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On the Néron-Severi torus of a rational surface

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Let X be a complete geometrically irreducible rational surface over an algebraic number field k. According to [1], X is birationally equivalent over k to a del Pezzo surface or admits a fibering into conics; $d = (K^2X)$ is called the degree of the surface; for a fibering into conics $d \leqslant 8$, for the del Pezzo surface $1 \leqslant d \leqslant 9$ [1], [2]. The Galois group of $\bar{k}/k$ acts in Pic $(X \otimes \bar{k})$, which is a torsion-free module of rank $10-d$. The Néron-Severi torus is an algebraic k-torus S with character module $\hat{S} = \text{Pic}\,(X \otimes \bar{k})$. We put

$$Ш(S) = \text{Ker}\,(H^1(k,\ S) \to \otimes\ H^1(k_v,\ S)).$$

Let L/k be a minimal normal extension over which all exceptional curves of the del Pezzo surface (or all degenerate fibres of the fibering into conics, respectively) are defined. We call L the splitting field of X. The group $G = \text{Gal}(L/k)$ is a subgroup of the Weyl group $W(R_{9-d})$, where R_{9-d} is a root system from the list E_8, E_7, E_6, D_5, A_4, $A_2 \times A_1$ in the case of a del Pezzo surface [2], and $G \subseteq W(D_r)$ in the case of a fibering into conics ($r = 8-d$ is the number of degenerate fibres, each of which is a pair of rational curves intersecting transversely in one point).

The study of the torus S is of interest, in particular, because of the existence of the homomorphism Φ: $A_0(X) \to H^1(k, X)$, which is an embedding if X is a fibering into conics; $A_0(X)$ denotes the group of classes of 0-cycles of degree 0, considered up to rational equivalence [3], [4].

We are interested in birational and arithmetic invariants of S and X (see also [4], [5], [6]).

***Proposition* 1.** *The Néron-Severi torus S of a rational surface of degree $d \geqslant 5$ is rational; for any $d \leqslant 4$ there exist rational surfaces of degree d with non-rational Néron-Severi tori.*

We consider in greater detail a del Pezzo surface of degree 4 having k-points. We remark that $W(D_5) = (\mathbf{Z}/2)^4 \cdot S_5 \subset (\mathbf{Z}/2)^5 \cdot S_5$. If we denote the i-th generator of $(\mathbf{Z}/2)^5$ by c_i, then the elements of $W(D_5)$ are distinguished by the fact that the decomposition of an element contains an even number of the c_i; here the element c_ic_j is interpreted as a simultaneous transposition of components of the i-th and j-th degenerate fibres.

In the case when on X there is a conic bundle, which corresponds to types II–IX and XII–XV in Manin's classification [2], we regard X as a fibering into conics of degree 4. If there is no conic bundle, then by blowing up a k-point in general position and projecting the resulting cubic surface from an embedded straight line, we obtain a fibering into conics Y of degree 3. If X is not relatively minimal or Y has the index (number of straight lines simultaneously contracted over the base field) 2 or more, then the torus S is stably rational (rational in the case of a biquadratic splitting field). From now on we consider only relatively minimal models of a conic bundle, assuming that their degree is $d \leqslant 4$. The corresponding splitting groups are called minimal.

Since $Ш(S) = 0$ for tori with a cyclic splitting field the first non-trivial case is $G = (\mathbf{Z}/2)^2$; the birational classification of such tori can be found in [7]. We describe the tori with biquadratic splitting fields that can arise from del Pezzo surfaces of degree 4. If the splitting field is fixed, then the Néron-Severi torus is uniquely determined up to isomorphism by the action of the Galois group on straight lines, that is, by the subgroup $G \subseteq W(D_5)$ up to conjugacy.

***Theorem* 1.** *The group $W(D_5)$ contains seven conjugacy classes of minimal subgroups of type* $(\mathbf{Z}/2)^2$:

$$1.\ \langle c_1c_2,\ c_3c_4\rangle;\ 2.\ \langle c_1c_2,\ c_2c_3c_4c_5\rangle;\ 3.\ \langle c_1c_2c_3c_4,\ (12)(34)\rangle;\ 4.\ \langle c_1c_2c_3c_4,\ 12)\rangle;$$

$$5.\ \langle c_1c_2c_3c_5,\ (12)c_3c_5\rangle;\ 6.\ \langle c_1c_2,\ (12)c_3c_4\rangle;\ 7.\ \langle (12)(34),\ (12)c_1c_2c_3c_4\rangle.$$

The rationality of the Néron-Severi torus is determined by the conjugacy class of $G \subseteq W(D_5)$ and is independant of the splitting field.

***Theorem* 2.** *Rational tori correspond to classes* 1–5, *irrational to classes* 6 *and* 7.

Remark. In the notation of [2] they correspond to types IX, X, VI, VII, XI, IX, VIII of partitioning of the set of exceptional curves into orbits. In particular, to type IX there may correspond both rational and irrational tori.

All possibilities enumerated in Theorem 1 can arise.

***Theorem* 3.** *Let $L = K(\sqrt{a}, \sqrt{b})$ be a biquadratic extension of k. Then for any type in Theorem 1 there exists a rational surface X over k with splitting field L and a torus of the given type.*

We give an example of such a realization for each type:

1. $y^2 - xz^2 = b(x-a)(x-b)$; 2. $y^2 - xz^2 = -b(x-a)(x-c^2a)(x-d^2a)$;
3. $y^2 - bz^2 = (x^2-a)(x^2-ac^2)$; 4. $y^2 - bz^2 = (x^2-a)(x-c)(x-d)$;
5. $y^2 - b(2x+a+1)z^2 = -a(x^2-a)(2x+a)$;
6. $y^2 - b(2x+a+1)z^2 = a(x^2-a)$; 7. $y^2 - abz^2 = (x^2-a)(x^2-b)$.

For fiberings into conics $Ш A_0(X) \subseteq Ш(S)$, where $Ш A_0(X) = \operatorname{Ker}(A_0(X) \to \bigoplus A_0(X_v))$. It has been conjectured [4] that $Ш A_0(X) = Ш(S)$. For surfaces of type 7 we have the following fact.

***Proposition* 2.** *Let X be a surface over k given by the affine equation $y^2 - abz^2 = (x^2-a)(x^2-b)$, where a and b are such that $[k(\sqrt{a}, \sqrt{b}), k] = 4$. Then either $Ш A_0(X) = Ш(S) = 0$ or $A_0(X) = Ш A_0(X) \subseteq Ш(S) = \mathbf{Z}/2$.*

The second case occurs if and only if the splitting groups of all valuations of k in $k(\sqrt{a}, \sqrt{b})/k$ are cyclic.

Using ideas of [5] we can give the following example.

***Proposition* 3.** *Let X be the surface over $\mathbf{Q}$ with the equation $y^2 - 221z^2 = (x^2-13)(x^2-17)$. Then $A_0(X) = Ш A_0(X) = Ш(S) = \mathbf{Z}/2$.*

Theorem 2 and both remarks were proved by B.E. Kunyavskii; Propositions 1, 2, 3 and Theorems 1 and 3 by M.A. Tsfasman.

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