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DISCRETE UNIVERSALITY OF THE RIEMANN ZETA-FUNCTION AND UNIFORM DISTRIBUTION MODULO 1

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It is proved that a wide class of analytic functions can be approximated by shifts $\zeta(s+i\varphi(k))$, $k \geqslant k_0$, $k \in \mathbb{N}$, of the Riemann zeta-function. Here the function $\varphi(t)$ has a continuous nonvanishing derivative on $[k_0, \infty)$ satisfying the estimate $\varphi(2t) \max_{t \leqslant u \leqslant 2t} (\varphi'(u))^{-1} \ll t$, and the sequence $\{a\varphi(k) : k \geqslant k_0\}$ with every real $a \neq 0$ is uniformly distributed modulo 1. Examples of $\varphi(t)$ are given.

§1. Introduction

As usual, let $\zeta(s)$, $s = \sigma + it$, be the Riemann zeta-function. In 1975, S. M. Voronin discovered [11] the universality property of the function $\zeta(s)$ in approximation of a wide class of analytic functions by shifts $\zeta(s+i\tau)$, $\tau \in \mathbb{R}$. For the precise statement of the Voronin theorem, it is convenient to use the following notation. Let

$$D = \{s \in \mathbb{C} : \tfrac{1}{2} < \sigma < 1\},$$

let $\mathcal{K}$ be the class of compact subsets of the strip D with connected complements, and let $H_0(K)$ with $K \in \mathcal{K}$ denote the class of continuous nonvanishing functions on K that are analytic in the interior of K. Then the last version of the Voronin theorem asserts that if $K \in \mathcal{K}$ and $f(s) \in H_0(K)$, then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\Big\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\Big\} > 0.$$

Here meas$A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. The proof of the above statement can be found, for example, in [1, 4, 6, 10].

The above version of the Voronin theorem is of continuous type: τ in $\zeta(s+i\tau)$ can take an arbitrary real value. Also, there exists a discrete version of the Voronin universality theorem when τ takes values in a certain discrete set.

Key words: Riemann zeta-function, uniform distribution modulo 1, universality, weak convergence.

Denote by $\#A$ the cardinality of the set A. Then, for the same K and $f(s)$ as above, for every $\varepsilon > 0$ and $h > 0$ we have

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon\Big\} > 0.$$

The discrete universality was proposed by A. Reich in [9]. He proved that the Dedekind zeta-functions of number fields have a discrete universality property. In the case of the field of rational numbers $\mathbb{Q}$, the Reich theorem gives the above statement for the Riemann zeta-function.

The discrete set $\{kh : h = 0, 1, 2, \ldots,\ h > 0\}$ is very simple. In [3], this set was replaced by $\{k^\alpha h : k = 0, 1, 2, \ldots, h > 0\}$ with a fixed $0 < \alpha < 1$. There exists a problem to describe the sets for which a discrete universality theorem is valid. Our aim in this paper is to generalize the result of [3]. We denote by $U_1(k_0)$, $k_0 \in \mathbb{N}$, the class of real monotone increasing functions $\varphi(t)$ having a continuous nonvanishing derivative on $[k_0, \infty)$ and satisfying the estimate

$$\varphi(2t) \max_{t\leqslant u\leqslant 2t} \big(\varphi'(u)\big)^{-1} \ll t$$

for $t \geqslant k_0$. Moreover, let $U_2(k_0)$ be the class of real functions $\varphi(t)$ on $[k_0, \infty)$ such that the sequence

$$\{a\varphi(k) : k \geqslant k_0\}$$

with every real $a \neq 0$ is uniformly distributed modulo 1. We remind the reader that the sequence

$$\{a_k : k \in \mathbb{N}\} \subset \mathbb{R}$$

is said to be uniformly distributed modulo 1 if for every real numbers a, b with $0 \leqslant a < b \leqslant 1$ we have

$$\lim_{N\to\infty} \frac{\#\{1 \leqslant k \leqslant N : \{a_k\} \in [a, b)\}}{N} = b - a,$$

where $\{u\}$ is the fractional part of $u \in \mathbb{R}$.

Theorem 1. *Suppose that*

$$\varphi(t) \in U_1(k_0) \cap U_2(k_0).$$

Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then for every $\varepsilon > 0$ and $h > 0$ we have

$$\liminf_{N\to\infty} \frac{1}{N-k_0+1} \#\Big\{k_0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ih\varphi(k)) - f(s)| < \varepsilon\Big\} > 0.$$

It is known (see [5, Excercise 3.9]) that the sequence $\{ak^\alpha : k \in \mathbb{N}\}$, where $\alpha > 0$ is not an integer and $a \neq 0$ is real, is uniformly distributed modulo 1. Therefore,

$$t^\alpha \in U_1(1) \cap U_2(1).$$

Let a and α be the same as above, and let $\beta \in \mathbb{R}^+$ be arbitrary. Then the sequence

$$\{ak^\alpha \log^\beta k : k \geqslant 2$$

is uniformly distributed modulo 1, see [5, Excercise 3.10]. It is easy to check that $t^\alpha \log^\beta t \in U_1(2)$. Therefore, Theorem 1 is valid with

$$\varphi(k) = k^\alpha \log^\beta k.$$

The functions

$$\varphi(t) = t^\alpha \log^\beta t,$$

with $\alpha \in \mathbb{N}$ and $\beta > 1$,

$$\varphi(t) = t \log^\beta t \quad \text{and} \quad \varphi(t) = t^2 \log^\beta t,$$

with $0 < \beta \leqslant 1$, $t \geqslant 2$, see [5, Excercises 3.11 and 3.14], also satisfy the hypothesis of Theorem 1.

Theorem 1 admits the following modification.

Theorem 2. *Suppose that*

$$\varphi(t) \in U_1(k_0) \cap U_2(k_0).$$

Let $K \in \mathcal{K}$, $f(s) \in H_0(K)$ and $h > 0$. Then the limit

$$\lim_{N\to\infty} \frac{1}{N-k_0+1} \#\Big\{k_0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ih\varphi(k)) - f(s)| < \varepsilon\Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

§2. Lemmas

Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and let

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p. Denote by $\mathcal{B}(X)$ the Borel σ-field of the space X. Since the torus Ω with the product topology and pointwise multiplication is a compact topological Abelian group, the space $(\Omega, \mathcal{B}(\Omega))$ admits the probability Haar measure m_H, and we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(p)$ be the projection of $\omega \in \Omega$ to the coordinate space γ_p, $p \in \mathbb{P}$, where $\mathbb{P}$ is the set of all prime numbers.

We start with a limit theorem on the torus Ω. For $A \in \mathcal{B}(\Omega)$, let

$$Q_N(A) = \frac{1}{N-k_0+1} \#\left\{k_0 \leqslant k \leqslant N : \left(p^{-ih\varphi(k)} : p \in \mathbb{P}\right) \in A\right\}.$$

Lemma 1. *Suppose that $\varphi(t) \in U_2(k_0)$. Then Q_N converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. We consider the Fourier transform $g_N(\underline{k})$, $\underline{k} = (k_p \in \mathbb{Z} : p \in \mathbb{P})$, of the measure Q_N. It is well known that

$$\begin{aligned} g_N(\underline{k}) &= \int\limits_{\Omega} \prod_p \omega^{k_p}(p)\,\mathrm{d}\,Q_N = \frac{1}{N-k_0+1}\sum_{k=k_0}^{N}\prod_p p^{-ik_p h\varphi(k)} \\ &= \frac{1}{N-k_0+1}\sum_{k=k_0}^{N}\exp\Big\{-ih\varphi(k)\sum_p k_p\log p\Big\}, \end{aligned} \tag{1}$$

where only a finite number of integers k_p are distinct from zero. Obviously,

$$g_N(\underline{0}) = 1. \tag{2}$$

Since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$, we have

$$\sum_p k_p \log p \neq 0$$

for $\underline{k} \neq \underline{0}$. Therefore, by the hypothesis of the lemma, the sequence

$$\Big\{-\frac{h}{2\pi}\varphi(k)\sum_p k_p\log p\Big\}$$

with $\underline{k} \neq \underline{0}$ is uniformly distributed modulo 1. Hence, by the Weyl criterion (see, e.g., [5, Theorem 2.1]), from (1) and (2) we see that

$$\lim_{N\to\infty} g_N(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases}$$

Since the right-hand side of the latter identity is the Fourier transform of the Haar measure m_H, we obtain the assertion of the lemma by using a continuity theorem for probability measures on compact groups. □

Lemma 2. *Suppose that $\varphi(t) \in U_1(k_0)$, $\frac{1}{2} < \sigma < 1$, $\tau \in \mathbb{R}$ and $h > 0$. Then*

$$\int\limits_{k_0}^{T} |\zeta(\sigma + i\tau + ih\varphi(t))|^2\,\mathrm{d}\,t \ll_{h,\sigma} T(1 + |\tau|).$$

Proof. The definition of the class $U_1(k_0)$ shows that, for $V \geqslant k_0$,

$$\int_V^{2V} |\zeta(\sigma + i\tau + ih\varphi(t))|^2 \,\mathrm{d}\,t = \int_V^{2V} \frac{1}{\varphi'(t)} |\zeta(\sigma + i\tau + ih\varphi(t))|^2 \,\mathrm{d}\,\varphi(t)$$

$$\ll_{h,\sigma} \max_{V \leqslant t \leqslant 2V} \frac{1}{\varphi'(t)} \int_V^{2V} \mathrm{d}\left(\int_1^{\tau + h\varphi(t)} |\zeta(\sigma + iu)|^2 \,\mathrm{d}\,u \right)$$

$$\ll_{h,\sigma} \max_{V \leqslant t \leqslant 2V} \frac{1}{\varphi'(t)} \int_1^{\tau + h\varphi(t)} |\zeta(\sigma + iu)|^2 \,\mathrm{d}\,u \Big|_V^{2V}.$$

Hence, using the well-known estimate

$$\int_1^T |\zeta(\sigma + it)|^2 \,\mathrm{d}\,t \ll_\sigma T,$$

which is valid for $\frac{1}{2} < \sigma < 1$, we obtain

$$\int_V^{2V} |\zeta(\sigma + i\tau + ih\varphi(t))|^2 \,\mathrm{d}\,t \ll_{h,\sigma} V + |\tau| \max_{V \leqslant t \leqslant 2V} \left(\varphi'(t)\right)^{-1}$$

$$\ll_{h,\sigma} V + \frac{|\tau|}{\varphi(2V)} \varphi(2V) \max_{V \leqslant t \leqslant 2V} \left(\varphi'(t)\right)^{-1}$$

$$\ll_{h,\sigma} V + V \frac{|\tau|}{\varphi(2V)} \ll_{h,\sigma} V + V|\tau|.$$

Now, taking $V = 2^{-k-1}T$ and summing over $k = 0, 1, \ldots$, we arrive at the estimate of the lemma. □

To estimate the discrete mean square

$$J(N, t) \stackrel{def}{=} \sum_{k=k_0}^{N} |\zeta(\sigma + it + ih\varphi(k)|^2, \qquad t \in \mathbb{R},$$

we need the following Gallagher lemma, see [8, Lemma 1.4].

Lemma 3. *Let T_0 and $T \geqslant \delta > 0$ be real numbers, and let $\mathcal{T}$ be a finite set in the interval $[T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}]$. Define*

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t-x| < \delta}} 1,$$

and let $S(x)$ be a complex-valued continuous function on $[T_0, T+T_0]$ having a continuous derivative on $(T_0, T+T_0)$. Then

$$\sum_{t\in\mathcal{T}} N_\delta^{-1}(t)|S(t)|^2 \leqslant \frac{1}{\delta}\int_{T_0}^{T_0+T} |S(x)|^2 \,\mathrm{d}\,x + \left(\int_{T_0}^{T_0+T} |S(x)|^2 \,\mathrm{d}\,x \int_{T_0}^{T_0+T} |S'(x)|^2 dx\right)^{\frac{1}{2}}.$$

Lemma 4. *Suppose that $\varphi(t) \in U_1(k_0)$. Then for $\frac{1}{2} < \sigma < 1$ and $h > 0$ we have*

$$J(N,t) \ll_\sigma N(1+|t|).$$

Proof. We take $\mathcal{T} = \{k : k_0 \leqslant k \leqslant N\}$ in Lemma 3. Then, after a suitable choice of δ, T_0 and T, we get

$$\begin{aligned} J(N,t) \ll &\int_{k_0+\frac{1}{2}}^{N+\frac{1}{2}} |\zeta(\sigma+it+ih\varphi(\tau))|^2 \,\mathrm{d}\,\tau \\ &+\left(\int_{k_0+\frac{1}{2}}^{N+\frac{1}{2}} |\zeta(\sigma+it+ih\varphi(\tau))|^2 \,\mathrm{d}\,\tau \int_{k_0+\frac{1}{2}}^{N+\frac{1}{2}} |\zeta'(\sigma+it+ih\varphi(\tau))|^2 \,\mathrm{d}\,\tau\right)^{\frac{1}{2}}. \end{aligned} \tag{3}$$

Lemma 2 implies the estimate

$$\int_{k_0+\frac{1}{2}}^{N+\frac{1}{2}} |\zeta(\sigma+it+ih\varphi(\tau))|^2 \,\mathrm{d}\,\tau \ll_{h,\sigma} N(1+|t|).$$

Moreover, by the Cauchy integral formula, for $\frac{1}{2} < \sigma < 1$,

$$\zeta'(\sigma+it+ih\varphi(\tau)) = \frac{1}{2\pi i}\int_{|z-\sigma|=\delta} \frac{\zeta(z+it+ih\varphi(\tau))}{(z-\sigma)^2}\,\mathrm{d}\,z \ll |\zeta(\sigma_1+it+ih\varphi(\tau))|$$

with a suitable $\delta > 0$ and σ_1 satisfying $\frac{1}{2} < \sigma_1 < 1$. Hence, in view of Lemma 2,

$$\int_{k_0+\frac{1}{2}}^{N+\frac{1}{2}} |\zeta'(\sigma+it+ih\varphi(\tau))|^2 \,\mathrm{d}\,\tau \ll_{h,\sigma} N(1+|t|).$$

We note that the last estimate also follows from the well-known bound

$$\int_0^T |\zeta'(\sigma+it)|^2 \,\mathrm{d}\,t \ll_\sigma T, \quad \frac{1}{2}<\sigma<1,$$

in the same way as that used in the proof of Lemma 2. Combined with (3), this proves the lemma. □

For a fixed $\theta > \frac{1}{2}$ and $m, n \in \mathbb{N}$, we put

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^\theta\right\},$$

and define the functions

$$\zeta_n(s) = \sum_{m=1}^\infty \frac{v_n(m)}{m^s}$$

and

$$\zeta_n(s,\omega) = \sum_{m=1}^\infty \frac{\omega(m)v_n(m)}{m^s},$$

where, for $\omega \in \Omega$,

$$\omega(m) = \prod_{\substack{p^l \mid m \\ p^{l+1} \nmid m}} \omega^l(p).$$

We note that the series for $\zeta_n(s)$ and $\zeta_n(s,\omega)$ are absolutely convergent for $\sigma > \frac{1}{2}$ [6].

Let $H(D)$ denote the space of analytic functions on D equipped with the topology of uniform convergence on compact sets.

Let $\{K_l : l \in \mathbb{N}\}$ be a sequence of compact subsets of the strip D such that

$$D = \bigcup_{l=1}^\infty K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact subset, then $K \subset K_l$ for some $l \in \mathbb{N}$. Then

$$\rho(g_1,g_2) = \sum_{l=1}^\infty 2^{-l} \frac{\sup_{s\in K_l} |g_1(s)-g_2(s)|}{1+\sup_{s\in K_l}|g_1(s)-g_2(s)|}, \qquad g_1, g_2 \in H(D),$$

is a metric in the space $H(D)$ inducing its topology of uniform convergence on compact sets.

Lemma 5. *Suppose that* $\varphi(t) \in U_1(k_0)$ *and* $h > 0$. *Then*

$$\liminf_{n\to\infty} \frac{1}{N-k_0+1} \sum_{k=k_0}^{N} \rho\left(\zeta(s+ih\varphi(k)), \zeta_n(s+ih\varphi(k))\right) = 0.$$

Proof. Let K be a compact subset of the strip D. For $\sigma > \frac{1}{2}$, the function $\zeta_n(s)$ has the representation

$$\zeta_n(s) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z) l_n(z) \frac{\mathrm{d}\,z}{z},$$

where

$$l_n(s) = \frac{s}{\theta}\Gamma\left(\tfrac{s}{\theta}\right) n^s, \qquad n \in \mathbb{N},$$

and $\Gamma(s)$ is the Euler gamma-function, see the proof of Theorem 5.4.2 in [6]. This and the residue theorem imply the estimate

$$\frac{1}{N-k_0+1} \sum_{k=k_0}^{N} \sup_{s\in K} |\zeta(s+ih\varphi(k)) - \zeta_n(s+ih\varphi(k))|$$
$$\ll \int_{-\infty}^{\infty} |l_n(\sigma_1+i\tau)| \left(\frac{1}{N-k_0+1} \sum_{k=k_0}^{N} |\zeta(\sigma+it+ih\varphi(k)+i\tau)| \right) \mathrm{d}\,\tau + o(1)$$

as $N \to \infty$, where $\sigma_1 < 0$, $\frac{1}{2} < \sigma < 1$, and t is bounded by a constant depending on K. Therefore, by Lemma 4, the left-hand side of the above relation is estimated as

$$\ll_K \int_{-\infty}^{\infty} l_n(\sigma_1+i\tau)(1+|\tau|)\,\mathrm{d}\,\tau + o(1)$$

as $N \to \infty$. Now, this and the properties of $l_n(s)$ give the identity

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N-k_0+1} \sum_{k=k_0}^{N} \sup_{s\in K} |\zeta(s+ih\varphi(k)) - \zeta_n(s+ih\varphi(k))| = 0.$$

Recalling the definition of the metric ρ, we get the assertion of the lemma. □

§3. A limit theorem

For $\omega \in \Omega$, define

$$\zeta(s,\omega) = \sum_{m=1}^{\infty} \frac{\omega(m)}{m^s}.$$

Then $\zeta(s,\omega)$ is an $H(D)$-valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, because for almost all $\omega \in \Omega$ the above series converges uniformly on compact subsets of the strip D, see [6]. Denote by P_ζ the distribution of $\zeta(s,\omega)$, i.e.,

$$P_\zeta(A) = m_H(\omega \in \Omega : \zeta(s,\omega) \in A), \qquad A \in \mathcal{B}(H(D)).$$

Theorem 3. *Suppose that $\varphi(t) \in U_1(k_0) \cap U_2(k_0)$ and $h > 0$. Then*

$$P_N(A) \overset{def}{=} \frac{1}{N-k_0+1} \# \left\{k_0 \leqslant k \leqslant N : \zeta(s+ih\varphi(k)) \in A\right\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to the measure P_ζ as $N \to \infty$. Moreover, the support of P_ζ is the set $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$.

Proof. We start with a limit theorem for

$$P_{N,n}(A) \overset{\text{def}}{=} \frac{1}{N-k_0+1} \# \left\{k_0 \leqslant k \leqslant N : \zeta_n(s+ih\varphi(k)) \in A\right\}, \ A \in \mathcal{B}(H(D)).$$

Let $u_n : \Omega \to H(D)$ be defined by the formula

$$u_n(\omega) = \zeta_n(s,\omega), \qquad \omega \in \Omega.$$

The absolute convergence of the series for $\zeta_n(s,\omega)$ ensures that the function u_n is continuous. Moreover, $P_{N,n} = Q_N u_n^{-1}$, where Q_N is from Lemma 1. Therefore, denoting $\widehat{P}_n = m_H u_n^{-1}$ and using Lemma 1 and the continuity of u_n, we see that $P_{N,n}$ converges weakly to $\widehat{P}_n$ as $N \to \infty$.

Let θ_N be a random variable defined on a certain probability space with measure μ and having the distribution

$$\mu\left\{\theta_N = h\varphi(k)\right\} = \frac{1}{N-k_0+1}, \qquad k = k_0, k_0+1, \ldots, N.$$

We introduce the $H(D)$-valued random elements

$$X_{N,n}(s) = \zeta_n(s+i\theta_N)$$

and

$$X_N(s) = \zeta(s+i\theta_N).$$

Using these random elements, the weak convergence for $P_{N,n}$, Lemma 5, and a standard method based on the Prokhorov theorem (see [2, Theorem 6.1] and Theorem 4.2 of [2]), we conclude that, on $(H(D), \mathcal{B}(H(D)))$, there exists a probability measure P such that P_N converges weakly to P as $N \to \infty$, and that $\widehat{P}_n$ also converges weakly to P as $n \to \infty$. On the other hand,

$$\frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \zeta(s+i\tau) \in A\right\}, \qquad A \in \mathcal{B}(H(D)),$$

also converges weakly to the limit measure P of $\widehat{P}_n$ as $n \to \infty$, and P coincides with P_ζ, see [3]. Therefore, P_N also converges weakly P_ζ as $N \to \infty$. Moreover, it is known that the support of P_ζ is the set S, see [6]. The theorem is proved. □

§4. Proof of universality theorems

Proof of Theorem 1. By the Mergelyan theorem on the approximation of analytic functions by polynomials, see [7] and [12], there exists a polynomial $p(s)$ such that

$$\sup_{s \in K}\left|f(s) - e^{p(s)}\right| < \frac{\varepsilon}{2}. \tag{4}$$

Define the set

$$G = \left\{g \in H(D) : \sup_{s \in K}\left|g(s) - e^{p(s)}\right| < \frac{\varepsilon}{2}\right\}.$$

Then G is an open set in $H(D)$. Moreover, since $e^{p(s)} \in S$, Theorem 3 implies that the set G is an open neighborhood of the support of the measure P_ζ. Thus, $P_\zeta(G) > 0$. Therefore, Theorem 3 together with the equivalent of the weak convergence of probability measure in terms of open sets, see [2, Theorem 2.1], yields the inequality

$$\liminf_{N \to \infty} P_N(G) \geqslant P_\zeta(G) > 0.$$

Hence, by the definitions of the set G and P_N, we have

$$\liminf_{N \to \infty} \frac{1}{N - k_0 + 1} \# \left\{k_0 \leqslant k \leqslant N : \sup_{s \in K}\left|\zeta(s + ih\varphi(k)) - e^{p(s)}\right| < \frac{\varepsilon}{2}\right\} > 0.$$

Combining this with (4) gives the assertion of Theorem 1. □

Proof of Theorem 2. For a given $\varepsilon > 0$, let

$$G_\varepsilon = \left\{g \in H(D) : \sup_{s \in K}|g(s) - f(s)| < \varepsilon\right\}.$$

Then we see that the boundary ∂G_ε is the set

$$\left\{g \in H(D) : \sup_{s \in K}|g(s) - f(s)| = \varepsilon\right\}.$$

Therefore, $\partial G_{\varepsilon_1} \cap \partial G_{\varepsilon_2} = \varnothing$ whenever the positive numbers ε_1 and ε_2 are different. It follows that $P_\zeta(\partial G_\varepsilon) > 0$ for an at most countable set of values of

$\varepsilon > 0$, i.e., the set G_ε is a continuity set of the measure P_ζ for all but at most countably many $\varepsilon > 0$. Therefore, using the equivalent of the weak convergence of probability measures in terms of continuity sets, see [2, Theorem 2.1], and recalling Theorem 3, we have

$$\lim_{N\to\infty} P_N(G_\varepsilon) = P_\zeta(G_\varepsilon)$$

for all but at most countably many $\varepsilon > 0$. This and the definitions of P_N and G_ε give the relation

$$\lim_{N\to\infty} \frac{1}{N-k_0+1} \#\Big\{k_0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ih\varphi(k)) \\ - f(s)| < \varepsilon\Big\} = P_\zeta(G_\varepsilon) \tag{5}$$

for all but at most countably many $\varepsilon > 0$. By Theorem 3, the function $e^{p(s)}$, where $p(s)$ is a polynomial, is an element of the support of the measure P_ζ. By the Mergelyan theorem, the polynomial $p(s)$ can be chosen to satisfy inequality (4). Define yet another set

$$\widehat{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s\in K} \left| g(s) - e^{p(s)} \right| < \frac{\varepsilon}{2} \right\}.$$

Then $P_\zeta(\widehat{G}_\varepsilon) > 0$. Moreover, inequality (4) shows that $\widehat{G}_\varepsilon \subset G_\varepsilon$. Thus,

$$P_\zeta(G_\varepsilon) \geqslant P_\zeta(\widehat{G}_\varepsilon) > 0,$$

and the assertion of Theorem 2 follows from (5). □

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