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FORMAL GROUPS AND THE ATIYAH-HIRZEBRUCH FORMULA

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I. M. KRIČEVER

Abstract. In this article, manifolds with actions of compact Lie groups are considered. For each rational Hirzebruch genus $h: \Omega_* \rightarrow Q$, an "equivariant genus" h^G , a homomorphism from the bordism ring of G -manifolds to the ring $K(BG) \otimes Q$, is constructed. With the aid of the language of formal groups, for some genera it is proved that for a connected compact Lie group G , the image of h^G belongs to the subring $Q \subset K(BG) \otimes Q$. As a consequence, extremely simple relations between the values of these genera on bordism classes of S^1 -manifolds and submanifolds of its fixed points are found. In particular, a new proof of the Atiyah-Hirzebruch formula is obtained.

Bibliography: 10 items.

In [1] it was proved that the signature of every S^1 -manifold X is equal to the sum of the signatures of the submanifolds F_s of its fixed points:

$$\text{Sign}([X]) = \sum_s \text{Sign}([F_s]).$$

As a unitary variant of this formula, there is a relation between the values of the classical T_y -genus of an S^1 -manifold, i.e. an almost complex manifold on which the action of S^1 preserves the complex structure in the stable tangent bundle, and its fixed submanifolds:

$$T_y([X]) = \sum_s (-y)^{\epsilon_s^-} T_y([F_s]).$$

Here ϵ_s^- is the number of summands in the decomposition of the representation of S^1 in the fibers of the normal bundle over the submanifold F_s into irreducible representations $\eta^{j_{si}}$ (the action of $z \in S^1$ in η^k is multiplication by z^k) with $j_{si} < 0$. We shall denote by ϵ_s^+ the number of the remaining summands.

The previous proof of both formulas in [1] is based on the index theorem of Atiyah and Singer. In this article they are obtained as a consequence of a fundamentally different approach, whose essence is reduced to the study of the analytic properties of the Conner-Floyd expressions.

We recall these expressions for actions with isolated fixed points (see [2], [3] and

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[4]; for an arbitrary action they were first obtained in [5], whose formulas were improved in [6].

Assume that the action of S^1 on X has only isolated fixed points p_s and that the representation of the group in the fibers of the tangent bundle over them is $\sum_{i=1}^n \eta^{i s i}$.

If $[u]_j$ is the j th power of u in the formal group of "geometric cobordisms"

$$f(u, v) = g^{-1}(g(u) + g(v)), \quad g(u) = \sum_{n=0}^{\infty} \frac{[CP^n]}{n+1} u^{n+1},$$

i.e. $[u]_j = g^{-1}(jg(u))$, then the equalities of Conner and Floyd assert that the Laurent series

$$\Phi(u) = \sum_s \prod_{i=1}^n \frac{1}{[u]_{i s i}},$$

with coefficients in $U^* \otimes Q$, contains only the right part, and the independent term is the bordism class of X .

To each rational Hirzebruch genus $h: U^* \rightarrow Q$ there corresponds a numerical realization $\Phi_h(u)$, a series in $1 - \eta$ with rational coefficients:

$$\Phi_h(\eta) = \sum_s \prod_{i=1}^n \frac{1}{g_h^{-1}(\ln \eta)^{i s i}},$$

where $g_h^{-1}(t)$ is functionally inverse to the logarithm

$$g_h(t) = \sum_{n=0}^{\infty} \frac{h([CP^n])}{n+1} t^{n+1}$$

of the formal group $f_h(u, v)$ corresponding to the homomorphism h . We note that $\Phi_h(\eta)$ is the image of the series $\Phi(u)$, to which there corresponds a cobordism class in $U^*(CP^\infty) \otimes Q = U^*[[u]] \otimes Q$, under the homomorphism $\tilde{h}: U^*(CP^\infty) \otimes Q \rightarrow K(CP^\infty) \otimes Q$ induced by the genus h . The existence of the transformation of functors \tilde{h} follows from [7].

We shall assume that the series $g_h^{-1}(\ln \eta)$ is the expansion about 1 of an analytic function in some neighborhood of that point; then from the Conner-Floyd equalities it follows that $\Phi_h(\eta)$ is analytic in some neighborhood of 1, and that $\Phi_h(1) = h([X])$.

Our task is to prove that if the function $g_h^{-1}(\ln \eta)$ is analytic in the disk $|\eta| < 2$ and does not have zeros there, except at 1, then $\Phi_h(\eta)$, which could have poles at the roots of 1, is also analytic in the disk $|\eta| < 2$. From this it follows that, for a genus h such that $g_h^{-1}(\ln \eta)$ is a rational function with one simple zero at 1, i.e. $g_h^{-1}(\ln \eta) = (\eta - 1)/(a\eta + b)$, $a + b = 1$, $\Phi_h(\eta)$ is analytic everywhere and hence is a constant. Then its value at 1, equal to $h([X])$, coincides with

$$\lim_{\eta \rightarrow \infty} \Phi_h(\eta) = \sum_s a^{\varepsilon_s^+} (-b)^{\varepsilon_s^-}.$$

Precisely in this way, a new proof of the Atiyah-Hirzebruch formula will be obtained; for the two-parameter genus $T_{x,y}$ (the value of $T_{x,y}$ on the bordism class

$[CP^n]$ is equal to $\sum_{i=0}^n x^{n-1}(-y)^i$; we note that $T_{1,y}$ coincides with the T_y -genus) we shall obtain the relation

$$T_{x,y}([X]) = \sum_s x_s^{e_s^+} (-y)^{e_s^-} T_{x,y}([F_s]).$$

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§1. "Characteristic" homomorphisms for G -bundles

To each characteristic class $\chi \in U^i(BU)$ of complex vector bundles in unitary cobordism there corresponds a "characteristic" homomorphism

$$\chi^*: U_n(BU(k)) \rightarrow U^{-n+i} = U^{-n+i}(pt),$$

which associates to a bundle ξ over the manifold X the image of the cobordism class $\chi(\xi)$ under the composite

$$U^i(X) \xrightarrow{D} U_{n-i}(X) \rightarrow U_{n-i} \cong U^{-n+i},$$

where D is the duality homomorphism.

In this section, an analogous homomorphism for complex G -bundles (here and in what follows G is a compact Lie group) will be constructed and studied.

1. Consider the category of complex G -bundles over unitary G -manifolds. Two such bundles ξ_1 and ξ_2 over G -manifolds X_1 and X_2 are bordant if there exists a G -bundle ζ over W such that $\partial W = X_1 \cup X_2$ and the restriction of ζ to X_i , $i = 1, 2$, coincides with ξ_i . The bordism groups obtained in this way will be denoted by $U_{n,k}^G$, where $k = \dim_{\mathbb{C}} \xi$ and $n = \dim_{\mathbb{R}} X$. In the case when $G = \{e\}$ is trivial, $U_{n,k}^e$ coincides with $U_n(BU(k))$. Then $U_{*,*}^G$ becomes a ring and a U_* -module in the usual way. The submodule $U_{*,0}^G$ is identified with the bordism module U_*^G of unitary G -manifolds.

We shall denote by X_G the space $(X \times EG)/G$, and by ξ_G the image of the G -bundle ξ under the homomorphism

$$\text{Vect}_G(X) \rightarrow \text{Vect}(X_G).$$

If $p_1: U^*(X_G) \rightarrow U^*(BG)$ is the Gysin homomorphism induced by the projection $p: X_G \rightarrow BG$, then the formula

$$\chi^G([\xi]) = p_1(\chi(\xi_G))$$

defines an "equivariant characteristic" homomorphism

$$\chi^G: U_{n,k}^G \rightarrow U^{-n+i}(BG).$$

Its relation with χ is given by the following lemma.

Lemma 1.1. *Let $U_{n,k}^G \rightarrow U_{n,k}^e$ be the homomorphism which "forgets" the G -action; then the diagram*

$$\begin{array}{ccc} U_{n,k}^G & \xrightarrow{\chi^G} & U^{-n+l}(BG) \\ \downarrow & & \downarrow \\ U_{n,k}^e & \xrightarrow{\chi^e} & U^{-n+l} \end{array}$$

is commutative.

The proof of the lemma follows immediately from the definition of the Gysin homomorphism and from the fact that the restriction to the fiber X of the fibering $X_G \rightarrow BG$ of ξ_G coincides with ξ .

In what follows, it will be convenient to denote the "characteristic" homomorphism 1^G corresponding to $1 \in U^0(BU)$ by

$$\chi_0^G: U^G \rightarrow U^*(BG).$$

2. Let H be a normal subgroup of G . The set of fixed points under the action of H on a unitary G -manifold X is the disjoint union of almost complex submanifolds F_s (not necessarily connected). The normal bundles ν_s over the submanifolds F_s have a natural complex G -bundle structure.

As is known, there exists an equivariant embedding of X into the space of a unitary representation $\tilde{\Delta}$ of the group G . (To avoid repeating conditions, we shall agree that in this section and in the next one, only unitary manifolds, bundles, representations, etc., will be considered.) Denote the restriction of the normal G -bundle over X in the space of the representation $\tilde{\Delta}$ to the submanifold F_s by $(-\tilde{\nu}_s)$. It is evident that the sum $\nu_s \oplus (-\tilde{\nu}_s)$ is a trivial G -bundle. Let Δ be a maximal direct summand of $\tilde{\Delta}$ whose restriction to the subgroup H does not contain trivial representations of H . In an analogous way we select a direct summand $(-\nu_s)$ in the G -bundle $(-\tilde{\nu}_s)$.

Theorem 1.1. *Let ξ_s be the restriction of a complex G -bundle ξ over a G -manifold X to a submanifold F_s . Then, with the notation introduced above,*

$$e(\Delta_G) \chi^G(\{\xi\}) = \sum_s p_{s!} (e((-\nu_s)_G) \cdot \chi(\xi_s)),$$

where $p_{s!}: U^*(F_{sG}) \rightarrow U^*(BG)$ is the Gysin homomorphism and, for an arbitrary bundle ζ , $e(\zeta)$ is the Euler class of ζ .

Proof. The composite of the embedding of X in the space of the representation $\tilde{\Delta}$ and the projection on the direct summand Δ defines an equivariant map $h: X \rightarrow \Delta$.

For each bundle ζ we shall denote by $E\zeta$ its total space and by $S\zeta$ its sphere bundle. $(\Delta \times EG)/G$ coincides with $E\Delta_G$ by definition. Here and in what follows, Δ denotes both the representation and its space.

Let $h \times \text{id}: X \times EG \rightarrow \Delta \times EG$; then the corresponding quotient map

$$\tilde{h}: X_G \rightarrow (\Delta \times EG)/G$$

induces a Gysin homomorphism

$$\tilde{h}_!: U^*(X_G) \rightarrow U^*(E\Delta_G, S\Delta_G).$$

Lemma 1.2. *Let $i^*: U^*(E\Delta_G, S\Delta_G) \rightarrow U^*(BG)$ be the homomorphism from the*

exact sequence of the pair; then for every $x \in U^*(X_G)$ the following equality is satisfied:

$$\theta(\Delta_G) p_1(x) = i^* \tilde{h}_1(x).$$

Proof. From the definition of the Gysin homomorphism it follows immediately that

$$h_1(x) = t(\Delta_G) p_1(x),$$

where $t(\Delta_G)$ is the Thom class of Δ_G . Apply to both sides of this equality the homomorphism i^* . The lemma follows from the fact that $i^* t(\Delta_G) = e(\Delta_G)$.

We shall state a simple corollary of Lemma 1.2. We denote by $I^*(G)$ the ideal of $U^*(BG)$ consisting of those cobordism classes which are annihilated by multiplication by the Euler classes of bundles associated with representations of G .

Corollary. *If the action of the group G on the manifold X has no fixed points, then the image of $p_1: U^*(X_G) \rightarrow U^*(BG)$ belongs to the ideal $I^*(G)$.*

Proof. If H coincides with G , then, by definition of \tilde{h} , the nonexistence of fixed points implies that the image of X_G belongs to $S\Delta_G$. This means that $i^* \tilde{h}_1$ is a trivial homomorphism. By Lemma 1.2, the image of p_1 is annihilated by multiplication by $e(\Delta_G)$.

If we return to the proof of the theorem, we note that for an arbitrary action of G on X , \tilde{h} maps the pair (X_G, N_G) to the pair $(E\Delta_G, S\Delta_G)$. Here N is the complement of tubular neighborhoods of the fixed points under the action of H . The restriction of \tilde{h} to a closed tubular neighborhood of F_s defines a map of pairs $\tilde{h}_s: (Ev_{sG}, Sv_{sG}) \rightarrow (E\Delta_G, S\Delta_G)$ which induces a Gysin homomorphism:

$$\tilde{h}_{s!}: U^*(Ev_{sG}) = U^*(F_{sG}) \rightarrow U^*(BG).$$

Lemma 1.3. *If $f_s: Ev_{sG} \rightarrow X_G$ is the inclusion, then*

$$\tilde{h}_{s!} \circ f_s^*: U^*(X_G) \rightarrow U^*(BG) \quad \text{and} \quad \sum_s \tilde{h}_{s!} \circ f_s^* = i^* \circ \tilde{h}_1.$$

Proof. The Gysin homomorphisms induced by the maps of the commutative diagram

$$\begin{array}{ccc} E\Delta_G & \xrightarrow{i} & (E\Delta_G, S\Delta_G) \\ \uparrow \tilde{h} & & \uparrow \\ X_G & \rightarrow & (X_G, N_G) \end{array}$$

also form a commutative diagram:

$$\begin{array}{ccc} U^*(E\Delta_G, S\Delta_G) & \xrightarrow{i^*} & U^*(BG) \\ \uparrow \tilde{h}_1 & & \uparrow \\ U^*(X_G) & \longrightarrow & U^*(X_G \setminus N_G) \end{array}$$

The natural identification of $X_G \setminus N_G$ with the disjoint union of the Ev_{sG} will yield the lemma.

Lemma 1.4. *For $x \in U^*(F_{sG})$ the following equality holds:*

$$\tilde{h}_{s!}(x) = p_{s!}(x e((-v_s)_G)).$$

Proof. The map \tilde{h}_s can be factored as the composite

$$\begin{array}{ccc} & (E(p_s^* \Delta_G), S(p^* \Delta_G)) & \\ & \nearrow g & \searrow p_s \\ (E\nu_{sG}, S\nu_{sG}) & \xrightarrow{\tilde{h}_s} & (E\Delta_G, S\Delta_G) \end{array}$$

where g is the quotient map of the equivariant map

$$E\nu_s \times EG \rightarrow F_s \times \Delta \times EG,$$

obtained from the projection $E\nu_s \rightarrow F_s$, the equivariant inclusion of $E\nu_s$ in Δ and the identity map of EG . This means that $\tilde{h}_{s!}(x) = p_{s!}g_!(x)$. The map g is the identity on the base of the bundles; therefore $g^*(x) = x$ and $g_!(x) = g_!(g^*(x)) = xg_!(1)$. It remains to show that $g_!(1) = e((- \nu_s)_G)$.

According to the definition of \tilde{h} , g is the embedding with normal bundle $(- \nu_s)_G$. However, for any bundles ζ_1 and ζ_2 over the common base, the "diagonal" section of $\pi^*\zeta_2$ ($\pi: E(\zeta_1 \oplus \zeta_2) \rightarrow Y$ is the projection on the base) is transversal to the zero section and their intersection is the image of the embedding $i: (E\zeta_1, S\zeta_1) \rightarrow (E(\zeta_1 \oplus \zeta_2), S(\zeta_1 \oplus \zeta_2))$. From the definition of the Euler class and of the homomorphism $i_!$ it follows that $i_! = e(\zeta_2)$, which concludes the proof of the lemma.

The theorem follows immediately from the preceding lemmas and from the fact that $f_s^*(\chi(\xi_G)) = \chi(\xi_{sG})$.

Remark. In the case when the subgroup H coincides with the group G , Theorem 1.1 gives a relation between the value of the homomorphism χ^G on the bordism class of the G -bundle ξ and invariants in the cobordisms of the fixed submanifolds.

3. Along with the expression for $\chi_0^{S^1}$ given by Theorem 1.1, in what follows we shall need a modification of it, which will be obtained precisely in this subsection.

Let F_s be a connected component of the set of fixed points under the action of S^1 on an S^1 -manifold X . The normal bundle ν_s , like every complex S^1 -bundle over a trivial S^1 -manifold, will be represented in the form $\sum_{j \neq 0} \nu_{s_j} \otimes \eta^j$, where η^j is the j th tensor power of the standard representation of $S^1\eta$, as in the Introduction (see [8]).

The collection of complex bundles ν_{s_j} , of which only a finite number are different from zero, defines a bordism class belonging to the group

$$R_n = \Sigma U_l \left(\prod_{j \neq 0} BU(n_j) \right).$$

The summation is taken over all collections of nonnegative integers n_j and l such that $2 \sum n_j + l = n$.

The sum over all the connected components of these classes gives the image of the bordism class of the S^1 -manifold X , $[X, S^1] \in U_n^{S^1}$, under the homomorphism $\beta: U_n^{S^1} \rightarrow R_*$.

We choose as generators of the U_* -module $U_*(CP^\infty) = U_*(BU(1))$ the bordism classes $(CP^n) \in U_{2n}(CP^\infty)$ corresponding to the inclusion of CP^n in CP^∞ or, what is the same, the canonical bundle $\eta_{(n)}$ over CP^n . The standard multiplication in R_* allows us to choose as generators of the U_* -module R_* , in this case, the monomials

$$(CP_{i_1}^{l_1}) \times \dots \times (CP_{i_r}^{l_r}).$$

It will be convenient to denote by η not only the canonical representation of S^1 but also the corresponding canonical bundle over CP^∞ ; that is, $\eta_{S^1} = \eta$. Then for the S^1 -bundle $\eta_{(n)}$ over CP^n , the bundle $(\eta_{(n)} \otimes \eta)_{S^1}$ over $CP^n \times CP^\infty$ is equal to $\eta_{(n)} \otimes \eta$. The Euler class of $(-\eta_{(n)}) \otimes \eta$, where $(-\eta_{(n)})$ is the n -dimensional bundle complementary to $\eta_{(n)}$, is defined by

$$e((-\eta_{(n)}) \otimes \eta) f(u, v) = u^{n+1},$$

where $f(u, v) = c(\eta \otimes \eta) = u + v + \sum \alpha_{ij} u^i v^j$, the formal group of "geometric" cobordisms. Hence, if

$$A_n(u, v) = e((\eta_{(n)}) \otimes \eta) \in U^*(CP^n \times CP^\infty) = U^*[[u, v]]/v^{n+1} = 0,$$

then

$$A_n(u, v) = \frac{u^n}{\frac{1}{u} f(u, v)}.$$

Let $B^n(u)$ be the image of $A_n(u, v)$ under the Gysin homomorphism $U^*(CP^n \times CP^\infty) \rightarrow U^*(CP^\infty)$ induced by the projection. We note that for $A_n(u, v)$ this homomorphism corresponds to the substitution of $[CP^{n-k}]$ for v^k .

Theorem 1.1 immediately yields the following assertion.

Theorem 1.2 *There exists a U_* -module homomorphism $\Psi: R_* \rightarrow U^*[[u]] \otimes Q[u^{-1}]$ such that $\Psi \circ \beta$ coincides with the composition of $\chi_0^{S^1}$ and the inclusion $U^*(CP^\infty) \rightarrow U^*[[u]] \otimes Q[u^{-1}]$. The values of Ψ on the generators of the U_* -module are given by the formula*

$$\Psi \left(\prod_{m=1}^r (CP_{j_m}^{l_m}) \right) = \prod_{m=1}^r \left(\frac{1}{[u]_{j_m}} \right)^{l_m+1} B_{l_m}([u]_{j_m}), \quad [u]_i = \theta(\eta^i).$$

4. Consider an arbitrary homomorphism $\alpha: G_1 \rightarrow G$ of Lie groups. It induces a map $\alpha_*: BG_1 \rightarrow BG$ of universal classifying spaces and hence a homomorphism $\alpha^*: U^*(BG) \rightarrow U^*(BG_1)$.

On the other hand, by means of α each G -bundle becomes a G_1 -bundle, i.e. there exists a homomorphism

$$\alpha^*: U_{\dots}^G \rightarrow U_{\dots}^{G_1}.$$

The commutative diagram

$$\begin{array}{ccc} X_{G_1} & \rightarrow & X_G \\ \downarrow & & \downarrow \\ BG_1 & \rightarrow & BG \end{array}$$

where X is an arbitrary G -manifold, easily yields

Theorem 1.3. *For every characteristic class χ , the diagram*

$$\begin{array}{ccc} U_{\dots}^G & \xrightarrow{\chi^G} & U^*(BG) \\ \alpha^* \downarrow & & \alpha^* \downarrow \\ U_{\dots}^{G_1} & \xrightarrow{\chi^{G_1}} & U^*(BG_1) \end{array}$$

is commutative.

§2. Equivariant Hirzebruch genera. Statement and proof of the main theorem

1. From the viewpoint of characteristic classes, a rational Hirzebruch genus, i.e. a homomorphism $h: U_* \rightarrow Q$, is given by a series $t/h(t)$ with $h(t) = t + \sum_{i>1} \lambda_i t^i$, $\lambda_i \in Q$. The "action" of such a series on the bordism class $[CP^n]$ is given by the formula

$$h([CP^n]) = \left[\left[\frac{t}{h(t)} \right]^{n+1} \right]_n,$$

where $[r(u)]_n$ denotes the n th coefficient of the series $r(u)$. In [2], S. P. Novikov proved that $h(t)$ coincides with the series $g_h^{-1}(t)$ functionally inverse to the logarithm

$$g_h(t) = \sum_{n=0}^{\infty} \frac{h([CP^n])}{n+1} t^{n+1}$$

of the formal group $f_h(u, v)$ which is the image of the formal group of "geometric" cobordisms under the homomorphism.

By a theorem of Dold [7], to each rational Hirzebruch genus h there corresponds a transformation of functors $\tilde{h}: U^*(Y) \rightarrow K^\#(Y) \otimes Q$ (where $K^\#$ is the Z_2 -graded K -functor) such that $\tilde{h}: U^* \rightarrow Q$ coincides with the composite $U^* \simeq U_* \xrightarrow{h} Q$.

The proof of the following lemma is analogous to the proofs of Theorem 6.4 and Corollary 6.5 in [9]:

Lemma 2.1. *The value of the homomorphism h at the generator $u \in U^2(CP^\infty)$ is equal to $ch^{-1}(g_h^{-1}(t))$, where ch is the Chern character; that is,*

$$\tilde{h}(u) = g_h^{-1}(\ln \eta) \in K(CP^\infty) \otimes Q = Q[[1 - \eta]].$$

Definition. An equivariant Hirzebruch genus corresponding to a rational genus $h: U_* \rightarrow Q$ is a homomorphism $h^G = \tilde{h} \circ \chi_0^G: U_{e\nu}^G \rightarrow K(BG) \otimes Q$.

Since Lemma 1.1 implies the commutativity of the diagram

$$\begin{array}{ccccc} U_*^G & \rightarrow & U^*(BG) & \rightarrow & K(BG) \otimes Q \\ \downarrow & & \downarrow & & \downarrow \\ U_* & \rightarrow & U^* & \xrightarrow{\epsilon} & Q \end{array}$$

where $\epsilon: K^\#(Y) \otimes Q \rightarrow Q$ is the "augmentation", we have

Lemma 2.2. *The value of a genus on the bordism class of a G -manifold X is equal to $\epsilon(h^G([X, G]))$.*

2. Now we proceed to prove the main result.

Theorem 2.1. *For a connected compact Lie group G , the image of the homomorphism $T_{x,y}^G: U_{e\nu}^G \rightarrow K(BG) \otimes Q$ belongs to the subring $QC \subset K(BG) \otimes Q$. Moreover, for an S^1 -manifold X ,*

$$T_{x,y}([X]) = \sum_s x^{\epsilon_s^+} (-y)^{\epsilon_s^-} T_{x,y}([F_s]).$$

The Hirzebruch genus $T_{x,y}$ and the nonnegative integers ϵ_s^+ and ϵ_s^- appearing in the statement are the same as in the Introduction.

Proof. First of all we shall show that the first part of the theorem ($\text{Im } T_{x,y}^G \subset Q$) is a simple consequence of Theorem 1.3 and Lemma 2.3.

Lemma 2.3. *The image of $T_{x,y}^{S^1}$ belongs to $Q \subset K(\mathbb{C}P^\infty) \otimes Q$.*

Indeed, for a connected compact Lie group G the homomorphism $\alpha^*: K(BG) \otimes Q \rightarrow K(BH) \otimes Q$ induced by the inclusion of a maximal torus H in G is a monomorphism. Therefore, if there exists a G -manifold X such that $T_{x,y}^G([X, G]) \notin Q$, then also $\alpha^*(T_{x,y}^G([X, G])) \notin Q \subset K(BH) \otimes Q$. Evidently, there is an embedding of S^1 in H , $\alpha_1: S^1 \rightarrow H$, such that $\alpha_1^*(\alpha^*(T_{x,y}^G([X, G])))$ does not belong to Q either. However, this contradicts Lemma 2.3 because by Theorem 1.3

$$\alpha_1^*(\alpha^*(T_{x,y}^G([X, G]))) = T_{x,y}^{S^1}([X, S^1]).$$

Proof of Lemma 2.3. Consider an S^1 -manifold X . Let

$$\beta([X, S^1]) = \sum_i [M_i] \prod_m (\mathbb{C}P_{i,m}^{l_{im}});$$

then by Theorem 1.2

$$T_{x,y}^{S^1}([X, S^1]) = \sum_i T_{x,y}([M_i]) \prod_m (\tilde{T}_{x,y}([u]_{i,m}))^{(l_{im}+1)} \tilde{T}_{x,y}(B_{i,m}([u]_{i,m})). \tag{1}$$

We shall calculate $\tilde{T}_{x,y}([u]_j)$ and $\tilde{T}_{x,y}(B_N(u))$. Since

$$T_{x,y}([\mathbb{C}P^n]) = \frac{x^{n+1} - (-y)^{n+1}}{x+y},$$

we have

$$g_{T_{x,y}}(t) = \sum_{n=0}^{\infty} \frac{x^{n+1} - (-y)^{n+1}}{(x+y)(n+1)} t^{n+1} = \frac{1}{x+y} \ln \left(\frac{1+yt}{1-xt} \right).$$

Therefore

$$g_{T_{x,y}}^{-1}(t) = \frac{e^{(x+y)t} - 1}{xe^{(x+y)t} + y}$$

and hence

$$\tilde{T}_{x,y}([u]_j) = g_{x,y}^{-1}(j \ln \eta) = \frac{\eta^{j(x+y)} - 1}{x\eta^{j(x+y)} + y}.$$

By definition of $B_N(u)$, to find $\tilde{T}_{x,y}(B_N(u))$ we have to apply $T_{x,y}$ to the coefficients of the series $A_N(u, v)$ and replace v^k by $T_{x,y}([\mathbb{C}P^{N-k}])$ in the resulting series $A_{NT_{x,y}}(u, v)$. Since

$$f_{T_{x,y}}(u, v) = g_{T_{x,y}}^{-1}(g_{T_{x,y}}(u) + g_{T_{x,y}}(v)) = \frac{u+v+(y-x)uv}{1+yxuv},$$

we have

$$A_{NT_{x,y}}(u, v) \equiv \frac{u^N(1+yxuv)}{1 + \frac{v}{u} + (y-x)v} \pmod{v^{N+1}}.$$

Therefore

$$A_{NT_{x,y}}(u, v) = \sum_{k=0}^N (-1)^k v^k u^{N-k} (1 + (y-x)u)^k + \sum_{k=0}^{N-1} (-1)^k v^{k+1} u^{N-k+1} xy (1 + (y-x)uv)^k.$$

Thus we obtain

$$\tilde{T}_{x,y}(B_N(u)) = \sum_{k=0}^N (-1)^k \frac{x^{N-k+1} - (-y)^{N-k+1}}{x+y} \left(\frac{\eta^{x+y} - 1}{x\eta^{x+y} + y} \right)^{N-k} \left(\frac{x+y\eta^{x+y}}{x\eta^{x+y} + y} \right)^k + \sum_{k=0}^{N-1} (-1)^k xy \frac{x^{N-k} - (-y)^{N-k}}{x+y} \left(\frac{\eta^{x+y} - 1}{x\eta^{x+y} + y} \right)^{N-k+1} \left(\frac{x+y\eta^{x+y}}{x\eta^{x+y} + y} \right)^k.$$

We denote by $r_{x,y}^{(N)}(\eta)$ the function of η given by the right side of this equality. With the preceding formulas, the equality (1) takes the form

$$T_{x,y}^{S^1}([X, S^1]) = \sum_i T_{x,y}([M_i]) \prod_m \left(\frac{x\eta^{i m_i(x+y)} + y}{\eta^{i m_i(x+y)} - y} \right)^{i m_i + 1} \tau_{x,y}^{(i m_i)}(\eta^{i m_i}). \tag{2}$$

Let us pause to consider in detail the meaning of the latter equality.

Let $\Phi_{x,y}(\eta)$ be the function of the complex variable η given by the right side of (2). It is easy to see that in a deleted neighborhood of 1 it is analytic; hence it has a Laurent series expansion in the variable $1 - \eta$ there. By (2), this series coincides with $T_{x,y}^{S^1}([X, S^1]) \in Q[[1 - \eta]]$. This implies that $\Phi_{x,y}(\eta)$ is analytic not only in a deleted neighborhood, but also at 1 itself.

Our immediate task will be to prove that there are no poles at roots of 1 and, as a consequence, that $\Phi_{x,y}(\eta)$ is analytic in the whole plane.

Lemma 2.4. *Let $\tilde{x} = x/(x+y)$ and $\tilde{y} = y/(x+y)$. Then*

$$(x+y)^N \Phi_{\tilde{x}, \tilde{y}}(\eta^{x+y}) = \Phi_{x,y}(\eta), \quad N = \dim_{\mathbb{C}} X.$$

The proof of the lemma can easily be obtained from the fact that

$$\chi_0^{S^1}([X, S^1]) \in U^{-2N}(CP^\infty), \quad \tilde{T}_{\tilde{x}, \tilde{y}}(u) = \frac{\eta-1}{x\eta+y} = (x+y) \frac{\eta-1}{x\eta+y},$$

$$(x+y)^n T_{\tilde{x}, \tilde{y}}([CP^n]) = T_{x,y}([CP^n]).$$

By this lemma it is enough to consider the case when $x+y=1$, what will be assumed till the conclusion of Lemma 2.3.

Assume that H (the normal subgroup appearing in §1.2) is a cyclic subgroup of S^1 of order n . In the notation of Theorem 1.1,

$$e(\Delta_{S^1}) \chi_0^{S^1}([X, S^1]) = \sum_s p_s (e(-v_s) S^1).$$

Since by definition of the representation Δ of S^1 its restriction to the subgroup Z_n does not contain trivial summands, we have $\Delta = \sum_m \eta^j m$, where none of the j_m is divisible by n . Hence

$$\left[\prod_m \left(\frac{\eta^{j_m-1}}{x\eta^{j_m} + y} \right) \right] T_{x,y}^{S^1}([X, S^1]) = \sum_s \tilde{T}_{x,y} [p_s! (e(-v_s)_{S^1})]. \tag{3}$$

Now we consider an arbitrary S^1 -bundle ζ over an S^1 -manifold F such that the action of Z_n is trivial on F . ζ can be represented as a sum of S^1 -bundles ζ_r , $0 \leq r \leq n-1$. The generator of Z_n acts on a fiber of ζ_r by multiplication by $\exp(2\pi ir/n)$. Hence, if the S^1 -bundle $\tilde{\zeta}_r$ is $\zeta_r \otimes \eta^{-r}$, then Z_n acts trivially on $\tilde{\zeta}_r$. Since $\zeta_r = \tilde{\zeta}_r \otimes \eta^r$, we have

$$e(\zeta_{S^1}) = \prod_{r=0}^{n-1} e(\tilde{\zeta}_{rS^1} \otimes p^*(\eta^r)),$$

where $p: F_{S^1} \rightarrow CP^\infty$.

Let $\mu_{r,k}$ be the Wu generators of $\tilde{\zeta}_{rS^1}$. Then

$$\tilde{T}_{x,y}(p_!(e(\zeta_{S^1}))) = \tilde{T}_{x,y} \left[p_! \left(\prod_{r,k} f(\mu_{r,k}, p^*([u]_r)) \right) \right].$$

The coefficient of $p^*([u]_r)^i$ in the series

$$\prod_k f_{\tilde{T}_{x,y}}(\mu_{r,k}, p^*([u]_r)) = \prod_k \frac{\mu_{r,k} + p^*([u]_r) + (y-x)\mu_{r,k}p^*([u]_r)}{1 + yx\mu_{r,k}p^*([u]_r)}$$

is a symmetric polynomial in $\mu_{r,k}$. We denote the corresponding polynomial in the Chern classes of $\tilde{\zeta}_{rS^1}$ by $P_{i,k}$. The dimension of its lowest term is not smaller than $i - \dim \zeta_r$.

Thus

$$\tilde{T}_{x,y}(p_!(e(\zeta_{S^1}))) = \sum_{\omega} \tilde{T}_{x,y} \left(\prod_{r=0}^{n-1} ([u]_r)^{i_r} \right) \tilde{T}_{x,y} \left(p_! \left(\prod_{r=0}^{n-1} P_{i_r,r} \right) \right), \quad \omega = (i_1, \dots, i_{n-1}). \tag{4}$$

The projection $\alpha: S^1 \rightarrow S^1/Z_n = S^1$ of S^1 onto the quotient group induces a map of classifying spaces

$$\alpha_*: CP^\infty \rightarrow CP^\infty,$$

with $\alpha^*(u) = [u]_n$. Since the S^1 -bundle $\tilde{\zeta}_r$ is the inverse image under $\alpha^\#$ of some S^1 -bundle $\tilde{\zeta}'_r$ (we recall that the subgroup Z_n acts trivially on the fibration space of $\tilde{\zeta}_r$), it follows from Theorem 1.3 that $p_!(\prod_{r=0}^{n-1} P_{i_r,r}) \in \text{Im } \alpha^*$.

Since the diagram

$$\begin{array}{ccc} U^*(CP^\infty) & \xrightarrow{\tilde{T}_{x,y}} & K(CP^\infty) \otimes Q \\ \alpha^* \downarrow & & \downarrow \alpha^* \\ U^*(CP^\infty) & \longrightarrow & K(CP^\infty) \otimes Q \end{array}$$

is commutative and $\alpha^*(\eta) = \eta^n$, we have that

$$\tilde{T}_{x,y} \left(p_! \left(\prod_{r=0}^{n-1} P_{i_r,r} \right) \right) \in \text{Im } \alpha^* = Q[[1 - \eta^n]].$$

From this and from (3) and (4) it follows that

$$T_{x,y}^{S^1}([X, S^1]) = \prod_m \left(\frac{x\eta^{j_m} + y}{\eta^{j_m} - 1} \right) \left(\sum_k P_k \cdot (1 - \eta^n)^k \right), \tag{5}$$

where P_k is a polynomial in $(\eta^n - 1)/(x\eta^n + y)$.

Let η_1 be the closest point to 1 at which there might be a pole of $\Phi_{x,y}(\eta)$; that is, the closest point to 1 of the form $\exp(2\pi ir/n)$, $r < n$ and $(r, n) = 1$, for which there is a j_{mi} divisible by n . The function $\Phi_{x,y}(\eta)$ is analytic in the disc $|\eta - 1| < |\eta_1 - 1|$; therefore the series $T_{x,y}^{S^1}([X, S^1])$ converges uniformly to it on every compact subset of this disc. From (5) it easily follows that the limit of $\Phi_{x,y}(\eta)$ for $\eta \rightarrow \eta_1$ exists. Hence $\Phi_{x,y}(\eta)$ is analytic in the disc $|\eta - 1| < |\eta_2 - 1|$, and the series $T_{x,y}^{S^1}([X, S^1])$ converges uniformly on every compact subset of that disc. Here η_2 is the next point at which there can be a pole of $\Phi_{x,y}(\eta)$. If we continue this process we obtain that $\Phi_{x,y}(\eta)$ is analytic in the whole closed complex plane. Therefore it is a constant. This concludes the proof of Lemma 2.3.

Now we pass to the second part of the theorem. By Lemma 2.2, $T_{x,y}([X]) = \Phi_{x,y}(1)$. Since, by the previous part of the proof, $\Phi_{x,y}(\eta)$ is constant, we have that $\Phi_{x,y}(1) = \lim_{\eta \rightarrow \infty} \Phi_{x,y}(\eta)$, and

$$\lim_{\eta \rightarrow \infty} \Phi_{x,y}(\eta) = \sum_i T_{x,y}([M_i]) \prod_m \lim_{\eta \rightarrow \infty} \left(\frac{x\eta^{j_{mi}(x+y)} + y}{\eta^{j_{mi}(x+y)} - 1} \right)^{l_{mi}+1} \tau_{x,y}^{(l_{mi})}(\eta^{l_{mi}}).$$

Remark. In what follows, all the limits are found under the assumption that $x + y > 0$. In the other case all the formulas are valid if we replace $\eta \rightarrow \infty$ by $\eta \rightarrow 0$.

Assume that $j_{mi} > 0$. Then

$$\lim_{\eta \rightarrow \infty} \left(\frac{x\eta^{j_{mi}(x+y)} + y}{\eta^{j_{mi}(x+y)} - 1} \right)^{l_{mi}+1} \tau_{x,y}^{(l_{mi})}(\eta^{l_{mi}}) = x^{l_{mi}+1} \lim_{\eta \rightarrow \infty} \tau_{x,y}^{(l_{mi})}(\eta^{l_{mi}}).$$

If we remember the definition of $\tau_{x,y}^N(\eta)$, we obtain

$$\begin{aligned} \lim_{\eta \rightarrow \infty} \tau_{x,y}^{(N)}(\eta^{l_{mi}}) &= \sum_{k=0}^N (-1)^k \frac{x^{N-k+1} - (-y)^{N-k+1}}{x+y} \frac{1}{x^{N-k}} \left(\frac{y}{x}\right)^k \\ &\quad + \sum_{k=0}^{N-1} (-1)^k xy \frac{x^{N-k} - (-y)^{N-k}}{x+y} \frac{1}{x^{N-k+1}} \left(\frac{y}{x}\right)^k \\ &= \frac{1}{x^N(x+y)} \left[\sum_{k=0}^N (-y)^k (x^{N-k+1} - (-y)^{N-k+1}) \right. \\ &\quad \left. - \sum_{k=0}^{N-1} (-y)^{k+1} (x^{N-k} - (-y)^{N-k}) \right] = \frac{1}{x^N(x+y)} (x^{N+1} - (-y)^{N+1}). \end{aligned}$$

In an analogous way, for $j_{mi} < 0$ we find that

$$\lim_{\eta \rightarrow \infty} \left(\frac{x\eta^{l_{mi}(x+y)} + y}{\eta^{l_{mi}(x+y)} - 1} \right)^{l_{mi}+1} \tau_{x,y}^{(l_{mi})}(\eta^{l_{mi}}) = (-y) \frac{x^{l_{mi}+1} - (-y)^{l_{mi}+1}}{x+y}.$$

Thus

$$\lim_{\eta \rightarrow \infty} \Phi_{x,y}(\eta) = \sum_i T_{x,y}(|V_i|) x^{\epsilon_i^+} (-y)^{\epsilon_i^-} \prod_m T_{x,y}(|CP^{l_{mi}}|),$$

where ϵ_i^+ is the number of positive integers among the j_{mi} and ϵ_i^- is the number of negative ones, respectively.

Let $\sum_{i,k} [M_{i,k}] \prod_m (CP_{j_{mi,k}}^{l_{mi,k}})$ be the part of $\beta([X, S^1]) = \sum_i [M_i] \prod_m (CP_{j_{mi}}^{l_{mi}})$ equal to the bordism class in R_* of the S^1 -bundle ν_s over a fixed submanifold F_s . Then for all the i_k we have $\epsilon_{i_k}^+ = \epsilon_s^+$ and $\epsilon_{i_k}^- = \epsilon_s^-$. Since $[F_s] = \sum_{i,k} [M_{i,k}] \prod_m [CP_{j_{mi,k}}^{l_{mi,k}}]$, the proof of Theorem 2.1 is complete.

§ 3. The orientable case

We shall consider orientation-preserving actions of compact Lie groups on manifolds and vector bundles. All the constructions and results of the preceding sections for unitary actions automatically carry over to the present case; for this reason we shall restrict ourselves to making statements only, with minimal explanations when necessary.

To each characteristic class $\chi \in \Omega^i(BSO)$ in the oriented cobordism of vector bundles there corresponds a homomorphism of the Ω_* -module of bordisms of oriented G -bundles over oriented G -manifolds to the cobordism ring of the universal classifying space BG :

$$\chi^G : \Omega_{n,k}^G \rightarrow \Omega^{-n+i}(BG).$$

Theorem 3.1. *For every characteristic class χ and every G -bundle ξ , the following equality holds:*

$$e(\Delta_G) \chi^G(|\xi|) = \sum_s \rho_s! (e(-\nu_s)_G \cdot \chi(\xi_s)).$$

The notation is the same as in Theorem 1.1, with the substitution of "orientable" for "unitary" bundles (representations).

Let χ_0^G be, as before, the "equivariant characteristic homomorphism" corresponding to the characteristic class $1 \in \Omega^0(BSO)$.

Let us consider an arbitrary orientable S^1 -manifold X . As we know, the structure group of the normal S^1 -bundle ν_s over a connected component F_s of the set of fixed points under the action of S^1 on X can be reduced to the unitary group and ν_s becomes a complex S^1 -bundle (see [10], § 38). We choose the complex structure in ν_s in such a way that the representation of S^1 in the fibers has the form $\sum_i \eta^{j_{si}} s_i$, $j_{si} > 0$. As before, we define a homomorphism of Ω_* -modules

$$\beta' : \Omega_n^{S^1} \rightarrow R_n = \sum \Omega_i \left(\prod_{j>0} BU(n_j) \right),$$

where the summation is taken over all the collections of nonnegative integers n_j and l such that $2\sum_j n_j + l = n$.

Theorem 3.2. *There exists a homomorphism $\Psi: R'_* \rightarrow \Omega^*[[u]] \otimes Q[u^{-1}]$ of Ω_* -modules such that $\Psi \circ \beta'$ coincides with the composite of the homomorphism $\chi_0^{S^1}$ with the homomorphism $\Omega^*[[u]] \rightarrow \Omega^*[[u]] \otimes Q[u^{-1}]$. The values of Ψ on the generators of the Ω_* -module are given by the formula*

$$\Psi \left(\prod_m (CP^m) \right) = \prod_n \left(\frac{1}{[u]_{l_m}} \right)^{l_m+1} B_{l_m}([u]_{l_m}).$$

Theorem 3.3. *If $\alpha: G_1 \rightarrow G$ is a homomorphism of Lie groups, then the diagram*

$$\begin{array}{ccc} \Omega_{*,*}^G & \xrightarrow{\chi^G} & \Omega^*(BG) \\ \downarrow \alpha_* & & \downarrow \alpha^* \\ \Omega_{*,*}^{G_1} & \xrightarrow{\chi^{G_1}} & \Omega^*(BG_1) \end{array}$$

is commutative.

As in § 2, for each rational Hirzebruch genus $h: \Omega_* \rightarrow Q$ we construct an equivariant Hirzebruch genus $h^G: \Omega_*^G \rightarrow K(BG) \otimes Q$.

The values of the classical T_y -genus for $y = 1$ on almost complex manifolds coincide with the signature of these manifolds. Therefore, exactly as for Theorem 2.1, one can prove

Theorem 3.4. *For a connected compact Lie group G , the image of the homomorphism $\text{Sign}^G: \Omega_*^G \rightarrow K(BG) \otimes Q$ belongs to the subring $Q \subset K(BG) \otimes Q$. For every oriented S^1 -manifold X we have*

$$\text{Sign}([X]) = \sum_s \text{sign}([F_s]).$$

Addendum. In a subsequent article, the proof of the following theorem will appear:

Theorem. *If on a manifold X whose first Chern class $c_1(X) \in H^2(X, Z)$ is divisible by k there exists a nontrivial action of S^1 , then $A_k([X]) = 0$.*

The proof is based on "analyticity" arguments connected with the equivariant series corresponding to the Hirzebruch genus A_k , $k = 2, 3, \dots$, given by the series $kt \cdot e^t / (e^{kt} - 1)$.

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