asymptotical expansions.

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INTRODUCTION

Our studies concern some aspects of scattering theory of the differential systems

$$y' - x^{-1}Ay - q(x)y = \rho By, \quad x > 0 \quad (1)$$

with $n \times n$ matrices $A, B, q(x)$, $x \in (0, \infty)$, where A, B are constant and ρ is a spectral parameter.

Differential equations with coefficients having non-integrable singularities at the end or inside the interval often appear in various areas of natural sciences and engineering. For $n = 2$, there exists an extensive literature devoted to different aspects of spectral theory of the radial Dirac operators, see, for instance [1–5].

Systems of the form (1) with $n > 2$ and arbitrary complex eigenvalues of the matrix B appear to be considerably more difficult for investigation even in the “regular” case $A = 0$ [6]. Some difficulties of principal matter also appear due to the presence of the singularity. Whereas the “regular” case $A = 0$ has been studied fairly completely to date [6–8], for the system (1) with $A \neq 0$ there are no similar general results.

The important role in scattering theory is played by a certain distinguished basis of generalized eigenfunctions for (1) (the so-called *Weyl-type solutions*, see, for instance [9]). In the presence of the singularity construction and investigation of this basis encounters some difficulties which do not appear in the “regular” case $A = 0$. In particular, one can not use the auxiliary Cauchy problems with the initial conditions



at $x = 0$. The approach presented in [10] (see also [11] and references therein) for the scalar differential operators

$$\ell y = y^{(n)} + \sum_{j=0}^{n-2} \left(\frac{\nu_j}{x^{n-j}} + q_j(x) \right) y^{(j)} \quad (2)$$

is based on using some special solutions of the equation $\ell y = \lambda y$ that also satisfy certain Volterra integral equations. This approach assumes some additional decay condition for the coefficients $q_j(x)$ as $x \rightarrow 0$, moreover, the required decay rate depends on eigenvalues of the matrix A . In this paper, we do not impose any additional restrictions of such a type. Instead, we use a modification of the approach first presented in [12] for the higher-order differential operators with *regular* coefficients on the whole line and recently adapted for differential systems of the form (1) on the semi-axis in [9].

In brief outline the approach can be described as follows. We consider some auxiliary systems with respect to the functions with values in the exterior algebra $\wedge \mathbb{C}^n$. Our study of these auxiliary systems centers on two families of their solutions that also satisfy some asymptotical conditions as $x \rightarrow 0$ and $x \rightarrow \infty$ respectively, and can be constructed as solutions of certain *Volterra* integral equations. As in [12] we call these distinguished tensor solutions the *fundamental tensors*. The main difference from the above-mentioned method used in [10] is that we use the integral equations to construct the fundamental tensors rather than the solutions for the original system. Since each of the fundamental tensors has minimal growth (as $x \rightarrow 0$ or $x \rightarrow \infty$) among solutions of the same auxiliary system, this step does not require any decay of $q(x)$ as $x \rightarrow 0$.

Construction and properties of the fundamental tensors were considered in details in our paper [9] provided that $q(\cdot)$ is absolutely continuous and both q, q' are integrable on the semi-axis $(0, \infty)$. In this paper, we consider the important special case $q(0) = 0$ and obtain the classical-type asymptotical expansions for the fundamental tensors as $\rho \rightarrow \infty$ with $o(\rho^{-1})$ rate remainder estimate.

1. ASSUMPTIONS AND NOTATIONS. FORMULATIONS OF THE RESULTS

We are to discuss first the unperturbed system:

$$y' - x^{-1}Ay = \rho By \quad (3)$$

and its particular case corresponding to the value $\rho = 1$ of the spectral parameter

$$y' - x^{-1}Ay = By \quad (4)$$

but to *complex* (in general) values of x .

Assumption 1. Matrix A is off-diagonal. The eigenvalues $\{\mu_j\}_{j=1}^n$ of the matrix A are distinct and such that $\mu_j - \mu_k \notin \mathbb{Z}$ for $j \neq k$, moreover, $\operatorname{Re}\mu_1 < \operatorname{Re}\mu_2 < \dots < \operatorname{Re}\mu_n$, $\operatorname{Re}\mu_k \neq 0$, $k = \overline{1, n}$.

Assumption 2. $B = \operatorname{diag}(b_1, \dots, b_n)$, the entries b_1, \dots, b_n are nonzero distinct points on the complex plane such that $\sum_{j=1}^n b_j = 0$ and such that any 3 points are noncolinear.

Under Assumption 1 system (4) has the fundamental matrix $c(x) = (c_1(x), \dots, c_n(x))$, where

$$c_k(x) = x^{\mu_k} \hat{c}_k(x),$$

$\det c(x) \equiv 1$ and all $\hat{c}_k(\cdot)$ are entire functions, $\hat{c}_k(0) = \mathfrak{h}_k$, \mathfrak{h}_k is an eigenvector of the matrix A corresponding to the eigenvalue μ_k . We define $C_k(x, \rho) := c_k(\rho x)$, $x \in (0, \infty)$,



$\rho \in \mathbb{C}$. We note that the matrix $C(x, \rho)$ is a solution of the unperturbed system (3) (with respect to x for the given spectral parameter ρ).

Let Σ be the following union of lines through the origin in \mathbb{C} :

$$\Sigma = \bigcup_{(k,j):j \neq k} \{z : \operatorname{Re}(zb_j) = \operatorname{Re}(zb_k)\}.$$

By virtue of Assumption 2 for any $z \in \mathbb{C} \setminus \Sigma$ there exists the ordering R_1, \dots, R_n of the numbers b_1, \dots, b_n such that $\operatorname{Re}(R_1 z) < \operatorname{Re}(R_2 z) \dots < \operatorname{Re}(R_n z)$. Let \mathcal{S} be a sector $\{z = r \exp(i\gamma), r \in (0, \infty), \gamma \in (\gamma_1, \gamma_2)\}$ lying in $\mathbb{C} \setminus \Sigma$. Then [13] the system (4) has the fundamental matrix $e(x) = (e_1(x), \dots, e_n(x))$ which is analytic in \mathcal{S} , continuous in $\overline{\mathcal{S}} \setminus \{0\}$ and admits the asymptotics:

$$e_k(x) = e^{xR_k}(\mathbf{f}_k + x^{-1}\eta_k(x)), \quad \eta_k(x) = O(1), \quad x \rightarrow \infty, \quad x \in \overline{\mathcal{S}},$$

where $(\mathbf{f}_1, \dots, \mathbf{f}_n) = \mathbf{f}$ is a permutation matrix such that $(R_1, \dots, R_n) = (b_1, \dots, b_n)\mathbf{f}$. We define $E(x, \rho) := e(\rho x)$.

Everywhere below we assume that the following additional condition is satisfied.

Condition 1. For all $k = \overline{2, n}$ the numbers

$$\Delta_k^0 := \det(e_1(x), \dots, e_{k-1}(x), c_k(x), \dots, c_n(x))$$

are not equal to 0.

Under Condition 1 the system (4) has the fundamental matrix $\psi^0(x) = (\psi_1^0(x), \dots, \psi_n^0(x))$ which is analytic in \mathcal{S} , continuous in $\overline{\mathcal{S}} \setminus \{0\}$ and admits the asymptotics:

$$\psi_k^0(xt) = \exp(xtR_k)(\mathbf{f}_k + o(1)), \quad t \rightarrow \infty, \quad x \in \mathcal{S}, \quad \psi_k^0(x) = O(x^{\mu_k}), \quad x \rightarrow 0.$$

We define $\Psi^0(x, \rho) := \psi^0(\rho x)$. As above, we note that the matrices $E(x, \rho)$, $\Psi^0(x, \rho)$ solve (3).

In the sequel we use the following notations:

- $\{\mathbf{e}_k\}_{k=1}^n$ is the standard basis in \mathbb{C}^n ;
- \mathcal{A}_m is the set of all ordered multi-indices $\alpha = (\alpha_1, \dots, \alpha_m)$, $\alpha_1 < \alpha_2 < \dots < \alpha_m$, $\alpha_j \in \{1, 2, \dots, n\}$;
- for a sequence $\{u_j\}$ of vectors and a multi-index $\alpha = (\alpha_1, \dots, \alpha_m)$ we define $u_\alpha := u_{\alpha_1} \wedge \dots \wedge u_{\alpha_m}$;
- for a numerical sequence $\{a_j\}$ and a multi-index α we define $a_\alpha := \sum_{j \in \alpha} a_j$,
- $a^\alpha := \prod_{j \in \alpha} a_j$;
- for a multi-index α the symbol α' denotes the ordered multi-index that complements α to $(1, 2, \dots, n)$;
- for $k = \overline{1, n}$ we denote

$$\vec{a}_k := \sum_{j=1}^k a_j, \quad \overleftarrow{a}_k := \sum_{j=k}^n a_j, \quad \vec{a}^k := \prod_{j=1}^k a_j, \quad \overleftarrow{a}^k := \prod_{j=k}^n a_j.$$

We note that Assumptions 1, 2 imply, in particular, $\sum_{k=1}^n \mu_k = \sum_{k=1}^n R_k = 0$ and therefore for any multi-index α one has $R_{\alpha'} = -R_\alpha$ and $\mu_{\alpha'} = -\mu_\alpha$;



– the symbol $V^{(m)}$, where V is $n \times n$ matrix, denotes the operator acting in $\wedge^m \mathbb{C}^n$ so that for any vectors u_1, \dots, u_m the following identity holds:

$$V^{(m)}(u_1 \wedge u_2 \wedge \dots \wedge u_m) = \sum_{j=1}^m u_1 \wedge u_2 \wedge \dots \wedge u_{j-1} \wedge V u_j \wedge u_{j+1} \wedge \dots \wedge u_m;$$

– if $h \in \wedge^n \mathbb{C}^n$ then $|h|$ is a number such that $h = |h| \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \dots \wedge \mathbf{e}_n$;

– for $h \in \wedge^m \mathbb{C}^n$ we set $\|h\| := \sum_{\alpha \in \mathcal{A}_m} |h_\alpha|$, where $\{h_\alpha\}$ are the coefficients from the

expansion $h = \sum_{\alpha \in \mathcal{A}_m} h_\alpha \mathbf{e}_\alpha$.

We use the same notation $L_p(a, b)$ for all the spaces of the form $L_p((a, b), \mathcal{E})$, where \mathcal{E} is a finite-dimensional space. The notation $C[a, b]$ for the spaces of continuous functions will be used in a similar way.

Everywhere below the symbol \mathcal{S} denotes some (arbitrary) open sector with the vertex at the origin lying in $\mathbb{C} \setminus \Sigma$.

For each fixed $\rho \in \overline{\mathcal{S}} \setminus \{0\} =: \mathcal{S}'$ we consider the following Volterra integral equations ($k = \overline{1, n}$):

$$Y(x) = T_k^0(x, \rho) + \int_0^x G_{n-k+1}(x, t, \rho) (q^{(n-k+1)}(t)Y(t)) dt, \quad (5)$$

$$Y(x) = F_k^0(x, \rho) - \int_x^\infty G_k(x, t, \rho) (q^{(k)}(t)Y(t)) dt, \quad (6)$$

where

$$T_k^0(x, \rho) := C_k(x, \rho) \wedge \dots \wedge C_n(x, \rho), \quad (7)$$

$$F_k^0(x, \rho) := E_1(x, \rho) \wedge \dots \wedge E_k(x, \rho) = \Psi_1^0(x, \rho) \wedge \dots \wedge \Psi_k^0(x, \rho) \quad (8)$$

and $G_m(x, t, \rho)$ is an operator acting in $\wedge^m \mathbb{C}^n$ as follows:

$$G_m(x, t, \rho)f = \sum_{\alpha \in \mathcal{A}_m} \sigma_\alpha |f \wedge C_{\alpha'}(t, \rho)| C_\alpha(x, \rho). \quad (9)$$

Here and below $\sigma_\alpha := |\mathfrak{h}_\alpha \wedge \mathfrak{h}_{\alpha'}|$.

For any $\rho \in \mathcal{S}'$ equations (5) and (6) were shown to have the unique solutions $T_k(x, \rho)$ and $F_k(x, \rho)$ respectively such that (see [9] for details):

$$\|T_k(x, \rho)\| \leq M \begin{cases} |(\rho x)^{\overleftarrow{\mu}_k}|, & |\rho x| \leq 1, \\ \left| \exp(\rho x \overleftarrow{R}_k) \right|, & |\rho x| > 1, \end{cases}$$

$$\|F_k(x, \rho)\| \leq M \begin{cases} |(\rho x)^{\overrightarrow{\mu}_k}|, & |\rho x| \leq 1, \\ \left| \exp(\rho x \overrightarrow{R}_k) \right|, & |\rho x| > 1. \end{cases}$$

We call the functions $F_k(x, \rho)$, $T_k(x, \rho)$ the *fundamental tensors*. Note that the fundamental tensors solve the auxiliary systems

$$Y' = Q^{(m)}(x, \rho)Y, \quad Q(x, \rho) := x^{-1}A + \rho B + q(x) \quad (10)$$

with $m = k$ and $m = n - k + 1$.



We note that the tensors $\{E_\alpha(x, \rho)\}_{\alpha \in \mathcal{A}_m}$ form the fundamental system of solutions for the system (10) in the “unperturbed” case. Therefore, the following representation holds:

$$T_k^0(x, \rho) = \sum_{\alpha \in \mathcal{A}_{n-k+1}} T_{k\alpha}^0 E_\alpha(x, \rho) \tag{11}$$

with x -independent coefficients $T_{k\alpha}^0$. Taking into account the special construction of the fundamental matrices $C(x, \rho)$, $E(x, \rho)$ one can conclude that the coefficients $T_{k\alpha}^0$ do not depend on ρ as well.

The $G_m(x, t, \rho)$ terms in equations (5), (6) are actually the Green operator functions for the nonhomogeneous systems:

$$Y' = Q^{(m)}(x, \rho)Y + f(x).$$

In order to construct them one can use various fundamental systems of solutions of the unperturbed system (3). In particular the following representations hold:

$$G_m(x, t, \rho)f = \sum_{\alpha \in \mathcal{A}_m} \chi_\alpha |f \wedge \Psi_{\alpha'}^0(t, \rho)| \Psi_\alpha^0(x, \rho) = \sum_{\alpha \in \mathcal{A}_m} \chi_\alpha |f \wedge E_{\alpha'}(t, \rho)| E_\alpha(x, \rho). \tag{12}$$

Here and below $\chi_\alpha := |f_\alpha \wedge f_{\alpha'}|$.

In the paper, we study the asymptotical behavior of the fundamental tensors for $\rho \rightarrow \infty$. In [9] the following expansions were obtained:

$$\begin{aligned} T_k(x, \rho) &= T_k^0(x, \rho) + O\left(\rho^{-\varepsilon} \exp\left(\rho x \overleftarrow{R}_k\right)\right), \quad \varepsilon \in (0, 1), \\ F_k(x, \rho) &= F_k^0(x, \rho) + O\left(\rho^{-1} \exp\left(\rho x \overrightarrow{R}_k\right)\right) \end{aligned}$$

for any fixed $x \in (0, \infty)$ and $\rho \rightarrow \infty$, $\rho \in \mathcal{S}'$. We show that under the additional condition $q(0) = 0$ more detailed expansion can be obtained.

Let $W_0(\xi)$ be the function defined as follows:

$$W_0(\xi) = (1 - |\xi|)\xi + |\xi|^2, \quad |\xi| \leq 1, \quad W_0(\xi) := (W_0(\xi^{-1}))^{-1}, \quad |\xi| > 1.$$

Notice that $W_0(\xi)$ is continuous in $\xi \in \mathbb{C}$, never vanishes for nonzero ξ and admits the estimate:

$$M_1|\xi| \leq |W_0(\xi)| \leq M_2|\xi|$$

for all $\xi \in \mathbb{C}$. Moreover, we have $W_0(\xi) = 1$ if $|\xi| = 1$ and the asymptotics $W_0(\xi) = \xi(1 + o(1))$ hold as $\xi \rightarrow 0$ and $\xi \rightarrow \infty$.

We introduce the following weight functions:

$$W_k(\xi) := \begin{cases} W_0(\xi^{\mu_k}) \exp(R_k \xi), & |\xi| \leq 1, \\ \exp(R_k \xi), & |\xi| > 1. \end{cases}$$

From the definition and the above-mentioned properties of $W_0(\cdot)$ it follows that the weight functions $W_k(\cdot)$, $k = \overline{1, n}$ are all continuous in \mathcal{S}' , never vanish and admit the asymptotics $W_k(\xi) = \xi^{\mu_k}(1 + o(1))$ as $\xi \rightarrow 0$. We define

$$\tilde{F}_k(x, \rho) := (\overrightarrow{W}^k(\rho x))^{-1} F_k(x, \rho), \quad \tilde{T}_k(x, \rho) := (\overleftarrow{W}^k(\rho x))^{-1} T_k(x, \rho).$$



Theorem 1. Suppose that $q(\cdot)$ is an absolutely continuous off-diagonal matrix function such that $q(0) = 0$. Denote by $\hat{q}_o(\cdot)$ the off-diagonal matrix function such that $[B, \hat{q}_o(x)] = -q(x)$ for all $x > 0$ (here $[\cdot, \cdot]$ denotes the matrix commutator). Define the diagonal matrix $d(x) = \text{diag}(d_1(x), \dots, d_n(x))$, where

$$d_k(x) := \int_x^\infty t^{-1} ([\hat{q}_o(t), A]_{kk}) dt$$

and set $\hat{q}(x) := \hat{q}_o(x) + d(x)$.

Suppose that all the functions $q_{ij}(\cdot), q'_{ij}(\cdot)$ and $\tilde{q}_{ij}(\cdot)$, where $\tilde{q}(x) := \hat{q}'(x) + x^{-1}[\hat{q}(x), A]$ are from $X_p := L_1(0, \infty) \cap L_p(0, \infty)$, $p > 2$.

Then for each fixed $x > 0$ and $\rho \rightarrow \infty$, $\rho \in \mathcal{S}'$ the following asymptotics hold:

$$\begin{aligned} \rho(\tilde{T}_k(x, \rho) - \tilde{T}_k^0(x, \rho)) &= d_{0k} \tilde{T}_k^0(x, \rho) + \sum_{\alpha, \beta \in \mathcal{A}_{n-k+1}} T_{k\beta}^0 g_{k\alpha\beta}(x) \exp(\rho x (R_\beta - \overleftarrow{R}_k)) \mathfrak{f}_\alpha + o(1), \\ \rho(\tilde{F}_k(x, \rho) - \tilde{F}_k^0(x, \rho)) &= \sum_{\alpha \in \mathcal{A}_k} f_{k\alpha}(x) \mathfrak{f}_\alpha + o(1). \end{aligned}$$

Here

$$d_{0k} = -\sigma_{\alpha^*(k)} |(d^{(n-k+1)}(0) \mathfrak{h}_{\alpha^*(k)}) \wedge \mathfrak{h}_{(\alpha^*(k))'}|,$$

$\alpha^*(k) := (k, \dots, n)$ and the coefficients in the representations are defined as follows:

$$f_{k\alpha}(x) = \chi_\alpha |(\hat{q}^{(k)}(x) \mathfrak{f}_{\alpha^*(k)}) \wedge \mathfrak{f}_{\alpha'}|$$

for $\alpha \neq \alpha_*(k) := (1, \dots, k)$,

$$\begin{aligned} f_{k, \alpha_*(k)}(x) &= - \sum_{\alpha \in \mathcal{A}_k} \int_x^\infty \chi_{\alpha_*(k)} |(q^{(k)}(t) \mathfrak{f}_\alpha) \wedge \mathfrak{f}_{\alpha_*(k)}| \chi_\alpha |(\hat{q}^{(k)}(t) \mathfrak{f}_{\alpha_*(k)}) \wedge \mathfrak{f}_{\alpha'}| dt; \\ g_{k\alpha\beta}(x) &= \chi_\alpha |(\hat{q}^{(n-k+1)}(x) \mathfrak{f}_\beta) \wedge \mathfrak{f}_{\alpha'}| \end{aligned}$$

for $\beta \neq \alpha$,

$$g_{k\beta\beta}(x) = \sum_{\alpha \in \mathcal{A}_{n-k+1}} \int_0^x \chi_\beta |(q^{(n-k+1)}(t) \mathfrak{f}_\alpha) \wedge \mathfrak{f}_{\beta'}| \chi_\alpha |(\hat{q}^{(n-k+1)}(t) \mathfrak{f}_\beta) \wedge \mathfrak{f}_{\alpha'}| dt.$$

2. PROOF OF THEOREM 1

We consider in details the function $T_k(x, \rho)$, for the function $F_k(x, \rho)$ similar arguments are valid.

For the function $\hat{T}_k(x, \rho) := \tilde{T}_k(x, \rho) - \tilde{T}_k^0(x, \rho)$ we have the representation $\hat{T}_k(\cdot, \rho) = (Id - \mathcal{K}(\rho))^{-1} v_k(\cdot, \rho)$, where $\mathcal{K}(\rho)$ is an operator of the form:

$$(\mathcal{K}(\rho)f)(x) := \int_0^x \mathcal{G}_{n-k+1}(x, t, \rho) (q^{(n-k+1)}(t) f(t)) dt$$

acting in $L_\infty(0, T)$, $T \in (0, \infty)$ is arbitrary. Here and below

$$\mathcal{G}_{n-k+1}(x, t, \rho) := \frac{\overleftarrow{W}^k(\rho t)}{\overleftarrow{W}^k(\rho x)} G_{n-k+1}(x, t, \rho),$$



$$v_k(x, \rho) = \int_0^x \mathcal{G}_{n-k+1}(x, t, \rho) \left(q^{(n-k+1)}(t) \tilde{T}_k^0(t, \rho) \right) dt.$$

Let us consider first the function $v_k(x, \rho)$. From the identity:

$$\begin{aligned} & \rho(q^{(n-k+1)}(t)T_k^0(t, \rho)) \wedge E_{\alpha'}(t, \rho) = \\ & = \frac{d}{dt} \left((\hat{q}^{(n-k+1)}(t)T_k^0(t, \rho)) \wedge E_{\alpha'}(t, \rho) \right) - (\tilde{q}^{(n-k+1)}(t)T_k^0(t, \rho)) \wedge E_{\alpha'}(t, \rho), \end{aligned}$$

where $\alpha \in \mathcal{A}_{n-k+1}$ is arbitrary it follows the relation:

$$\begin{aligned} & \rho \int_{x_0}^x G_{n-k+1}(x, t, \rho) \left(q^{(n-k+1)}(t)T_k^0(t, \rho) \right) dt = \\ & = G_{n-k+1}(x, t, \rho) \left(\hat{q}^{(n-k+1)}(t)T_k^0(t, \rho) \right) \Big|_{t=x_0}^{t=x} - \int_{x_0}^x G_{n-k+1}(x, t, \rho) \left(\tilde{q}^{(n-k+1)}(t)T_k^0(t, \rho) \right) dt. \end{aligned}$$

Passing to the limits as $x_0 \rightarrow 0$ and taking into account that $\hat{q}_o(0) = 0$ we arrive at the relation:

$$\begin{aligned} \rho v_k(x, \rho) & = d_{0k} \tilde{T}_k^0(x, \rho) + \sum_{\alpha \in \mathcal{A}_{n-k+1}} \chi_\alpha \left| (\hat{q}^{(n-k+1)}(x) \tilde{T}_k^0(x, \rho)) \wedge E_{\alpha'}(x, \rho) \right| E_\alpha(x, \rho) - \\ & \int_0^x \mathcal{G}_{n-k+1}(x, t, \rho) \left(\tilde{q}^{(n-k+1)}(t) \tilde{T}_k^0(t, \rho) \right) dt. \end{aligned} \quad (13)$$

Since $\tilde{q}_{jj} = 0$, $j = \overline{1, n}$, from (13) and [14] we obtain (in particular) the estimate:

$$\|v_k(\cdot, \rho)\|_{BC[0, \infty)} = O(\rho^{-1}), \quad \rho \in \mathcal{S}'. \quad (14)$$

In what follows if $V = V(x, \rho)$ is some matrix function then \tilde{V} denotes the matrix function $\tilde{V}(x, \rho) := V(x, \rho)(W(\rho x))^{-1}$, where $W = \text{diag}(W_1, \dots, W_n)$. Since $\tilde{\Psi}^0(x, \rho)$ is continuous and bounded in $[0, \infty) \times \overline{\mathcal{S}}$ we have:

$$\|\mathcal{G}_{n-k+1}(x, t, \rho)\| \leq M, \quad 0 < t \leq x < \infty, \quad \rho \in \mathcal{S}' \quad (15)$$

with some absolute constant M .

Using the boundedness of $\mathcal{G}_{n-k+1}(x, t, \rho)$ one can obtain the estimate (see also the proof of [9, Theorem 3.1]):

$$\|\mathcal{K}^r(\rho)\| \leq M_0 \frac{M_1^r}{r!} \left(\int_0^T \|q(t)\| dt \right)^r,$$

where the norm $\|\mathcal{K}^r(\rho)\|$ assumes the norm of the operator acting in $L_\infty(0, T)$ for arbitrary $T > 0$ and the constants M_0, M_1 do not depend on T . This yields the estimate $\|(Id - \mathcal{K}(\rho))^{-1}\| = O(1)$ uniformly in $\rho \in \mathcal{S}'$. Thus (with taking into account (14)), we obtain the auxiliary prior estimate for \hat{T}_k :

$$\|\hat{T}_k(\cdot, \rho)\|_{L_\infty(0, T)} = O(\rho^{-1}), \quad \rho \in \mathcal{S}' \quad (16)$$



for any $T > 0$.

In order to make a more detailed study we represent the operator $\mathcal{K}(\rho)$ in the form $\mathcal{K}(\rho) = \mathcal{K}_0(\rho) + \mathcal{K}_1(\rho)$, where:

$$\begin{aligned} & \mathcal{K}_0(\rho)f(x) := \\ & = \theta^+(|\rho x| - 1) \sum_{\alpha \in \mathcal{A}_{n-k+1}} \chi_\alpha \int_{|\rho|^{-1}}^x \exp(\rho(x-t)(R_\alpha - \overleftarrow{R}_k)) |(q^{(n-k+1)}(t)f(t)) \wedge \mathfrak{f}_{\alpha'}| \mathfrak{f}_\alpha dt. \end{aligned}$$

Here and below the symbols $\theta^\pm(\cdot)$ denote the Heaviside step functions:

$$\theta^+(\xi) = \begin{cases} 0, & \xi \leq 0, \\ 1, & \xi > 0, \end{cases} \quad \theta^-(\xi) = \begin{cases} 1, & \xi \leq 0 \\ 0, & \xi > 0 \end{cases} = 1 - \theta^+(\xi).$$

Lemma 1. *Under the conditions of Theorem 1 one has the estimate $\|\mathcal{K}_1(\rho)\| = O(\rho^{-1})$.*

Proof. We split the operator as follows: $\mathcal{K}_1 = \mathcal{K}_0^{(1)} + \mathcal{K}_1^{(1)} + \mathcal{K}_2^{(1)}$, where:

$$\begin{aligned} (\mathcal{K}_0^{(1)}f)(x) &= \theta^-(|\rho x| - 1) \int_0^x \mathcal{G}_{n-k+1}(x, t, \rho) (q^{(n-k+1)}(t)f(t)) dt, \\ (\mathcal{K}_1^{(1)}f)(x) &= \theta^+(|\rho x| - 1) \int_0^{|\rho|^{-1}} \mathcal{G}_{n-k+1}(x, t, \rho) (q^{(n-k+1)}(t)f(t)) dt. \end{aligned}$$

By virtue of (15) we have:

$$\|\mathcal{K}_1^{(1)}f\| \leq M\|f\| \cdot \int_0^{|\rho|^{-1}} \|q(t)\| dt \leq M|\rho|^{-1}\|f\| \cdot \|q(\cdot)\|_{L_\infty(0,T)}.$$

Proceeding in a similar way and taking into account that $(\mathcal{K}_0^{(1)}f)(x) \neq 0$ only if $|\rho x| \leq 1$ one can obtain the similar estimate for $\|\mathcal{K}_0^{(1)}f\|$.

Let us consider $\mathcal{K}_2^{(1)}$. Using the representation (9) for $G_{n-k+1}(x, t, \rho)$, the asymptotics

$$E_\alpha(x, \rho) = \exp(\rho x R_\alpha)(\mathfrak{f}_\alpha + O((\rho x)^{-1})),$$

which is uniform in $|\rho x| \geq 1$ and taking into account that $\text{Re}(\rho(x-t)(R_\alpha - \overleftarrow{R}_k)) \leq 0$ for any $0 \leq t \leq x$, $\rho \in \mathcal{S}'$, $\alpha \in \mathcal{A}_{n-k+1}$ we obtain the estimate:

$$\begin{aligned} & \theta^+(|\rho x| - 1)\theta^+(|\rho t| - 1)\theta^+(x-t) \|\mathcal{G}_{n-k+1}(x, t, \rho) (q^{(n-k+1)}(t)f(t)) - \\ & \sum_{\alpha \in \mathcal{A}_{n-k+1}} \chi_\alpha \exp(\rho(x-t)(R_\alpha - \overleftarrow{R}_k)) |(q^{(n-k+1)}(t)f(t)) \wedge \mathfrak{f}_{\alpha'}| \mathfrak{f}_\alpha \Big\| \leq \frac{M}{|\rho t|} \|q(t)\| \end{aligned}$$

with some absolute constant M . Since under the conditions of Theorem 1 $t^{-1}q(t) \in L_1(0, \infty)$ the estimate above yields

$$\|\mathcal{K}_2^{(1)}f\| \leq M|\rho|^{-1}\|f\| \cdot \int_0^\infty t^{-1}\|q(t)\| dt$$

and therefore $\|\mathcal{K}_2^{(1)}\| = O(\rho^{-1})$. □



Lemma 2. Under the conditions of Theorem 1 one has the estimate $\|\mathcal{K}_0^2(\rho)\| = O(\rho^{-1})$.

Proof. We have:

$$\begin{aligned} (\mathcal{K}_0^2 f)(x) &= \theta^+(|\rho x| - 1) \sum_{\alpha \in \mathcal{A}_{n-k+1}} \int_{|\rho|^{-1}}^x \exp(\rho(x-t)(R_\alpha - \overleftarrow{R}_k)) \chi_\alpha \times \\ &\quad \times |(q^{(n-k+1)}(t)(\mathcal{K}_0 f)(t)) \wedge \mathfrak{f}_{\alpha'}| \mathfrak{f}_\alpha dt, \\ &\quad \chi_\alpha |(q^{(n-k+1)}(t)(\mathcal{K}_0 f)(t)) \wedge \mathfrak{f}_{\alpha'}| = \theta^+(|\rho t| - 1) \times \\ &\times \sum_{\beta \in \mathcal{A}_{n-k+1}} \chi_\beta \int_{|\rho|^{-1}}^t \exp(\rho(t-\tau)(R_\beta - \overleftarrow{R}_k)) |(q^{(n-k+1)}(\tau)f(\tau)) \wedge \mathfrak{f}_{\beta'}| Q_{\alpha\beta}(t) d\tau, \end{aligned}$$

where $Q_{\alpha\beta}(t) := \chi_\alpha |(q^{(n-k+1)}(t)\mathfrak{f}_\beta) \wedge \mathfrak{f}_{\alpha'}|$.

Thus, we can rewrite:

$$(\mathcal{K}_0^2 f)(x) = \theta^+(|\rho x| - 1) \sum_{\alpha\beta \in \mathcal{A}_{n-k+1}} \int_{|\rho|^{-1}}^x |(q^{(n-k+1)}(\tau)f(\tau)) \wedge \mathfrak{f}_{\beta'}| H_{\alpha\beta}(x, \tau, \rho) d\tau,$$

where:

$$H_{\alpha\beta}(x, \tau, \rho) = \int_\tau^x Q_{\alpha\beta}(t) \exp(\rho(x-t)(R_\alpha - \overleftarrow{R}_k) + \rho(t-\tau)(R_\beta - \overleftarrow{R}_k)) \mathfrak{f}_\alpha dt.$$

We notice again that $\text{Re}(\rho(x-t)(R_\alpha - \overleftarrow{R}_k) + \rho(t-\tau)(R_\beta - \overleftarrow{R}_k)) \leq 0$ for any $0 \leq \tau \leq t \leq x$, $\rho \in \mathcal{S}'$, $\alpha, \beta \in \mathcal{A}_{n-k+1}$. Moreover, under the conditions of Theorem 1 $Q_{\alpha\beta}(\cdot)$ are absolutely continuous and $Q_{\alpha\beta}(t) \equiv 0$ if $\alpha = \beta$. This yields the estimate

$$\theta^+(|\rho\tau| - 1)H_{\alpha\beta}(x, \tau, \rho) = O(\rho^{-1}),$$

which is uniform in $0 \leq \tau \leq x$, $\rho \in \mathcal{S}'$. The estimate implies the required assertion. \square

Proof of Theorem 1. We have $\hat{T}_k(\cdot, \rho) = v_k(\cdot, \rho) + \mathcal{K}(\rho)v_k(\cdot, \rho) + \mathcal{K}^2(\rho)\hat{T}_k(\cdot, \rho)$.

We note that

$$(\mathcal{K}(\rho)\tilde{T}_k^0(\cdot, \rho))(x) = \int_0^x \mathcal{G}_{n-k+1}(x, t, \rho) \left(q^{(n-k+1)}(t)\tilde{T}_k^0(t, \rho) \right) = v_k(x, \rho) = O(\rho^{-1})$$

uniformly for $\rho \in \mathcal{S}'$, $x \in (0, T)$.

This, prior estimate (16), (14) and Lemmas 1, 2 yield:

$$\hat{T}_k(\cdot, \rho) = v_k(\cdot, \rho) + \mathcal{K}_0(\rho)\omega_k(\cdot, \rho) + O(\rho^{-2}), \quad (17)$$

where

$$\begin{aligned} \rho\omega_k(x, \rho) &= \sum_{\alpha \in \mathcal{A}_{n-k+1}} \chi_\alpha \left| (\hat{q}^{(n-k+1)}(x)\tilde{T}_k^0(x, \rho)) \wedge E_{\alpha'}(x, \rho) \right| E_\alpha(x, \rho) - \\ &\quad - \int_0^x \mathcal{G}_{n-k+1}(x, t, \rho) \left(\tilde{q}^{(n-k+1)}(t)\tilde{T}_k^0(t, \rho) \right) dt. \end{aligned} \quad (18)$$

and the $O(\cdot)$ term assumes an estimate in $L_\infty(0, T)$ norm.



From [14, Theorem 1] and (18) we have:

$$\rho\omega_k(x, \rho) = \sum_{\alpha \in \mathcal{A}_{n-k+1}} \chi_\alpha \left| (\hat{q}^{(n-k+1)}(x) \tilde{T}_k^0(x, \rho)) \wedge E_{\alpha'}(x, \rho) \right| E_\alpha(x, \rho) + o(1),$$

that yields:

$$\begin{aligned} & \theta^+(|\rho t| - 1) \rho\omega_k(t, \rho) = \\ & = \theta^+(|\rho t| - 1) \sum_{\alpha, \beta \in \mathcal{A}_{n-k+1}} T_{k\beta}^0 \exp(\rho t(R_\beta - \overleftarrow{R}_k)) \hat{Q}_{\alpha\beta}(t) \mathbf{f}_\alpha + \rho^{-1} \hat{\omega}_k(t, \rho) + o(1), \end{aligned}$$

where $\hat{Q}_{\alpha\beta}(t) = \chi_\alpha \left| (\hat{q}^{(n-k+1)}(t) \mathbf{f}_\beta) \wedge \mathbf{f}_{\alpha'} \right|$, the $o(\cdot)$ term assumes an estimate in $L_\infty(0, T)$ norm and $t\hat{\omega}_k(t, \rho)$ is uniformly bounded in $\{|\rho t| \geq 1\}$.

Under the conditions of Theorem 1 we have $t^{-1}q(t) \in L_1(0, \infty)$. This yields $\mathcal{K}_0(\rho)\hat{\omega}_k(\cdot, \rho) = O(1)$ and thus from the representation above we obtain:

$$\begin{aligned} (\mathcal{K}_0(\rho)\omega_k(\cdot, \rho))(x) &= \rho^{-1} \theta^+(|\rho x| - 1) \sum_{\alpha, \beta, \gamma \in \mathcal{A}_{n-k+1}} \chi_\gamma T_{k\beta}^0 \int_{|\rho|^{-1}}^x \exp(\rho(x-t)(R_\gamma - \overleftarrow{R}_k) + \\ & + \rho t(R_\beta - \overleftarrow{R}_k)) \hat{Q}_{\alpha\beta}(t) \left| (q^{(n-k+1)}(t) \mathbf{f}_\alpha) \wedge \mathbf{f}_{\gamma'} \right| \mathbf{f}_\gamma dt + o(\rho^{-1}) = \rho^{-1} \theta^+(|\rho x| - 1) \times \\ & \times \sum_{\beta, \gamma \in \mathcal{A}_{n-k+1}} T_{k\beta}^0 \int_{|\rho|^{-1}}^x \exp(\rho(x-t)(R_\gamma - \overleftarrow{R}_k) + \rho t(R_\beta - \overleftarrow{R}_k)) \tilde{Q}_{\gamma\beta}(t) \mathbf{f}_\gamma dt + o(\rho^{-1}), \end{aligned}$$

where:

$$\tilde{Q}_{\gamma\beta}(t) := \sum_{\alpha \in \mathcal{A}_{n-k+1}} Q_{\gamma\alpha}(t) \hat{Q}_{\alpha\beta}(t), \quad Q_{\gamma\alpha}(t) = \chi_\gamma \left| (q^{(n-k+1)}(t) \mathbf{f}_\alpha) \wedge \mathbf{f}_{\gamma'} \right|$$

and the $o(\cdot)$ term assumes an estimate in $L_\infty(0, T)$. Under the conditions of Theorem 1 the functions $Q_{\alpha\beta}$ и $\hat{Q}_{\alpha\beta}$ (for any pair of multi-indices α, β) are absolutely continuous. Therefore, we have for $\gamma \neq \beta$:

$$\int_{|\rho|^{-1}}^x \exp(\rho(x-t)(R_\gamma - \overleftarrow{R}_k) + \rho t(R_\beta - \overleftarrow{R}_k)) \tilde{Q}_{\gamma\beta}(t) dt = O(\rho^{-1}),$$

that yields:

$$\begin{aligned} & (\mathcal{K}_0(\rho)\omega_k(q, \cdot, \rho))(x) = \\ & = \rho^{-1} \theta^+(|\rho x| - 1) \sum_{\beta \in \mathcal{A}_{n-k+1}} T_{k\beta}^0 \exp(\rho x(R_\beta - \overleftarrow{R}_k)) \int_{|\rho|^{-1}}^x \tilde{Q}_{\beta\beta}(t) dt \mathbf{f}_\beta + o(\rho^{-1}). \end{aligned}$$

Substituting the obtained asymptotics to the representation (17) we arrive at:

$$\begin{aligned} \hat{T}_k(x, \rho) &= v_k(x, \rho) + \rho^{-1} \theta^+(|\rho x| - 1) \times \\ & \times \sum_{\beta \in \mathcal{A}_{n-k+1}} T_{k\beta}^0 \exp(\rho x(R_\beta - \overleftarrow{R}_k)) \int_{|\rho|^{-1}}^x \tilde{Q}_{\beta\beta}(t) dt \mathbf{f}_\beta + o(\rho^{-1}). \end{aligned} \tag{19}$$



Here, as above, the $o(\cdot)$ term assumes an estimate in $L_\infty(0, T)$ norm. But all the terms in (19) are actually continuous with respect to $x \in (|\rho^{-1}|, T)$. This means that the expansion can be considered in point-wise sense as $\rho \rightarrow \infty$ while $x > 0$ is arbitrary fixed.

Now we notice that

$$\int_{|\rho|^{-1}}^x \tilde{Q}_{\beta\beta}(t) dt \rightarrow \int_0^x \tilde{Q}_{\beta\beta}(t) dt = g_{k\beta\beta}(x)$$

as $\rho \rightarrow \infty$. Then we use the representation (13) for $v_k(x, \rho)$ and thus we obtain the required asymptotics. \square

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Асимптотики решений некоторых интегральных уравнений, связанных с дифференциальными системами с особенностью

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В работе изучаются некоторые аспекты теории рассеяния для сингулярных систем дифференциальных уравнений $y' - x^{-1}Ay - q(x)y = \rho By$, $x > 0$ со спектральным параметром ρ , где $A, B, q(x)$, $x \in (0, \infty)$ — $n \times n$ матрицы, причем матрицы A, B постоянны. Основным предметом исследования являются некоторые вольтерровские интегральные уравнения относительно тензорно-значных функций. Решения этих уравнений играют центральную роль в построении так называемых решений типа Вейля для исходной системы дифференциальных уравнений. Поскольку классические методы при наличии особенности оказываются неприменимыми, изучение рассматриваемых интегральных уравнений становится в этом случае ключевым этапом исследования аналитических и асимптотических свойств решений типа Вейля. В данной работе мы рассматриваем важный частный случай, когда матрица-функция $q(\cdot)$ является гладкой и $q(0) = 0$. В этом случае для решений рассматриваемых интегральных уравнений удается получить асимптотические разложения при $\rho \rightarrow \infty$ с оценкой остаточного члена $o(\rho^{-1})$. Полученный результат позволяет получить асимптотики для решений типа Вейля, играющие, в свою очередь, важную роль при исследовании обратной задачи рассеяния.

Ключевые слова: дифференциальные системы, особенности, интегральные уравнения, асимптотические разложения.

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