N. V. Velichko, On the theory of spaces of continuous functions, *Russian Mathematical Surveys*, 1982, Volume 37, Issue 4, 149–150

DOI: 10.1070/RM1982v037n04ABEH003972

On the theory of spaces of continuous functions

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The set of real-valued continuous functions on a topological space X is denoted by $C(X)$. When this set is equipped with the topology of ordinary convergence, then we write $C_p(X)$; if it is equipped with the compact-open topology, we keep to the general notation $C(X)$. In the last case there is also the weak topology, for which we use the notation $C_\omega(X)$.

Our main result is the following theorem.

Theorem. *When X is compact and $C_p(X)$ is normal then X has countable tightness.*

Proof. As a preliminary we establish the following assertion: (A) *Let $C_p(X)$ be normal, let F and Φ be two closed disjoint sets in $C_p(X)$. Then there is a countable set $Y \subseteq X$ for which $[\pi_Y(F)]_M \cap \cap [\pi_Y(\Phi)]_M = \phi$, where $\pi_Y(f) = f\,|_Y$ and $M = \pi_Y(C_p(X))$.*

Let H and V be disjoint neighbourhoods of F and Φ in $C_p(X)$. Since $C_p(X)$ is dense in R^X, we can use the theorem of Bockstein [1] according to which there exists a countable set $Z \subseteq X$ for which $\pi_Z(H) \cap \pi_Z(V) = \varnothing$. Suppose that $\varnothing = K' = [\pi_Z(F)] \cap [\pi_Z(\Phi)]$ and let $K = \pi_Z^{-1}(K')$. Now K is closed in $C_p(X)$, and since $\varnothing = K' \cap \pi_Z(H) = K' \cap \pi_Z(V)$, we see that $K \cap F = = K \cap \Phi = \varnothing$. Then we can choose disjoint neighbourhoods W_1, W_2, and W_3 of F, Φ, and K in $C_p(X)$. Again by Bockstein's theorem, there exists a countable set $Y \supseteq Z$ such that the family $\{\pi_Y(W_i)\colon i = 1, 2, 3\}$ is disjoint. Then $\varnothing = [\pi_Y(F)] \cap [\pi_Y(\Phi)]$. For if there were a function $f \in C(X)$ such that $\pi_Y(f) \in [\pi_Y(F)] \cap [\pi_Y(\Phi)]$, then we would have $\pi_Z(f) \in [\pi_Z(F)] \cap [\pi_Z(\Phi)] = = K'$, and $f \in K$. But then $\pi_Y(f) \in \pi_Y(W_3)$, and $\pi_Y(W_3) \cap \pi_Y(F) = \varnothing = \pi_Y(W_3) \cap \pi_Y(\Phi)$, which gives a contradiction. (A) is now proved.

Let $t(X) > \aleph_0$. In X there is a free sequence $\mathcal{F} = \{x_\alpha\colon \alpha < \omega_1\}$ of length $\aleph_1$. Let F be the set of condensation points of $\mathcal{F}$. Then $\mathcal{F}$ converges with respect to cardinal to F, that is, any neighbourhood of F contains all elements of $\mathcal{F}$ from some one onwards.

Each ordinal $\alpha < \omega_1$ can be written as $\alpha_0 + n$, where α_0 is a limit ordinal. Hence we can speak of the parity of α (namely, that of n).

We choose a point $x_0 \in \mathcal{F}$ arbitrarily and a function $f_0 \in C(X, [0, 1])$ such that $f_0\,|_F \equiv 0$ and $f_0(x_0) = 1$. We choose closed sets F_{0n} so that $CZ(f_0) = \{y\colon f_0(y) > 0\} = \bigcup\{F_{0n}\colon n = 1, \ldots\}$, $F_{0n} \supseteq \{x\colon f_0(x) = 1\}$ and $F_{0n} \subseteq F_{0,n+1}$ for all $n = 1, 2, \ldots$.

Next we choose for every $\alpha < \omega_1$ a point $x_\alpha \in \mathcal{F}$, a function $f_\alpha \in C(X, [0, 1])$ and closed sets $F_{\alpha n}$ so that the following conditions hold:

1) $f_\alpha(x_\beta) = 1$ for $\beta \leqslant \alpha$, $f_\alpha(x_\beta) = 0$ for $\beta > \alpha$, and $f_\alpha\,|_F \equiv 0$,

2) $CZ(f_\alpha) = \bigcup\{F_{\alpha n}\colon n = 1, \ldots\}$, $F_{\alpha n} \supseteq \{x\colon f_\alpha(x) = 1\}$, $F_{\alpha n} \subseteq F_{\alpha, n+1}$,

3) $x_\alpha \in \bigcap\{Z(f_\beta) = \{y\colon f_\beta(y) = 0\}\colon \beta < \alpha\}$, $Z(f_\alpha) \subseteq \bigcap\{Z(f_\beta)\colon \beta < \alpha\}$,

4) for any $F_{\alpha n}$ there exists a $\beta > \alpha$ of the other parity for which $f_\beta\,|F_{\alpha n} \equiv f_{\alpha n}|F_{\alpha n}$.

Suppose that the construction has been carried out for all β less than α and that α is a limit ordinal.

The family $\{F_{\beta n}\colon \beta < \alpha\}$ is countable and so can be numbered: $\{F_i\colon i = 1, \ldots\}$.

Let $Z = \bigcap\{Z(f_\beta)\colon \beta < \alpha\} \cap \mathcal{F}$. Then $F \cap [\{x_\beta\colon \beta < \alpha\}] = \varnothing$ and $|\, Z \setminus [\bigcup\{x_\beta\colon \beta < \alpha\}]\,| > 1$ (by the property of free sequences and by 1)). We choose the point x_α arbitrarily from $Z \setminus [\{x_\beta\colon \beta < \alpha\}]$ and the function f_α in $C(X, [0, 1])$ so that $f_\alpha\,|_F \equiv 0$, $f_\alpha\,|_{T_\alpha} \equiv 1$, and $f_\alpha\,|_{F_k} \equiv f_\beta\,|_{F_k}$, where $T_\alpha = [\{x_\beta\colon \beta < \alpha\}] \cup \{x_\alpha\}$, k is minimal with $F_k = F_{\beta n}$, and n is odd. We then choose $F_{\alpha n}$ to satisfy 2).

Suppose that $k > 0$ is arbitrary and that the objects have been constructed for all $\alpha + n$ with $n < k$. Then we choose $x_{\alpha+k}$ arbitrarily in $\mathcal{F} \cap \bigcap\{Z(f_\beta)\colon \beta < \alpha + k\}$ and we choose $f_{\alpha+k}$ from 1) and 3) so that $f_{\alpha+k}\,|_{F_m} \equiv f_\sigma\,|_{F_m}$, where m is minimal with $F_m = F_{\sigma n}$ and n and k are of opposite parity. We choose the sets $F_{\alpha+k,n}$ using 2).

It is obvious that with this construction the objects satisfy all the requirements.

We put $\Phi = \{f_\alpha: \alpha < \omega_1\}$ and $T = \{x_\alpha: \alpha < \omega_1\}$. We claim that Φ is discrete in $C_p(X)$.

Let $f \in [\Phi]$. Then $f|_{\bigcap\{Z(f_\alpha):\ \alpha<\omega_1\}} \equiv 0$ and f can take only the values 0 and 1 on T. Let α be minimal with $f(x_\alpha) = 0$. The set $V = \{g: |g(x_\alpha)| < 2^{-1}\}$ is a neighbourhood of f that does not contain f_β for $\beta \geqslant \alpha$. Suppose that α is a limit ordinal. Then there exists a $y \in [\{x_\beta: \beta < \alpha\}]$ such that $y \notin [\{x_\gamma: \gamma < \beta\}], \beta < \alpha$ (this is easy to verify). But $f(y) = 1$. If $\beta < \alpha$, then $f_\beta(y) = 0$ by 3), and $y \in [\{x_\gamma: \gamma < \alpha\}]$. Hence, the neighbourhood $W = \{g: |g(y) - 1| < 2^{-1}\}$ of f does not contain f_β for $\beta < \alpha$, which contradicts the condition that $f \in [\Phi]$.

Thus, $\alpha = \gamma + k$. Then the neighbourhood $U = \{g: |g(x_{\gamma+k-1}) - 1| < 2^{-1}\}$ of f does not contain f_β for $\beta < \alpha - 1$, so that it is clear that $f = f_{\alpha-1} \in \Phi$, and Φ is closed in $C_p(X)$.

It is easy to verify that Φ is discrete in itself.

We put $K = \{f_\alpha: \alpha \text{ odd}\}$ and $L = \{f_\alpha: \alpha \text{ even}\}$.

Let Y be an arbitrary countable set in X, and $Y' = Y \setminus \bigcap \{Z(f_\beta): \beta < (\omega_1)\}$. There exists an $\alpha < \omega_1$ such that $Y' \subseteq CZ(f_\alpha)$. Let $V = \{g: |f_\alpha(y_i) - g(y_i)| < \varepsilon\}$ be a neighbourhood of f_α for which $y_i \in Y$. For the set $Q = \{y_i: y_i \in Y'\}$ we can find an n for which $Q \subseteq F_{\alpha n}$. By 4) there is a $\gamma > \alpha$ of the other parity such that $f_\gamma|_{F_{\alpha n}} \equiv f_\alpha|_{F_{\alpha n}}$. If $f_\alpha \in K$, then $f_\gamma \in L$, and $f_\alpha \in V$, $f_\gamma \in V$. Consequently, $\pi_Y(f_\alpha) \in [\pi_Y(K)] \cap [\pi_Y(L)]$. By (A), $C_p(X)$ cannot be normal.

The theorem is now proved.

For $C_\omega(X)$ the following estimates have been obtained (X being compact):

1) $|X| \leqslant \chi(C_\omega(X)) = w(C_\omega(X)) \leqslant \exp W(X)$,

2) $\psi X(C_\omega(X)) = d(X)$ if X is an Eberlein compactum,

3) $hd(C_\omega(X)) = nw(C_\omega(X)) = d(C_\omega(X)) = w(X)$, where χ is the character, w the weight, d the density, ψX the pseudocharacter, nw the net weight, $h\varphi = \sup\{\varphi(Y): Y \subseteq X\}$ for any cardinal-valued invariant φ.

Reference

[1] M. Bockstein, Un théorème de séparabilité pour les produits topologiques, Fund. Math. 35 (1948), 242-246. MR 10-316.

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Received by the Editors 11 July 1981