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Groups which are isomorphic to their subgroups with non-modular subgroup lattice

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Abstract. This paper describes the structure of groups G containing proper subgroups with non-modular subgroup lattice and such that all such subgroups are isomorphic to G.

Introduction

A subgroup of a group G is called *modular* if it is a modular element of the lattice $\mathfrak{L}(G)$ of all subgroups of G. It is clear that every normal subgroup of a group is modular, but arbitrary modular subgroups need not be normal. Lattices with modular elements are also called *modular*. Abelian groups and the so-called Tarski groups (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) are obvious examples of groups with modular subgroup lattice. Recall also that a subgroup H of a group G is said to be *permutable* if $HK = KH$ for every subgroup K of G, and a group is called *quasihamiltonian* if all its subgroups are permutable; it is well known that a subgroup of a group G is permutable if and only if it is modular and ascendant (see [22], Theorem 6.2.10), so that quasihamiltonian groups coincide with locally nilpotent groups having modular subgroup lattice. The structure of groups with modular subgroup lattice and that of quasihamiltonian groups have been completely described by K. Iwasawa [9],[10] and R. Schmidt [21]. In particular, it turns out that groups with modular subgroup lattice are metabelian, provided that they have no Tarski sections,

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and that non-periodic groups with modular subgroup lattice are always quasihamiltonian.

In recent years various results concerning normal subgroups have been lattice-theoretically interpretated, using modularity as the lattice analogue of normality. In particular, in [3], [5], [7], [8] it has been shown that some important theorems of B.H. Neumann on the structure of infinite groups in which all subgroups are close to be normal (see [2], [16]), have a lattice corresponding. Moreover, lattice-theoretic interpretations of some results of Romalis and Sesekin ([18], [19], [20]) on groups with abelian non-normal subgroups have recently been obtained in [4].

The aim of this article is to give the lattice traslation of another group theoretical result. The structure of groups G containing proper non-abelian subgroups all of which are isomorphic to G has been investigated by Smith and Wiegold [24], and the corresponding problem for the class of nilpotent groups has also been considered (see [14], [25], [26]). Here we will consider the class $\mathfrak{X}$ of groups G for which there exist proper subgroups with non-modular subgroup lattice and all such subgroups are isomorphic to G.

Let $\mathfrak{M}$ denote the class consisting of all groups G such that the subgroup lattice of G is not modular, while all proper subgroups of G have modular subgroup lattice. Finite $\mathfrak{M}$-groups have been classified by F. Napolitani [15]; an example of an infinite $\mathfrak{M}$-group is given in [6], where it is also proved that locally graded $\mathfrak{M}$-groups are finite (recall here that a group is *locally graded* if every finitely generated non-trivial subgroup has a proper subgroup of finite index). In this article we shall prove that every $\mathfrak{X}$-group G contains a normal subgroup N with modular subgroup lattice such that the factor group G/N is either simple or infinite cyclic, and this should be seen in relation with the fact that a group G containing proper non-abelian subgroups all of which are isomorphic to G, has an abelian normal subgroup A such that G/A is simple. Moreover, we will prove that a group G containing proper non-quasihamiltonian subgroups all of which are isomorphic to G has a quasihamiltonian normal subgroup with simple factor group. Finally, an example will be given to show that $\mathfrak{X}$ is different from the class studied by Smith and Wiegold.

Most of our notation is standard and can be found in [17]. We will use the monograph [22] as a general reference for results on subgroup lattices.

1. Soluble groups

Of course, every $\mathfrak{X}$-group is infinite. Moreover, since modularity is a local property (see [8], Lemma 5.1), groups in the class $\mathfrak{X}$ must be finitely generated.

Lemma 1. *Let G be a group and let T be a finite abelian normal subgroup of G such that the factor group G/T is infinite cyclic. If a is any element of infinite order of G such that $G = \langle a\rangle T$, then $|\langle a\rangle : \langle a\rangle \cap Z(G)| = |G : Z(G)|/|T : T \cap Z(G)|$.*

Proof. Let $a^t x$ be any element of $Z(G)$ where t is an integer and $x \in T$; clearly $[a^t, T] = \{1\}$, so that $a^t \in Z(G)$, and also x is an element of $Z(G)$. Therefore

$$Z(G) = (\langle a\rangle \cap Z(G)) \times (T \cap Z(G)).$$

It follows easily that

$$|G : Z(G)| = |\langle a\rangle : \langle a\rangle \cap Z(G)|\, |T : T \cap Z(G)|,$$

and the lemma is proved. □

Lemma 2. *Let G be a nilpotent $\mathfrak{X}$-group. Then G contains a quasihamiltonian subgroup of prime index.*

Proof. Assume that the statement is false, and let G be a counterexample for which the order m of the subgroup T consisting of all elements of finite order of G is minimal. Since every non-quasihamiltonian subgroup of G is isomorphic to G, it contains m elements of finite order. Thus T is contained in any non-quasihamiltonian subgroup of G.

If H/T is any non-abelian subgroup of G/T, then H is not quasihamiltonian (see [22], Theorem 2.4.11), so that $H \simeq G$ and $H/T \simeq G/T$. It follows that G/T contains an abelian subgroup of prime index (see [24], Theorem 2 and [1], Corollary 2), and hence it is abelian (see for instance [12], 2.3.9). Assume that G/T is cyclic, and let a be an element of infinite order of G such that $G = \langle a, T\rangle$. Consider a positive integer n such that $\langle a\rangle \cap Z(G) = \langle a^n\rangle$; if T_1 is a subgroup of prime index of T, the subgroup $\langle a^n\rangle \times T_1$ is quasihamiltonian, since it does not contain T. Then $\langle a^n\rangle \times T$ contains a quasihamiltonian subgroup of prime index, so that it is not isomorphic to G, and hence it is quasihamiltonian. In particular, T is abelian (see [22], Theorem 2.4.11) and the index $n = |G : \langle a^n\rangle \times T|$ is not a prime number. Let p be a prime divisor of n, and put $L = \langle a^p, T\rangle$; clearly $\langle a^p\rangle \cap Z(L) = \langle a^n\rangle$, so that $|\langle a\rangle : \langle a\rangle \cap Z(G)| \neq |\langle a^p\rangle : \langle a^p\rangle \cap Z(L)|$, and by Lemma 1 L is not isomorphic to G, which is impossible since L has prime index in G. This contradiction proves that the finitely generated torsion-free abelian group G/T has rank greater than 1, so that if g is any element of infinite order of G, the subgroup $\langle g, T\rangle$ is quasihamiltonian; in particular, T is abelian and all subgroups of T are normal in G, since G is generated by its elements of infinite order.

Put

$$G/T = \langle a_1 T\rangle \times \cdots \times \langle a_r T\rangle,$$

and suppose that $A = \langle a_1, \ldots, a_r\rangle$ is abelian, so that there exists a subgroup L of A such that $A = L \times (A \cap T)$, and $G = L \ltimes T$. If T_1 is a subgroup of prime index of T, the quasihamiltonian subgroup LT_1 has prime index in G, and this contradiction shows that there exist $i, j \leq r$ such that $[a_i, a_j] \neq 1$. Therefore the subgroup $\langle a_i, a_j\rangle$ is not quasihamiltonian, and hence it is isomorphic to G; it follows that G is generated by elements a and b of infinite order such that $G/T = \langle aT\rangle \times \langle bT\rangle$. Moreover, $\langle [a, b]\rangle$ is a subgroup of T and hence it is normal in G, so that $G/\langle [a, b]\rangle$ is abelian, and $G' = \langle [a, b]\rangle$. Let p be a prime number dividing the order of G' and let N be the unique subgroup of index p of G'. If H/N is any non-quasihamiltonian subgroup of G/N, then $H \simeq G$, so that $H' = G'$, and $H/N \simeq G/N$. This shows that either G/N is an $\mathfrak{X}$-group or it is quasihamiltonian (see [6], Proposition 1). Assume by contradiction that $N \neq \{1\}$. Then by the minimality of the order of T, G/N contains in any case a quasihamiltonian subgroup A/N of prime index. Clearly, A/N has torsion-free rank greater than 1, and hence A' is a subgroup of N, which is properly contained in G'. Therefore A is quasihamiltonian, a contradiction which shows that G' has order p, and so it is contained in the centre of G; in particular, $\langle a^p, b\rangle G'$ is abelian. Let L be a subgroup of G such that $G/G' = L/G' \times T/G'$. Then T/G' must be trivial, since otherwise it would contain a subgroup of prime index, and $G = LT$ would have a quasihamiltonian subgroup of prime index. Finally, $\langle a^p, b\rangle G' = \langle a^p, b\rangle T$ has index p in G, and this last contradiction proves the lemma. □

A group G is called a *P^*-group* if it is the semidirect product of an abelian normal subgroup A of prime exponent by a cyclic group $\langle x\rangle$ of prime-power order such that x induces on A a power automorphism of prime order (recall here that a *power automorphism* of a group G is an automorphism mapping every subgroup of G onto itself). It is easy to see that the subgroup lattice of any P^*-group is modular, and Iwasawa ([9],[10]) proved that a locally finite group has modular subgroup lattice if and only if it is a direct product

$$G = \operatorname*{Dr}_{i \in I} G_i,$$

where each G_i is either a P^*-group or a primary locally finite group with modular subgroup lattice, and elements of different factors have coprime orders.

We can now prove the main result for soluble groups in the class $\mathfrak{X}$.

Theorem 1. *Let G be a soluble $\mathfrak{X}$-group. Then G contains a subgroup of prime index having modular subgroup lattice.*

Proof. Assume that the statement is false, and let R be the Hirsch-Plotkin radical of G. By Lemma 2, G is not locally nilpotent, and so R has modular subgroup lattice. Moreover, R contains every non-periodic normal subgroup of G with modular subgroup lattice (see [22], Theorem 2.4.11); in particular, either G' is periodic or it is contained in R, and it follows that in any case the elements of finite order of G/R form a subgroup X/R. Suppose that G/R is finite. If N is any proper normal subgroup of G containing R, N is a non-periodic group with modular subgroup lattice, so that it is contained in R. Thus G/R has prime order, and this contradiction shows that G/R is infinite. Clearly, X is not isomorphic to G, so that it has modular subgroup lattice. Since G/X is torsion-free, the torsion subgroup T of X coincides with the set of all elements of finite order of G. Write $T = Dr_{i\in I}T_i$, where each T_i is either a P^*-group or a primary locally finite group with modular subgroup lattice, and $\pi(T_i) \cap \pi(T_j) = \emptyset$ if $i \neq j$. Assume that for some $i \in I$ there exists $a \in T_i$ such that $H = \langle a, g\rangle$ is not quasihamiltonian, where g is a suitable element of infinite order of G. Then G must be a 2-generator group.

Clearly $H = \langle g\rangle\langle a\rangle^{\langle g\rangle}$ and $H/\langle a\rangle^{\langle g\rangle}$ is torsion-free, so that there exists an element x of infinite order such that $G = \langle x\rangle \ltimes T$ and $T \simeq \langle a\rangle^{\langle g\rangle}$ has finite exponent (see [22], Theorem 2.4.13). Since G/R is infinite, also T is infinite, and so G has infinite Prüfer rank. It follows that there exist a 2-generator subgroup U of G and a normal subgroup V of U such that $U/V \simeq C_p \wr C_\infty$ (see [11]). Let B/V be the base group of U/V and let $u \in U$ such that $U = B\langle u\rangle$; since U/V is not nilpotent-by-finite, the subgroup $B\langle u^2\rangle$ is not quasihamiltonian and therefore it is 2-generated. On the other hand, $B\langle u^2\rangle/V$ cannot be 2-generated and this contradiction proves that $\langle a, g\rangle$ is quasihamiltonian for each $a \in T_i$, whenever g is an element of infinite order of G. Since G is generated by its elements of infinite order, all subgroups of T are normal in G and all elements of G with order a prime or 4 are central (see [22], Theorem 2.4.11). Therefore T is abelian. Clearly, the factor group G/T is isomorphic to all its non-abelian subgroups, and hence it contains an abelian subgroup B/T of prime index (see [24], Theorem 2 and [1], Corollary 2); then B is not quasihamiltonian, and so G/T is abelian. Moreover, G/T has rank greater than 1 (see [22], Theorem 2.4.11); it follows that for every element of infinite order g of G

the subgroup $\langle g, G' \rangle$ is quasihamiltonian, and hence it is contained in R. Therefore $G = R$, and this last contradiction completes the proof of the theorem. □

Corollary 1. *Let G be a soluble group containing proper non-quasihamiltonian subgroups, all of which are isomorphic to G. Then G contains a quasihamiltonian subgroup of prime index.*

Proof. Obviously, G is a finitely generated infinite group, so that in particular it is not periodic. It follows from Proposition 1 of [6] that G is an $\mathfrak{X}$-group, so that Theorem 1 yields that G contains a quasihamiltonian subgroup of prime index. □

2. Insoluble groups

We begin this section with a lemma concerning groups in which the join of normal subgroups with modular subgroup lattice has likewise modular subgroup lattice.

Lemma 3. *Let G be an $\mathfrak{X}$-group in which the subgroup M generated by all normal subgroups with modular subgroup lattice has itself modular subgroup lattice. Then the factor group G/M is either simple or infinite cyclic.*

Proof. Suppose that G/M is not simple, and assume for a contradiction that G/M has no finite non-trivial homomorphic image. Let L/M be a proper non-trivial normal subgroup of G/M and consider an element $g \in G \setminus L$. As the lattice $\mathfrak{L}(L)$ is not modular, $L\langle g \rangle$ is isomorphic to G, and hence G has a finite non-trivial homomorphic image G/Y. Then $G = YM$ and G/Y has modular subgroup lattice. In particular, G/Y is soluble, and so $G' < G$. As $G/G^{(3)}M$ is a finitely generated soluble group, we have $G = G^{(3)}M$, so that $G/G^{(3)}$ is metabelian, and $G^{(2)} = G^{(3)}$. On the other hand, $\mathfrak{L}(G^{(3)})$ is not modular, so that $G^{(3)} \simeq G$ and $G^{(4)} < G^{(3)}$. This contradiction shows that G/M has a finite non-trivial homomorphic image. If X/M is any non-trivial normal subgroup of G/M, the lattice $\mathfrak{L}(X)$ is not modular, so that $X \simeq G$ and $X/M \simeq G/M$. Therefore G/M is isomorphic to all its non-trivial normal subgroups, and hence it is infinite cyclic (see [13]). □

A group G is called an *extended Tarski group* if it contains a cyclic non-trivial normal subgroup N with prime-power order such that G/N is a Tarski group and either $H \leq N$ or $N \leq H$ for every subgroup H of G. It was proved by R. Schmidt that a periodic group G has modular subgroup lattice if and only if $G = M \times T$, where $\pi(M) \cap \pi(T) = \emptyset$, M is a locally finite group with modular subgroup lattice and $T = \mathrm{Dr}_i\, T_i$ is a direct product of Tarski and extended Tarski groups such that $\pi(T_i) \cap \pi(T_j) = \emptyset$ if $i \neq j$ (see [22], Theorem 2.4.16).

Theorem 2. *Let G be an insoluble $\mathfrak{X}$-group. Then either G contains a subgroup M with modular subgroup lattice such that the factor group G/M is simple or $G = \langle g \rangle \ltimes T$, where g is an element of infinite order, T is a Tarski or an extended Tarski group and $C_{\langle g \rangle}(T) = \{1\}$.*

Proof. Suppose that G is not an extension of a group with modular subgroup lattice by a simple group. Let $\mathfrak{T}$ be the set of all subgroups of G which are Tarski or extended Tarski groups, and let S be the soluble radical of G. Clearly, S has modular subgroup lattice, so that it is soluble and G/S is not simple. It follows from Lemma 3 that there exists a normal subgroup of G with modular subgroup lattice containing a G-invariant subgroup T of G which is either a Tarski or an extended Tarski group (see [22], Theorems 2.4.11 and 2.4.16). Assume by contradiction that G is periodic, and let x be any element of G such that $\mathfrak{L}(\langle x \rangle T)$ is not modular. Then $\langle x \rangle T \simeq G$ and so $\langle x \rangle T$ is not an extension of a group with modular subgroup lattice by a simple group. Let H/T be a subgroup of prime index of $\langle x \rangle T/T$. Then $\mathfrak{L}(H)$ is not modular, but H is not isomorphic to $\langle x \rangle T \simeq G$, a contradiction. Therefore $\langle x \rangle T$ is a group with modular subgroup lattice for every $x \in G$. It follows from Theorem 2.4.16 of [22] that T is the set of all π-elements of G, with $\pi = \pi(T)$ and it is centralized by all π'-elements, so that $G = TC$ where $C = C_G(T)$. Moreover, $Z = Z(T)$ is the set of all π-elements of C, so that $C \not\simeq G$ and C has modular subgroup lattice; then there exists a π'-subgroup L of C such that $C = Z \times L$, since $Z \leq Z(C)$ (see [22], Theorem 2.4.16). Therefore $G = T \times L$ has modular subgroup lattice, and this contradiction proves that G contains an element g of infinite order. Clearly, the lattice $\mathfrak{L}(\langle g \rangle \ltimes T)$ is not modular, so that $G \simeq \langle g \rangle \ltimes T$. Finally, let g^n be any element of $C_{\langle g \rangle}(T)$; the direct product $\langle g^n \rangle \times T$ is the extension of the abelian group $\langle g^n \rangle \times Z(T)$ by the Tarski group $T/Z(T)$, and so it is not isomorphic to G. Thus $n = 0$, and $C_{\langle g \rangle}(T) = \{1\}$. □

Recall that a subgroup H of a group G is *almost modular* if it is modular in a subgroup of finite index of G. The structure of groups in which every subgroup is almost modular has been studied in [8]. Here we will use a property of groups whose cyclic subgroups are almost modular.

Lemma 4. *Let G be a quasihamiltonian-by-finite group which is isomorphic to its non-quasihamiltonian subgroups. Then G is soluble with derived length at most* 3.

Proof. We may obviously suppose that G is not quasihamiltonian, so that it is finitely generated and contains proper non-quasihamiltonian subgroups (see [6], Proposition 1); in particular, G is an infinite group sartisfying the maximal condition on subgroups.

Let N be a quasihamiltonian subgroup of finite index of G. If for some $g \in G$ the subgroup $N\langle g\rangle$ is not quasihamiltonian, then $G \simeq N\langle g\rangle$ is soluble with derived length at most 3. Thus it can be assumed that $N\langle g\rangle$ is quasihamiltonian for all $g \in G$. It follows that all cyclic subgroups of G are almost modular, and hence the set T of all elements of finite order of G is a finite subgroup (see [8], Theorem 3.3). Moreover, NT/T is contained in the centre of G/T, and hence G/T is finite-by-abelian. Therefore the derived subgroup G' of G is finite, so that G' is quasihamiltonian and G is soluble with derived length at most 3. □

It has been proved in [24] that if G is an insoluble group containing proper non-abelian subgroups all of which are isomorphic to G, then the factor group $G/Z(G)$ is simple. In our situation, we obtain a corresponding result, replacing the centre by a suitable relevant subgroup. If G is a group, we shall denote by $Q(G)$ the set of all elements a of G such that $\langle a, g\rangle$ is quasihamiltonian for each $g \in G$. Clearly, if $a \in Q(G)$, the subgroup $\langle a\rangle$ is permutable in G. Therefore $Q(G)$ is a subset of the Hirsch-Plotkin radical of G.

Theorem 3. *Let G be an insoluble group containing proper non-quasihamiltonian subgroups. If G is isomorphic to all its non-quasihamiltonian subgroups, then $Q(G)$ is a quasihamiltonian subgroup of G and the factor group $G/Q(G)$ is simple.*

Proof. Let R be the Hirsch-Plotkin radical of G. As G is not locally nilpotent, R is quasihamiltonian; moreover, if g is any element of G, it follows that $R\langle g\rangle$ is soluble and so quasihamiltonian. Therefore R is contained in $Q(G)$, and hence $Q(G) = R$. Assume for a contradiction that $G/Q(G)$ is not simple, and let N be a normal subgroup of G such that $Q(G) < N < G$. If $g \in G \setminus N$, the subgroup $\langle g, N\rangle \simeq G$ and G has a finite non-trivial homomorphic image. Since every finitely generated subgroup of G is either nilpotent or isomorphic to G, it follows that G is locally graded and so $G/Q(G)$ has a finite non-trivial homomorphic image (see [23], Lemma 1). Moreover, N is isomorphic to G, so that

$N/Q(G) \simeq G/Q(G)$ and $G/Q(G)$ is isomorphic to all its non-trivial normal subgroups. Thus $G/Q(G)$ is cyclic (see [13]) and this contradiction completes the proof of the theorem. $\square$

Finally, the following example shows that there exists an $\mathfrak{X}$-group G containing a proper non-abelian subgroup which is not isomorphic to G. Let $G = \langle a, b\rangle$ be the nilpotent group of class 2 generated by two elements a and b subject to the additional relations

$$b^{81} = 1, \ \ [a, b]^3 = b^{27}.$$

It follows from Theorem 2.4.11 of [22] that the subgroup lattice of G is not modular; moreover, G is not an $\mathfrak{M}$-group (see [6], Proposition 1). It is easy to show that $\langle a^3, b, [a, b]\rangle$ is a non-abelian subgroup with modular subgroup lattice, so that in particular, it is not isomorphic to G. Moreover, it can be proved that every subgroup of G with non-modular subgroup lattice has the form $H = \langle a^n, b\rangle$, where the integer n is not divisible by 3. Then $[a^n, b]$ has order 9 and either $b^{27} = [a^n, b]^3$ or $b^{27} = [a^{-n}, b]^3$, so that H is isomorphic to G. Therefore G belongs to the class $\mathfrak{X}$.

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