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Solution of the Dirichlet problem in a subspace of a Hilbert space using potentials

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It is known that a bounded solution of the Dirichlet problem for functions in a half-space of a finite-dimensional space is unique, and can be represented as the potential of a double layer from the boundary value ([1], Ch. 18, §9, formual (3)). In this paper we show that this assertion remains valid in a Hilbert space if the boundary value satisfies a Lipschitz condition (Theorem 1). The corresponding potential is understood here as indicated in Definition 2. In the infinite-dimensional case it is natural to consider the Dirichlet problem also for measures. The conjugate potential of a double layer introduced in the note gives a unique solution of this problem for a certain class of measures (Theorem 2).

Let us give exact formulations. Let H be a real separable Hilbert space with the scalar product $(\cdot, \cdot)$, norm $|\cdot|$, and orthonormal basis $\{e_i\}_{i=1}^{\infty}$. We fix in H a linear operator B for which $Be_i = b_ie_i$, $b_i > 0$, $b_1 = 1, \sum_{i=1}^{\infty} b_i < \infty$, and we denote by p_t the Gaussian measure with the characteristic functional $\tilde{p}_t(x) = \exp\{-2t(Bx, x)\}$ defined on the σ-algebra $\mathscr{B}(H)$ of Borel subsets of H. We define the Laplace operator Δ by the chain of equalities

$$\Delta u = \Delta_B u = \operatorname{tr}(Bu'') = \sum_{i=1}^{\infty} b_i d_{e_i}^2 u,$$

where d_h is the operator of differentiation with respect to a vector $h \in H$. In addition, we use the following notation: $\mu^x = \mu(dy - x)$ is the shift of the measure μ by the vector $x \in H$, and μ_Γ is the surface measure in the sense of Uglanov [2], generated on the surface Γ by μ: $\mathscr{B}(H) \to \mathbf{R}$; standard coordinates $x_i = (x, e_i)$, $x' = x - x_1e_1$; $\Gamma_t = \{x_1 = t\}$, $U_t = \{x_1 > t\}$, $\Gamma = \Gamma_0$, $U = U_0$, $B_r(x) = \{|y - x| < r\}$, $B_r = B_r(0)$, $D_r^t = B_r \cap U_t$; $L^\infty(A)$ is the space of measurable bounded functions on a Borel set $A \subset H$; $C^n(U)$ (or $C_p^n(U)$) is the space of functions having in U at least n Fréchet derivatives (or partial derivatives of order at most n, respectively), bounded on each bounded subspace $A \subset U$ by a common constant $C = C(A)$, $C^\infty(U) = \bigcap_{n=1}^{\infty} C^n(U)$, $D(U) = \{f \in C^\infty(U)$: $\exists\, t, r > 0$, $\operatorname{supp} f \subset D_r^t\}$; $M^n(U)$ is the space of measures defined on $\mathscr{B}(U) = \mathscr{B}(H) \cap \{A \subset U\}$ and n times differentiable with respect to the subspace $H_1 = \sqrt{B} \cdot H$ [3].

Definition 1. A function f: $U \to \mathbf{R}$ is called a solution of the Dirichlet problem in a domain U with boundary value ρ: $\Gamma \to \mathbf{R}$ if $f \in C^1(U) \cap C_4^2(U)$, $\Delta f(x) = 0$, $x \in U$, and $f(x + te_1) \to \rho(x)$ for all r and for $t \searrow 0$ uniformly in $x \in B_r \cap \Gamma$.

Definition 2. The potential of a double layer from $\rho \in L^\infty(\Gamma)$ is a function W_ρ: $U \to \mathbf{R}$:

$$W_\rho(x) = \int_\Gamma \rho(y) K_{x_1}(dy - x'),$$

where K_{x_1} is the Cauchy measure defined by

$$K_{x_1}(A) = (2\pi)^{-1/2} \cdot x_1 \int_0^\infty (p_t)_\Gamma \cdot (A) \cdot t^{-3/2} \cdot e^{-x_1^2/} \, dt.$$

It is not difficult to verify that the finite-dimensional densities K_{x_1} have the form

$$K_{x_1}^n(x_2, x_3, \ldots, x_{n+1}) = \pi^{-(n+1)/2} \cdot \Gamma\left(\frac{n+1}{2}\right) \cdot x_1 \cdot \left(\sum_{i=1}^{n+1} b_i^{-1} x_i^2\right)^{-(n+1)/2}.$$

***Theorem* 1.** *If the boundary value ρ satisfies the Lipschitz condition $|\rho(x) - \rho(y)| < C|x - y|$, $0 < C < \infty$, then the potential W_ρ is the unique solution of the Dirichlet problem.*

Definition 3. A measure μ: $\mathscr{B}(U) \to \mathbf{R}$ is called a solution of the Dirichlet problem (for measures) with the boundary value μ_0: $\mathscr{B}(\Gamma) \to \mathbf{R}$ if: 1) μ is three times differentiable with respect to the subspace H_1 [3] on each Borel subset $A \subset U$ whose projection of Γ along e_1 is bounded; 2) $\int_U \Delta\varphi\cdot\mu = 0$ for all $\varphi \in D(U)$; 3) the surface measure $\mu_t = \mu_{\Gamma_t}$ (this exists, by 1)) weakly converges to μ_0 as $t \searrow 0$ (we identify Γ_t with Γ); 4) for any fixed $r > 0$ the variation of μ_{x_1} on the ball $B_r(x) \cap \Gamma_{x_1}$ is bounded uniformly in $x \in U$.

Definition 4. The conjugate potential of the double layer from μ_0: $\mathscr{B}(\Gamma) \to \mathbf{R}$ is the measure Π_{μ_0}: $\mathscr{B}(U) \to \mathbf{R}$:

$$\Pi_{\mu_0}(A) = \int_\Gamma d_{e_1}Q(A - x)\,\mu_0(dx),$$

where $Q = d_{e_1}Q^2_\infty$ [3] and

$$Q^2(A) = \int_0^\infty p_t(A)\,dt.$$

The measure Q^2 is the fundamental solution of the Laplace operator [4].

***Theorem* 2.** *The conjugate potential of the double layer* Π_{μ_0} *gives the unique solution of the Dirichlet problem (in the sense of Definition* 3).

In proving uniqueness in Theorem 1 we use the following lemma.

***Lemma* 1.** *There exist constants* α, $\beta > 1$ *such that for any solution f of the Dirichlet problem with zero boundary value*

$$\alpha M_f(r) \leqslant M_f(\beta r) \quad \text{for all} \quad r > 0,$$

where $M_f(r) = \sup\{|f(x)|: |x| = r,\ x \in U\}$.

We remark that the Dirichlet problem for measures in a half-space was studied earlier by Bentkus [5]. However, the solutions he found turned out to be generalized measures (distributions), whereas the solutions given here are completely analogous to the classical ones.

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