F. Alberto Grünbaum, Band-time-band limiting integral operators and commuting differential operators,
Algebra i Analiz, 1996, Volume 8, Issue 1, 122–126

Алгебра и анализ
Том 8 (1996), вып. 1

BAND-TIME-BAND LIMITING INTEGRAL OPERATORS AND COMMUTING DIFFERENTIAL OPERATORS

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Given a differential operator L defined (for instance) in $L^2(0,\infty)$ with properly chosen eigenfunctions $f(x,k)$, namely,

$$Lf = -D_x^2 f + Vf = k^2 f,$$

one is often led naturally to consider the band-time-band limiting operator acting on $L^2(-G,G)$ as an integral operator K with kernel given by

$$K_T(k_1,k_2) = \int_0^T f(x,k_1)f(x,k_2)dx.$$

The issue of computing accurately and economically the eigenfunctions of this compact selfadjoint operator, for several values of G and T (the band and time limits), arises in practice. It is impossible to do this in terms of classical functions, and the task of doing an approximate computation of lots of these eigenfunctions becomes an impossible task given the "full" nature of the matrix involved.

In a remarkable series of papers dealing with work of Shannon in Communication Theory, the Bell Labs team of Landau, Pollak, and Slepian (see [9] and references there) discovered and exploited the amazing fact that when $V = 0$ (this corresponds to the case of Fourier analysis on $\mathbb{R}$) the appropriate integral kernel given by the "sine function"

$$K_T(k_1,k_2) = \sin(T(k_1-k_2))/(k_1-k_2)$$

admits a COMMUTING differential operator, namely, if

$$A = -D_{k_1}(G^2-k_1^2)D_{k_1} + T^2k_1^2$$

Partially supported by NSF Grant DMS91-01224 and AFOSR Contract AFO F49629-92. Part of this work was carried out during a stay at the Universite du Languedoc, Montpellier. The hospitality of Professor Sabatier and other members of the Laboratoire de Phys-Mathematique is greatly appreciated.

then on $L^2(-G, G)$ we have

$$KA = AK.$$

This was later extended by Slepian to the case of (radial) Fourier analysis in $\mathbb{R}^n$; see [9]. In this case the interval is $(0, \infty)$ and $V(x) = c/x^2$ with the constant c determined by the dimension n. He found that a commuting differential operator exists again. He also studied the problem on the circle. Slepian was dealing with

$$L_\nu = -D_x^2 + (\nu^2 - 1/4)/x^2.$$

The eigenfunctions in this case are given by

$$f_\nu(x, k) = \sqrt{xk} J_\nu(xk)$$

with J_ν the usual Bessel function, and the commuting differential operator is

$$A_\nu = -D_{k_1}(G^2 - k_1^2)D_{k_1} + k_1^2 T^2 + G^2(\nu^2 - 1/4)/k_1^2.$$

In all these cases the differential operator A has simple spectrum and thus every eigenfunction of it automatically becomes one for the integral operator. In terms of practical computation the existence of such a commuting differential operator can hardly be exagerated: we are now dealing with a local operator which allows for a small band good discretization. Moreover, the differential operator has very "well separated" eigenvalues, making the computation of the eigenfunctions a numerically stable problem. Finally, it is important to note that this is a very exceptional situation; see the remark at the end of this note.

It would be nice to understand the roots of this unexpected bonanza, and to extend the class of potentials V for which L leads to such a nice set-up. We consider this our MAIN PROBLEM.

In the context of Random Matrices, Mehta was led to consider the same sine kernel and independently proved and put to use the same commutativity property; see [7]. Recently other kernels have surfaced, also in the context of random matrices, namely the Bessel case and the Airy case; see [10].

In connection with the "limited angle" problem in X-ray Tomography I became interested several years ago in extending the class of potentials V for which this MIRACLE would hold. The case of "parallel beam tomography" leads one directly to the kernel on the circle that Slepian had considered in [9].

This led me to certain examples dealing with non-Abelian groups [5] and to the observation that the same result holds for harmonic analysis based on the Classical Orthogonal polynomials [2]. Perlstadt then studied the case of discrete classical orthogonal polynomials [8].

Noticing that all the "good examples" listed above had eigenfunctions that satisfy a second order differential or a second order difference equation in the SPECTRAL Parameter, and feeling that this was an important element in the story, led me to the following PROBLEM:

Find all instances of operators $L = -D_x^2 + V(x)$ such that some family of eigenfunctions should satisfy a "differential equation in the spectral parameter" of the form

$$\sum_{i=0}^{m} b_i(k) D_k^i f(k,x) = r(x) f(k,x).$$

This problem has been completely settled in joint work with Hans Duistermaat; see [1]. The complete list of these "bispectral potentials" consists of the Bessel ones (c/x^2), the Airy ones $(ax+b)$, and two families associated with the Korteweg-deVries hierarchy and its "master symmetries", respectively. While the description of ALL the potentials as well as the association of one family with the KdV hierarchy was completed in [1], it was only in [6] that the authors found the relevance of the master symmetries of KdV to the bispectral problem.

The paper [3] contains a number of open problems in connection with the "bispectral problem", as well as a statement to the effect that I had made some progress unraveling the relation between the notion of "bispectrality" and the existence of a commuting differential operator. No details of this progress are given in [3]. So far I have managed to add a few examples to the celebrated list in [9].

The purpose of this note is to report on two of these examples. We start with the Bessel operators L_1 and L_2 defined above and look at the non-Bessel operators obtained by applying one step of the Darboux process (i.e. we are one step away from Bessel, in the terminology of [1]) to these L's.

We get the deformations

$$LL_1(t) = -D_x^2 + (15x^4 - 18tx^2 - t^2)/(4x^6 + 8tx^4 + 4t^2x^2)$$

and

$$LL_2(t) = -D_x^2 + (35x^8 - 90tx^4 + 3t^2)/(4x^{10} + 8tx^6 + 4t^2x^2)$$

with eigenfunctions given by

$$f^{(1)}(x,k,t) = (1/k)(D_x + (t - 3x^2)/(2x(x^2+t)))\sqrt{kx}\,J_1(kx)$$

and

$$f^{(2)}(x,k,t) = (1/k)(D_x + (3t - 5x^2)/(2x(x^4+t)))\sqrt{kx}\,J_2(kx).$$

Notice that these families of operators do not include L_1 and L_2, respectively, rather they are "one step away" from them (see [1]), and are deformations joining (L_0, L_2) and (L_1, L_3), respectively, since

$$LL_1(0) = L_2, \quad LL_2(0) = L_3, \quad LL_1(\infty) = L_0, \quad LL_2(\infty) = L_1.$$

Main result. We can show that the corresponding commuting differential operators are

$$\begin{aligned} &AA^{(1)}(t) \\ &\quad = D_{k_1}^2 (k_1^2 - G^2)^2 D_{k_1}^2 \\ &\qquad + D_{k_1}(k_1^2 - G^2)(2k_1^2T^2 + (2G^2t + 1/2) + 15G^2/2k_1^2)D_{k_1} \\ &\qquad + k_1^4T^4 + 2G^2k_1^2tT^2 + 9/2k_1^2T^2 \\ &\qquad - G^4t/2k_1^2 + 15G^2/8k_1^2 - 135G^4/16k_1^4 \end{aligned}$$

and

$$\begin{aligned} &AA^{(2)}(t) \\ &\quad = D_{k_1}^3 (k_1^2 - G^2)^3 D_{k_1}^3 + D_{k_1}^2 (k_1^2 - G^2)^2 a D_{k_1}^2 + D_{k_1}(k_1^2 - G^2) b D_{k_1} + c, \end{aligned}$$

with a, b, c given by the expressions

$$\begin{aligned} a &= 3k_1^2T^2 + \frac{105G^2}{4k_1^2} + \frac{3}{4}, \\ b &= 3k_1^4T^4 + \frac{39k_1^2T^2}{2} + \frac{57G^2T^2}{2} + 3G^4t - \frac{315G^2}{8k_1^2} + \frac{315G^4}{16k_1^4} + \frac{27}{16}, \\ c &= k_1^6T^6 + \frac{75k_1^4T^4}{4} - \frac{15G^2k_1^2T^4}{4} + 3G^4k_1^2tT^2 + \frac{75k_1^2T^2}{16} + \frac{315G^4T^2}{16k_1^2} \\ &\quad + \frac{9G^6t}{4k_1^2} + \frac{945G^2}{64k_1^2} + \frac{10395G^4}{64k_1^4} - \frac{17325G^6}{64k_1^6}. \end{aligned}$$

At this point the only proof that I have is computational. One would like to do better than this, but one should keep in mind that even in the classical case of the sine kernel the only proof of the commutativity is computational too.

These results give, to the best of my knowledge, the first nontrivial instances of commuting differential operators of order higher than two.

Notice the limiting relations given by

$$AA^{(1)}(\infty) = 2G^2 A_0$$

and

$$AA^{(1)}(0) = A_2^2 - 3/2A_2 - 11/2G^2T^2,$$
$$AA^{(2)}(\infty) = 3G^4A_1$$

and

$$A^{(2)}(0) = A_3^3 - 29/4A_3^2 + (195 - 256G^2T^2)/16A_3 + 435G^2T^2/4$$

and the remarkable fact that

$$AA^{(1)}(t) = AA^{(1)}(0) + tAA^{(1)}(\infty) \quad \text{and} \quad AA^{(2)}(t) = AA^{(2)}(0) + tAA^{(2)}(\infty).$$

Remark. It is in fact very rare for an integral operator of our kind to admit a commuting differential operator. For convolution type integral operators this was seen by J. Morrison in the 60's. See [4] for references as well as some of the techniques required to prove this result. Thus, the examples above are very exceptional even if the orders of $AA^{(1)}$ and $AA^{(2)}$ are higher than two. For the first instance of "two steps away from Bessel" (namely, $V_4(t_2, t_3)$; see page 226 of [1]), we have an operator of order 10.

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Поступило 13 сентября 1995 г.