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ON IMPROVING AN ERROR ESTIMATE FOR A NONLINEAR PROJECTIVE REGULARIZATION METHOD WHEN SOLVING AN INVERSE BOUNDARY VALUE PROBLEM

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Abstract The paper suggests a solution to a combined initial boundary value problem for the heat equation, in which, the heating takes place in the interval from 0 to T, and then, starting with T, the free heat exchange with the surrounding medium occurs. Such a statement is an adequate mathematical model describing the temperature field of a heated object. The error estimation of the approximate solution to the problem is obtained in terms of the modulus of continuity of the inverse operator.

Key words: error estimation, modulus of continuity, Fourier transform, ill-posed problem

AMS Mathematics Subject Classification: 45Q05, 45B05, 65J20

1 Introduction

One of the branches of computational mathematics is the creation of effective methods for solving inverse problems, justification of these methods, and their investigation [1]-[20]. Justification of the methods, as a rule, is reduced either to the proof of the convergence of approximate solutions to the exact one, as the error of initial data tends to zero, or to the estimation of the accuracy of the methods. This paper gives the obtained accuracy estimation of the nonlinear projection regularization method for solving the inverse boundary value problem for the heat equation. Note that the solution of this class of problems cannot be limited to a single convergence of the approximate solution to the exact one.

Since the function $q_0(t)$ determines the exact temperature in the combustion chamber and, for a fixed pair of initial data (f_δ, δ), the approximate solution to the problem $q_\delta(t)$ can be far from $q_0(t)$, in spite of the convergence of $q_\delta(t)$ to $q_0(t)$ with $\delta \to 0$. Therefore, without an error estimate, the approximate solution $q_\delta(t)$ to the inverse problem does not give an adequate representation of the temperature in the combustion chamber. Estimating the errors in solving the inverse problem of thermal diagnostics started in [21] and based on the Fourier transform with respect to t.

In [22], a technique was developed to obtain an error estimate but, in this paper, we used the condition $q_0(t) = 0$ with $t \geq T$ to justify the Fourier transform with respect to t.

Note that this condition is not feasible in practice. Therefore, in this paper we propose a combined problem in which a combustion chamber is being heated at an interval from 0 to T, and then, starting from T, heating is cancelled and followed by free heat exchange with the environment. This leads to a combined initial boundary value problem for the heat equation. For this problem, the article gives a validation of the Fourier transform with respect to t. An approximate solution to the inverse boundary value problem is obtained by the nonlinear method of projection regularization [23], and an error estimate exact in an order of magnitude is obtained for the nonlinear method of projection regularization. We note that the problem considered in the article is most closely approximated to the real one.

2 The statement of a direct problem on the time interval $[0, T]$

$$\frac{\partial u_1(x,t)}{\partial t} = \frac{\partial^2 u_1(x,t)}{\partial x^2}, \quad 0 \le x \le 1, \quad 0 < t \le T, \tag{1}$$

$$u_1(x,0) = 0, \quad 0 \le x \le 1, \tag{2}$$

$$\frac{\partial u_1(0,t)}{\partial x} = 0, \quad 0 \le t \le T, \tag{3}$$

$$u_1(1,t) = q_1(t), \quad 0 \le t \le T. \tag{4}$$

Suppose that the function $q_1(t)$ is such that

$$q_1(t) \in H^3[0,T], \quad q_1(0) = q_1'(0) = q_1'(T) = q_1''(0) = 0, \tag{5}$$

is given, and the function $u_1(x,t)$ satisfying the conditions (1)–(5) is to be determined.

Solving the problem (1)–(4) by the method of the variable separation, we obtain

$$u_1(x,t) = \sum_{n=0}^{\infty} C_n(t) \cdot \cos\left(n + \frac{1}{2}\right)\pi x + q_1(t), \tag{6}$$

where $x \in [0,1]$, $t \in [0,T]$, and

$$C_n(t) = \frac{2e^{-(n+\frac{1}{2})^2\pi^2 t}}{(n+\frac{1}{2})\pi} \int_0^t q_1'(\tau) \cdot e^{(n+\frac{1}{2})^2\pi^2\tau} d\tau. \tag{7}$$

Lemma 2.1. *Let $q_1(t)$ satisfy the condition (5). Then there exists a solution $u_1(x,t)$ to the problem (1)–(4) such that $u_1(x,t)$ satisfies the equation (1) on the set $[0,1) \times (0,T]$, initial condition (2), boundary conditions (3) and (4), and $u_1(x,t) \in C([0,1] \times [0,T]) \cap C^{2,1}([0,1) \times [0,T])$.*

Proof

Integrating the right-hand side of (7) by parts twice, we obtain

$$C_n(t) = \frac{2}{(n+\frac{1}{2})^5\pi^5} \int_0^t q_1'''(\tau) \cdot e^{-(n+\frac{1}{2})^2\pi^2(t-\tau)} d\tau + \frac{2q_1'(t)}{(n+\frac{1}{2})^3\pi^3} - \frac{2q_1''(t)}{(n+\frac{1}{2})^5\pi^5}. \tag{8}$$

Since for any n $|C_n(t)\cos(n+\frac{1}{2})\pi x| \le |C_n(t)|$, and, from the Cauchy-Bunyakovsky inequality

$$\left|\int_0^t q_1'''(\tau)e^{-(n+\frac{1}{2})^2\pi^2(t-\tau)}d\tau\right| \le \|q_1'''(t)\|_{L_2[0,T]} \cdot \frac{1}{\sqrt{2}(n+\frac{1}{2})\pi}. \tag{9}$$

Then, according to (5), (8), and (9), for any $t \in [0,T]$ and any n

$$|C_n(t)| \le \frac{\sqrt{2}\|q_1'''(t)\|_{L_2[0,T]}}{(n+\frac{1}{2})^6\pi^6} + \frac{2q_1'(t)}{(n+\frac{1}{2})^3\pi^3} + \frac{2q_1''(t)}{(n+\frac{1}{2})^5\pi^5} \tag{10}$$

From the convergence of series $\sum_{n=1}^{\infty}\frac{1}{n^3}$, $\sum_{n=1}^{\infty}\frac{1}{n^5}$ and $\sum_{n=1}^{\infty}\frac{1}{n^6}$ and formulas (8)–(10), using the Weierstrass criterion, we obtain the uniform convergence of the series (6) on $[0,1]\times[0,T]$.

Since the function $q_1'''(\tau)e^{(n+\frac{1}{2})^2\pi^2\tau} \in L_2[0,T]$, then

$$\int_0^t q_1'''(\tau)e^{-(n+\frac{1}{2})^2\pi^2(t-\tau)}d\tau$$

$$= e^{-(n+\frac{1}{2})^2\pi^2 t}\int_0^t q_1'''(\tau)e^{(n+\frac{1}{2})^2\pi^2\tau}d\tau \in C[0,T]. \tag{11}$$

Thus, taking into account that $q_1(t)$, $q_1'(t)$ and $q_1''(t) \in C[0,T]$ and also the formulas (8) and (11), we obtain

$$C_n(t) \in C[0,T]. \tag{12}$$

From (12) and the uniform convergence of the series (6) by $[0,1]\times[0,T]$, it follows that $u_1(x,T) \in C([0,1]\times[0,T])$. Having differentiated the function $C_n(t)\cos(n+1/2)\pi x$ at x and taking (10) into account, we get

$$|(C_n(t)\cos(n+1/2)\cdot\pi x)_x'| \le \frac{\sqrt{2}\|q_1'''(t)\|_{L_2}}{(n+\frac{1}{2})^5\pi^5} + \frac{2q_1'(t)}{(n+\frac{1}{2})^2\pi^2} + \frac{2q_1''(t)}{(n+\frac{1}{2})^4\pi^4}. \tag{13}$$

The convergences of series $\sum_{n=1}^{\infty}\frac{1}{n^2}$, $\sum_{n=1}^{\infty}\frac{1}{n^4}$, $\sum_{n=1}^{\infty}\frac{1}{n^5}$ and (13) imply the uniform convergence of the series

$$\sum_{n=0}^{\infty}(C_n(t)\cos(n+1/2)\pi x)_x' \quad \text{on} \quad [0,1]\times[0,T]. \tag{14}$$

It follows from (6) and (14) that $\dfrac{\partial u_1(x,t)}{\partial x} = \sum_{n=0}^{\infty}(C_n(t)\cos(n+1/2)\pi x)_x'$ on $[0,1]\times[0,T]$ and

$$\frac{\partial u_1(x,t)}{\partial x} \in C([0,1]\times[0,T]).$$

Now let us study the function $\dfrac{\partial^2 u_1(x,t)}{\partial x^2}$.

Having differentiated the function $C_n(t)\cos(n+1/2)\pi x$ at x twice and using (8), we obtain

$$(C_n(t)\cos(n+1/2)\pi x)''_{xx} = \frac{-2\cos(n+\frac{1}{2})\pi x}{(n+\frac{1}{2})^3\pi^3}\int_0^t q_1'''(\tau)e^{-(n+\frac{1}{2})^2\pi^2(t-\tau)}d\tau$$

$$-\frac{2\cos(n+\frac{1}{2})\pi x}{(n+\frac{1}{2})\pi}q_1'(t)+\frac{2\cos(n+\frac{1}{2})\pi x}{(n+\frac{1}{2})^3\pi^3}q_1''(t).$$

Since the numerical series $\sum\limits_{n=1}^{\infty}\dfrac{1}{(n+\frac{1}{2})^3\pi^3}$, $\sum\limits_{n=1}^{\infty}e^{-(n+\frac{1}{2})^2\pi^2}$ converge, then, according to the Weierstrass criterion, the functional series

$$2\sum_{n=0}^{\infty}\int_0^t q_1'''(\tau)e^{-(n+\frac{1}{2})^2\pi^2(t-\tau)}d\tau\cdot\frac{\cos(n+\frac{1}{2})\pi x}{(n+\frac{1}{2})^3\pi^3},\ 2\sum_{n=0}^{\infty}\frac{\cos(n+\frac{1}{2})\pi x}{(n+\frac{1}{2})^3\pi^3}q_1''(t)$$

converge absolutely and uniformly on $[0,1]\times[0,T]$.

It remains to verify the uniform convergence of the series $\sum\limits_{n=0}^{\infty}\dfrac{\cos(n+\frac{1}{2})\pi x}{(n+\frac{1}{2})\pi}$.

For any sufficiently small $\eta>0$, according to the Dirichlet criterion, this series converges uniformly on the set $[0,1-\eta]$. Since for any $\eta>0$ the series $\sum\limits_{n=0}^{\infty}(C_n(t)\cos(n+1/2)\pi x)''_{xx}$ converge uniformly on $[0,1-\eta]\times[0,T]$, and the summands of this series are continuous, then

$$\frac{\partial^2 u_1(x,t)}{\partial x^2}=\sum_{n=0}^{\infty}(C_n(t)\cos(n+1/2)\pi x)''_{xx}\text{ on }\ [0,1)\times[0,T]\text{ and}$$

$$\frac{\partial^2 u_1(x,t)}{\partial x^2}\in C([0,1)\times[0,T]),\ 0\le x<1,\ 0\le t\le T.$$

This proves the lemma. Now let us examine the function $u_1(x,T)$.

Lemma 2.2. *The function $u_1(x,T)$ defined by formula* (6), (8) *belongs to the area* $H^4[0,1]$.

Proof

It follows from (5), (6) and (8) that

$$u_1(x,T)=\sum_{n=0}^{\infty}C_n(T)\cos(n+1/2)\pi x+q_1(T), \tag{15}$$

where

$$C_n(T)=\frac{2}{(n+\frac{1}{2})^5\pi^5}\int_0^T q_1'''(\tau)e^{-(n+\frac{1}{2})^2\pi^2(T-\tau)}d\tau-\frac{2q_1''(T)}{(n+\frac{1}{2})^5\pi^5}. \tag{16}$$

It follows from (15) that

$$\frac{\partial u_1(x,T)}{\partial x} = -\sum_{n=0}^{\infty}(n+1/2)\pi C_n(T)\sin(n+1/2)\pi x, \tag{17}$$

$$\frac{\partial^2 u_1(x,T)}{\partial x^2} = -\sum_{n=0}^{\infty}(n+1/2)^2\pi^2 C_n(T)\cos(n+1/2)\pi x, \tag{18}$$

$$\frac{\partial^3 u_1(x,T)}{\partial x^3} = \sum_{n=0}^{\infty}(n+1/2)^3\pi^3 C_n(T)\sin(n+1/2)\pi x, \tag{19}$$

$$\frac{\partial^4 u_1(x,T)}{\partial x^4} = \sum_{n=0}^{\infty}(n+1/2)^4\pi^4 C_n(T)\cos(n+1/2)\pi x. \tag{20}$$

It follows from (16)–(19) that

$$u_1(x,T),\quad \frac{\partial u_1(x,T)}{\partial x},\quad \frac{\partial^2 u_1(x,T)}{\partial x^2} \quad \text{and} \quad \frac{\partial^3 u_1(x,T)}{\partial x^3} \in C[0,1]. \tag{21}$$

We show that $\dfrac{\partial^4 u_1(x,T)}{\partial x^4} \in L_2[0,1]$.

From (16) and (20) we have

$$(n+1/2)^4\cdot\pi^4\cdot C_n(T) = \frac{2}{(n+\frac{1}{2})\pi}\int_0^T q_1'''(\tau)e^{-(n+\frac{1}{2})^2\pi^2(T-\tau)}d\tau - \frac{2q_1''(T)}{(n+\frac{1}{2})\pi}. \tag{22}$$

It follows from (9) that

$$\left|\frac{2}{(n+\frac{1}{2})\pi}\int_0^T q_1'''(\tau)e^{-(n+\frac{1}{2})^2\pi^2(T-\tau)}d\tau\right| \le \frac{\sqrt{2}}{(n+\frac{1}{2})^2\pi^2}\|q_1'''(t)\|_{L_2[0,T]}, \quad \text{and}$$

$$\sum_{n=0}^{\infty}\left|\frac{2}{(n+\frac{1}{2})\pi}\int_0^T q_1'''(\tau)e^{-(n+\frac{1}{2})^2\pi^2(T-\tau)}d\tau\right| < \infty.$$

Let us turn our attention to the last series $2q_1''(T)\displaystyle\sum_{n=0}^{\infty}\frac{\cos(n+\frac{1}{2})\pi x}{(n+\frac{1}{2})\pi}$ determined by the formulas (20) and (22).

For this kind of the series, according to the Dirichlet test, it follows that

$$2q_1''(T)\sum_{n=0}^{\infty}\frac{\cos(n+\frac{1}{2})\pi x}{(n+\frac{1}{2})\pi} \in C[0,1),$$

as well as $\dfrac{\partial^4 u_1(x,T)}{\partial x^4} \in L_2[0,1]$.

This completes the proof of the lemma.

Note that the point $x = 1$ was heated at the time $t = T$ and from that moment the rod began to cool due to the interaction with the environment. Therefore, we got a new problem on the time interval $[T, \infty)$, in which on the boundary x=1 we used the third-type boundary conditions. This approach reproduces the real situation more adequately than the paper [22], where on the time interval $[T, \infty)$ it was assumed that $q(t) = 0$. In [22] such strong conditions stem from the difficulty of the justification of the Fourier transform with respect to t.

3 An extension of the direct problem (1)–(5) to the time interval $[T, \infty)$

$$\frac{\partial u_2(x,t)}{\partial t} = \frac{\partial^2 u_2(x,t)}{\partial x^2}; \quad 0 \le x < 1, \quad t > T, \tag{23}$$

$$u_2(x,T) = u_1(x,T) = u_0(x); \quad 0 \le x \le 1, \tag{24}$$

$$\frac{\partial u_2(0,t)}{\partial x} = 0 \quad t \ge T, \tag{25}$$

$$\frac{\partial u_2(1,t)}{\partial x} + k u_2(1,t) = 0; \quad t \ge T, \ k > 0. \tag{26}$$

Solving the problem (23)–(26) by the method of the variable separation, we obtain

$$u_2(x,t) = \sum_{n=0}^{\infty} A_n e^{-\lambda_n^2 (t-T)} \cos \lambda_n x, \quad 0 \le x \le 1, t \ge T, \tag{27}$$

where, for any n

$$A_n = \frac{4\lambda_n}{2\lambda_n + \sin 2\lambda_n} \int_0^1 u_0(x) \cos \lambda_n x dx, \tag{28}$$

and λ_n are all positive solutions of the equation $ctg\lambda = \dfrac{\lambda}{\kappa}$.

Then $\cos\lambda - \dfrac{\lambda}{\kappa} \sin\lambda = 0$, and

$$\left(1 + \frac{\lambda^2}{\kappa^2}\right)^{-\frac{1}{2}} \cos\lambda - \frac{\lambda}{\kappa}\left(1 + \frac{\lambda^2}{\kappa^2}\right)^{-\frac{1}{2}} \sin\lambda = 0. \tag{29}$$

Let us define

$$\sin\theta = \left(1 + \frac{\lambda^2}{\kappa^2}\right)^{-\frac{1}{2}}, \quad \text{and} \quad \cos\theta = \frac{\lambda}{\kappa}\left(1 + \frac{\lambda^2}{\kappa^2}\right)^{-\frac{1}{2}}. \tag{30}$$

Taking (29) and (30) into account, we obtain

$$\sin(\theta - \lambda) = 0, \tag{31}$$

and from (31),

$$\theta - \lambda = -\pi n. \tag{32}$$

It follows from (30) and (32) that

$$\sin(\lambda_n + \pi n) = \left(1 + \frac{\lambda_n^2}{\kappa^2}\right)^{-\frac{1}{2}},$$

and

$$\lambda_n = \pi n + \theta_n, \tag{33}$$

where

$$\theta_n \to +0 \quad \text{as} \quad n \to \infty, \tag{34}$$

decreasing monotonically. It follows from (30), (33) and (34) that

$$|\sin \theta_n| \leq \frac{\kappa}{\pi n}. \tag{35}$$

From (35) , the existence of numbers d_1 and d_2 >0 follows, such that, for any $n > 0$

$$d_1 n \leq \lambda_n \leq d_2 n. \tag{36}$$

Lemma 3.1. *Let A_n be defined by the formula* (28)*. Then*

$$A_n = \frac{4\lambda_n}{2\lambda_n + sin2\lambda_n}\left\{\frac{1}{\lambda_n^4}\left[\int_0^1 u_0^{(4)}(x)\cos\lambda_n x dx - u_0'''(1)\sin\lambda_n\right]\right.$$
$$\left. -\frac{1}{\lambda_n^3}u_0'''(1)\sin\lambda_n\right\},$$

where $u_0^{(4)}(x)$ is the fourth generalized x-derivative of the function $u_0(x)$.

Proof

Since A_n is defined by (28), then, having integrated $\int_0^1 u_0(x)\cos\lambda_n x dx$ by parts twice, we obtain

$$\int_0^1 u_0(x)\cos\lambda_n x dx$$

$$= \frac{\sin\lambda_n}{\lambda_n^2}[\lambda_n u_0(1) + \frac{\lambda_n}{\kappa}u_0'(1)] - \frac{1}{\lambda_n^2}\int_0^1 u_0''(x)\cos\lambda_n x dx. \tag{37}$$

Considering that

$$\frac{\sin\lambda_n}{\lambda_n^2}[\lambda_n u_0(1)+\frac{\lambda_n}{\kappa}u_0'(1)]=\frac{\sin\lambda_n}{\kappa\lambda_n}[u_0'(1)+\kappa u_0(1)],$$

and, according to (26),

$$u_0'(1)+\kappa u_0(1)=0,$$

then, taking into account (37), we obtain

$$\int_0^1 u_0(x)\cos\lambda_n x dx=-\frac{1}{\lambda_n^2}\int_0^1 u_0''(x)\cos\lambda_n x dx. \tag{38}$$

Having integrated the right-hand side of (38) by parts twice, we get

$$\int_0^1 u_0(x)\cos\lambda_n x dx=\frac{1}{\lambda_n^4}\left[\int_0^1 u_0^{(4)}(x)\cos\lambda_n x dx-u_0'''(1)\sin\lambda_n\right]-\frac{1}{\lambda_n^3}u_0''(1)\sin\lambda_n.$$

This completes the proof of the lemma.

Lemma 3.2. *The numerical series* $\sum_{n=0}^{\infty}\left|\frac{\sin\lambda_n}{\lambda_n}\right|<\infty.$

Proof

From (33) it follows that

$$\sin\lambda_n=\sin(\pi n+\theta_n)$$

$$=\sin\pi n\cdot\cos\theta_n+\sin\theta_n\cdot\cos\pi n=(-1)^n\sin\theta_n, \tag{39}$$

and from (35) and (39)

$$|\sin\lambda_n|\le\frac{\kappa}{\pi n}. \tag{40}$$

Thus, it follows from (40) that

$$\left|\frac{\sin\lambda_n}{\lambda_n}\right|\le\frac{\kappa}{\pi n\lambda_n}, \tag{41}$$

and from (41), according to the comparison criteria, that

$$\sum_{n=0}^{\infty}\left|\frac{\sin\lambda_n}{\lambda_n}\right|<\infty.$$

That establishes the lemma.

The Lemmas 3.1 and 3.2 imply that

$$\sum_{n=0}^{\infty}|\lambda_n|^2|A_n|<\infty, \tag{42}$$

and from (27) and (42) it follows that

$$\frac{\partial u_2(1,t)}{\partial t} = -\sum_{n=0}^{\infty} \lambda_n^2 A_n e^{-\lambda_n^2 (t-T)} \cos \lambda_n, \text{ at } t \geq T \tag{43}$$

and

$$\frac{\partial u_2(1,t)}{\partial t} \in C[T, \infty). \tag{44}$$

Lemma 3.3. *Let the function* $\frac{\partial u_2(1,t)}{\partial t}$ *be defined by the formula* (43). *Then there is* $d_3 > 0$ *such that for any* $t \geq T+1$

$$\left| \frac{\partial u_2(1,t)}{\partial t} \right| \leq d_3 e^{-(t-T-1)} + A_0 e^{-\lambda_0^2 (t-T)}.$$

Proof

From (28) and (43) it follows that $\left| \frac{\partial u_2(1,t)}{\partial t} \right| \leq d_3 \sum_{n=0}^{\infty} \lambda_n^2 e^{\lambda_n^2 (t-T)}$, where d_3 is some number.

Suppose that $t \geq T+1$, and $n > 0$, then $\lambda_n^2 e^{-\lambda_n^2} e^{-\lambda_n^2 (t-T-1)} \leq \lambda_n^2 e^{-\lambda_n^2} e^{-(t-T-1)}$.

It follows from (36) that, for $n > 0$

$$\lambda_n^2 e^{-\lambda_n^2} \leq d_2 n^2 [e^{-d_1^2}]^n. \tag{45}$$

It follows from (45) that $\sum_{n=1}^{\infty} n^2 [e^{-d_1^2}]^n < \infty$ and therefore, there is a number d_4 such that, for any $t \geq T+2$ $\left| \frac{\partial u_2(1,t)}{\partial t} \right| \leq d_4 e^{-(t-T-1)} + A_0 e^{-\lambda_0^2 (t-T)}$.

This completes the proof of the lemma.

It follows from (44) and Lemma 3.3 that $\frac{\partial u_2(1,t)}{\partial t} \in C[T, \infty) \cap L_1[T, \infty) \cap L_2[T, \infty)$.

From (1) and (21) we get

$$\frac{\partial u_1(x,T)}{\partial t} \in C[0,1]. \tag{46}$$

Let us examine the behavior of $\frac{\partial u_2(x,t)}{\partial t}$

It follows from (27) that

$$\frac{\partial u_2(x,t)}{\partial t} = -\sum_{n=0}^{\infty} \lambda_n^2 A_n e^{-\lambda_n^2 (t-T)} \cos \lambda_n x, \tag{47}$$

where, according to Lemma 3.1

$$\lambda_n^2 A_n = \frac{4\lambda_n}{2\lambda_n + \sin 2\lambda_n} \left\{ \frac{1}{\lambda_n^2} \left[\int_0^1 u_0^{(4)} \cos \lambda_n x dx - u_0'''(1) \sin \lambda_n \right] - \right.$$

$$\frac{1}{\lambda_n} u_0''(1) \sin \lambda_n \Big\}. \tag{48}$$

It follows from (36), (48) and Lemma 3.2 that the series $\sum_{n=0}^{\infty} \lambda_n^2 |A_n| < \infty$, and then, according to the Weierstrass criterion, the series on the right side of (47) converges uniformly on the set $[T, \infty)$ and therefore the following formula is correct

$$\frac{\partial u_2(x,t)}{\partial t} \in C([0,1) \times [T, \infty)). \tag{49}$$

Now we estimate the rate of decrease for the functions

$$u_2(x,t), \frac{\partial u_2(x,t)}{\partial x} \text{ and } \frac{\partial^2 u_2(x,t)}{\partial x^2} \text{ with } t \to \infty$$

From (27) and (28) it follows for any $\eta > 0$ that a number $d_5(\eta)$ exists such that, for any $t \geq T+2$ and any $x \in [0, 1-\eta]$

$$\left| \frac{\partial^2 u_2(x,t)}{\partial x^2} \right| \leq d_5(\eta) \sum_{n=1}^{\infty} \lambda_n^2 e^{-\lambda_n^2 e^{-(t-T-1)}}, \tag{50}$$

and also a number $\bar{d}_5 > 0$ exists such that for any $t \geq T+2$ and $x \in [0,1]$

$$\max_{0 \leq x \leq 1} \left\{ |u_2(x,t)|, \left| \frac{\partial u_2(x,t)}{\partial x} \right| \right\} \leq \bar{d}_5(\eta) e^{-(t-T-1)}. \tag{51}$$

Since (36) implies that $e^{-\lambda_n^2} \leq [e^{d_1^2}]^{-n}$, then it follows from (50) and (51) that that there is a number $d_6(\eta) = d_5(\eta) \sum_{n=1}^{\infty} \lambda_n^2 e^{-\lambda_n^2}$ such that, for any $t \geq T+2$ and $x \in [0, 1-\eta]$

$$\left| \frac{\partial^2 u_2(x,t)}{\partial x^2} \right| \leq d_6(\eta) e^{-(t-T-1)},$$

and $\bar{d}_6 > 0$ exists such that for any $t \geq T+2$ and any $x \in [0, 1-\eta]$

$$\max_{0 \leq x \leq 1} \left\{ |u_2(x,t)|, \left| \frac{\partial u_2(x,t)}{\partial x} \right| \right\} \leq \bar{d}_6(\eta) e^{-(t-T-1)}.$$

In what follows, we introduce the following notation

$$q(t) = \begin{cases} q_1(t), & t \in [0,T], \\ q_2(t), & t > T \end{cases} \tag{52}$$

and

$$u(x,t) = \begin{cases} u_1(x,t), & t \in [0,T], \\ u_2(x,t), & t > T, \end{cases} \tag{53}$$

and $x \in [0,1]$.

Then from (46), (49), (52), and (53) it follows for any $\eta > 0$ that the function $\chi_\eta(t)$ exists such that, for any $t \geq 0$

$$\sup_{0\leq x\leq 1-\eta} \{|u(x,t)|, |u'_x(x,t)|, |u''_{xx}(x,t)|\} \leq \chi_\eta(t), \tag{54}$$

where

$$\chi_\eta(t) = \begin{cases} d_7(\eta), & 0 \leq t \leq T+2, \\ d_6(\eta)\cdot e^{-(t-T-1)}, & t > T+2. \end{cases}$$

Since $\chi_\eta(t) \in L_1[0,\infty)$, then for the combined direct problem (1)–(5) and (23)–(26) the Fourier transform with respect to t can be applied.

From Lemma 2.1 and (54) it follows that

Theorem 3.1. *Suppose that $\Phi(t) \in C[0,\infty)$ and is bounded on this line. Then the following relations hold*

$$\int_0^\infty u'_x(x,t)\Phi(t)dt = \frac{\partial}{\partial x}\left[\int_0^\infty u(x,t)\Phi(t)dt\right],$$

$$\int_0^\infty u''_{xx}(x,t)\Phi(t)dt = \frac{\partial^2}{\partial x^2}\left[\int_0^\infty u(x,t)\Phi(t)dt\right].$$

4 Formulation of the inverse boundary value problem

Suppose that the function $q(t)$ used in the combined problem (1)–(5), (23)–(26) is unknown and instead of it we are given an approximate value $f_\delta(t)$ of the function

$$f_0(t) = u_0(x_0, t), \tag{55}$$

where $x_0 \in (0,1)$, $t \geq 0$, and an error level $\delta > 0$ is such that

$$\|f_\delta(t) - f_0(t)\|_{L_2} \leq \delta. \tag{56}$$

It is required, using the initial data $f_\delta(t)$ and δ of the inverse boundary problem (1)–(3), (23)–(26), (55), and (56), to find its approximate solution $q_\delta(t)$ and obtain the error estimate $\|q_\delta(t) - q_0(t)\|$.

In [22] for the similar problem we used the Fourier transform acting from a space $L_2[0,\infty)$ to a space $L_2[0,\infty)$.

The lemma 4 in [22] proved the continuity of the operators F and F^{-1} in these spaces. The lemma allowed us to consider these spaces, though $\hat{f}_\delta(\sigma)$ could not belong to $R(F)$, since the range of values $R(F)$ of the operator F, generally speaking, is not equal to $L_2[0,\infty)$. Then we need an additional operation in the space $L_2[0,\infty)$ orthogonal projection of the function $\hat{f}_\delta(\sigma)$ to $R(F)$, which considerably complicate the solution. For these reasons in this paper we used the Fourier transform F from $L_2[0,\infty)$ to $L_2(-\infty,\infty)$. Because we chose this pair of spaces we could use the Plancherel theorem. As a result, the operator F^{-1} could be implied to all the functions from $L_2(-\infty,\infty)$ and transform $L_2(-\infty,\infty)$ to $L_2(-\infty,\infty)$. After the procedure

$$\overline{q}_1(t)=\begin{cases} Re[\hat{q}_1(\sigma)] \text{ at } t\geq 0, \\ 0, \quad \text{at } t<0, \end{cases}$$

similarly,

$$\overline{q}_2(t)=\begin{cases} Re[\hat{q}_2(\sigma)] \text{ at } t\geq 0, \\ 0, \quad \text{at } t<0, \end{cases}$$

we obtain that

$$\|\overline{q}_1(t)-\overline{q}_2(t)\|_{L_2[0,\infty)}\leq \|\hat{q}_1(\sigma)-\hat{q}_2(\sigma)\|_{L_2(-\infty,\infty)}.$$

Suppose $\overline{H}=L_2(-\infty;\infty)+iL_2(-\infty;\infty)$ is a space over the field of complex numbers, and the set M_r is defined by the formula

$$M_r=\left\{q(t) \quad : \quad q(t)\in L_2[0,\infty), \ \int_0^{+\infty}|q(t)|^2dt+\int_0^{+\infty}|q'(t)|^2dt\leq r^2\right\},$$

where r is known positive number.

To solve the problem (1)–(3), (23)–(26), (55) and (56), we introduce the operator F acting from $\overline{H}$ to $\overline{H}$ and called the Fourier transform.

$$\hat{q}(\sigma)=F[q(t)]=\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}q(t)\cdot e^{-i\sigma t}dt, \quad \sigma\in R, \ q(t)\in L_2[0,\infty)\bigcap L_1(0,\infty).$$

We denote the expansion of the operator $\overline{F}$ to the space $\overline{H}$ by F. It follows from the Plancherel theorem that the operator $\overline{F}$ isometrically maps $\overline{H}$ onto $\overline{H}$.

Let $\hat{q}(\sigma)\in\overline{H}$, then we get the formula

$$\overline{F}^{-1}[\hat{q}(\sigma)]=\frac{1}{\sqrt{2\pi}}\lim_{N\to\infty}\int_{-N}^{N}\hat{q}(\sigma)e^{it\sigma}d\sigma, \quad -\infty<t<\infty,$$

where the limit is thought of as mean square convergence.

Theorem 3.1 implies the applicability of the Fourier transform to the problem (1)–(3), (23)–(26)) solution.

Using the transformation F, we reduce problem (1)–(3), (23)–(26) to the following one:

$$\frac{\partial^2\hat{u}(x,\sigma)}{\partial x^2}=i\sigma\hat{u}(x,\sigma), \quad x\in(0,1), \quad -\infty<\sigma<\infty, \tag{57}$$

$$\frac{\partial\hat{u}(0,\sigma)}{\partial x}=0, \quad -\infty<\sigma<\infty, \tag{58}$$

$$\hat{u}(x_0,\sigma)=\hat{f}(\sigma), \quad -\infty<\sigma<\infty, \tag{59}$$

where $\hat{u}(x,\sigma) = F[u(x,t)]$, $\hat{f}(\sigma) = F[f(t)]$. The solution (57)–(59) is given by

$$\hat{u}(x,\sigma) = \begin{cases} E_1(\sigma) \cdot e^{\mu_0\sqrt{\sigma}x} + E_2(\sigma) \cdot e^{-\mu_0\sqrt{\sigma}x}, & \sigma \geq 0, \\ E_1(\sigma) \cdot e^{-\overline{\mu}_0\sqrt{|\sigma|}x} + E_2(\sigma) \cdot e^{\overline{\mu}_0\sqrt{|\sigma|}x}, & \sigma < 0, \end{cases}$$

where $\mu_0 = \dfrac{1}{\sqrt{2}}(1+i)$, $\overline{\mu}_0 = \dfrac{1}{\sqrt{2}}(1-i)$, $E_1(\sigma) = E_2(\sigma) = \dfrac{\hat{f}(\sigma)}{2\cosh \mu_0\sqrt{\sigma}x_0}$.

Using the condition $\hat{u}(1,\sigma) = \hat{q}(\sigma)$, we obtain

$$\hat{q}(\sigma) = \begin{cases} \dfrac{\cosh \mu_0\sqrt{\sigma}}{\cosh \mu_0\sqrt{\sigma}x_0} \cdot \hat{f}(\sigma), & \sigma \geq 0, \\ \dfrac{\cosh \overline{\mu}_0\sqrt{|\sigma|}}{\cosh \overline{\mu}_0\sqrt{|\sigma|}x_0} \cdot \hat{f}(\sigma), & \sigma < 0, \end{cases}$$

Thus, the problem (57)–(59) is reduced to the equation

$$A\hat{q}(\sigma) = \hat{f}(\sigma), \quad -\infty < \sigma < \infty, \tag{60}$$

where

$$A\hat{q}(\sigma) = \begin{cases} \dfrac{\cosh \mu_0\sqrt{\sigma}x_0}{\cosh \mu_0\sqrt{\sigma}}, & \sigma \geq 0, \\ \dfrac{\cosh \overline{\mu}_0\sqrt{|\sigma|}x_0}{\cosh \overline{\mu}_0\sqrt{|\sigma|}}, & \sigma < 0. \end{cases}$$

Let $\hat{f}_0(\sigma) = F[f_0(t)]$, $\hat{f}_\delta(\sigma) = F[f_\delta(t)]$. Then

$$\|\hat{f}_\delta(\sigma) - \hat{f}_0(\sigma)\|_{\overline{H}} \leq \delta. \tag{61}$$

We denote by $\hat{M}_r$ the set from $\overline{H}$ such that $\hat{M}_r \supset F[M_r]$ and

$$\hat{M}_r = \left\{ \hat{q}(\sigma) \ : \ \hat{q}(\sigma) \in \overline{H}, \quad \int_{-\infty}^{\infty} (1+\sigma^2)|\hat{q}(\sigma)|d\sigma \leq 2r^2 \right\}. \tag{62}$$

From $q_0(t) \in M_r$ it follows that $\hat{q}_0(\sigma) \in \hat{M}_r$.

To solve the problem (60)–(62), we use the regularized family of operators defined by formula

$$\hat{q}_\delta^\gamma(\sigma) = P_\gamma \hat{f}_\delta(\sigma) = \begin{cases} \dfrac{\cosh \mu_0\sqrt{\sigma}}{\cosh \mu_0\sqrt{\sigma}x_0} \cdot \hat{f}(\sigma), & 0 \leq \sigma \leq \gamma, \\ \dfrac{\cosh \overline{\mu}_0\sqrt{|\sigma|}}{\cosh \overline{\mu}_0\sqrt{|\sigma|}x_0} \cdot \hat{f}(\sigma), & -\gamma \leq \sigma < 0, \\ 0, & |\sigma| > \gamma. \end{cases} \tag{63}$$

To select the regularization parameter $\hat{\gamma} = \hat{\gamma}(\hat{f}_\delta, \delta)$ in (63) according to the initial data $(\hat{f}_\delta, \delta)$, we use the residual principle, that is, the equation

$$\|A\hat{q}_\delta^\gamma(\sigma) - \hat{f}_\delta(\sigma)\|^2 = \nu^2\delta^2, \tag{64}$$

where $\nu \geq 1$.

The lemma 1 proved in [24] implies an existence of the solution $\gamma(\hat{f}_\delta, \delta)$ to the equation (64) uniquely defining an element $\hat{q}_\delta(\sigma) = \hat{q}^{\gamma(\hat{f}_\delta,\delta)}(\sigma)$.

The element $\hat{q}_\delta(\sigma)$ hereinafter is referred to as an approximate solution to (60).

To estimate an error of the method $\{P_{\gamma(\hat{f}_\delta,\delta)}\}$, we assume that $\nu = 4$ in (64) . Such a large constant in (64) is a result of a quite difficult estimation procedure (see the statement on pp. 283-284 [23]).

The theorem formulated in [23] on page 284 implies that

$$\|\hat{q}_\delta(\sigma) - \hat{q}_0(\sigma)\| \leq 7\omega(\delta, r), \tag{65}$$

$\omega(\delta, r) = \sup\{\|\hat{q}(\sigma)\| \ : \ \hat{q}(\sigma) \in \hat{M}_r, \ \ \|A\hat{q}(\sigma)\| \leq \delta\}$.

Let $\{P_{\hat{\gamma}(\hat{f}_\delta,\delta)} \ : \ 0 < \delta \leq \delta_0\}$ be the method of approximate solution to the equation (60) on the class $\hat{M}_r$. Then for any $\delta \in (0, \delta_0]$ we introduce a quantitative characteristic of the accuracy of this method on the set $\hat{M}_r$.

$$\Delta_\delta[P_{\hat{\gamma}(\hat{f}_\delta,\delta)}] = \sup_{\hat{q}_0, \hat{f}_\delta}\{\|P_{\hat{\gamma}(\hat{f}_\delta,\delta)}\hat{f}_\delta(\sigma) - \hat{q}_0(\sigma)\| \ : $$

$$\hat{q}_0(\sigma) \in \hat{M}_r, \ \hat{f}_\delta(\sigma) \in \overline{H}, \ \|\hat{f}_\delta(\sigma) - A\hat{q}_0(\sigma)\| \leq \delta\}.$$

From the theorem proved in [25], the validity of the estimate follows:

$$\Delta_\delta[P_{\hat{\gamma}(\hat{f}_\delta,\delta)}] \geq \omega(\delta, r). \tag{66}$$

Let

$$\Phi^2(\gamma) = \sup\left\{\int_\gamma^\infty |\hat{q}_0(\sigma)|^2 d\sigma + \int_{-\infty}^{-\gamma} |\hat{q}_0(\sigma)|^2 d\sigma \ : \ \hat{q}_0(\sigma) \in \hat{M}_r\right\}. \tag{67}$$

From (62) and (67) we obtain that under the condition $\hat{q}_0(\sigma) \in \hat{M}_r$ $\Phi^2(\gamma) = \dfrac{2r^2}{1+\gamma^2}$.

Lemma 4.1. *Suppose* $\gamma_0 = \dfrac{1}{2x_0^2}\ln^2 2$. *Then, for* $\gamma \geq \gamma_0$, *the following relation is valid:*

$$\frac{1}{4}e^{(1-x_0)\sqrt{\frac{\gamma}{2}}} \leq \|P_\gamma\| \leq 4e^{(1-x_0)\sqrt{\frac{\gamma}{2}}}. \tag{68}$$

The proof of the lemma follows from the definition of the norm of the operator.

According to Lemma 2 from [26], to calculate the modulus of continuity $\omega(\delta, r)$, it is necessary to solve the equation

$$r \cdot \gamma \cdot G(\gamma) = \delta, \tag{69}$$

After that, the solution $\overline{\gamma}(\delta)$ of this equation is substituted into the function $G(\gamma)$, defined parametrically by the formulas

$$\overline{G}(\beta) = \frac{1}{\sqrt{1+\beta^2}}, \quad \gamma = e^{(1-x_0)\sqrt{\frac{\beta}{2}}}. \tag{70}$$

Then, it follows from (69) and (70) that

$$\omega(\delta, r) = rG(\overline{\gamma}(\delta)). \tag{71}$$

Thus, from (65), (70), and (71) we obtain the estimate

$$\|\hat{q}_\delta(\sigma) - \hat{q}_0(\sigma)\| \leq 7r \cdot G(\overline{\gamma}(\delta)). \tag{72}$$

To simplify the estimate (72)), we consider two equations

$$e^{(x_0-1)\sqrt{\frac{\gamma}{2}}} = \frac{r}{\delta} \quad \text{and} \quad e^{2(x_0-1)\sqrt{\frac{\gamma}{2}}} = \frac{r}{\delta}. \tag{73}$$

We denote the solutions to the equations (73) by $\overline{\gamma}_1(\delta)$ and $\overline{\gamma}_2(\delta)$.

Then, from (69), (73) we find that, for sufficiently small values of δ defined by $\overline{\gamma}(\delta)$, the following relations are true:

$$\overline{\gamma}_2(\delta) \leq \overline{\gamma}(\delta) \leq \overline{\gamma}_1(\delta), \tag{74}$$

where $\overline{\gamma}_1(\delta) = \dfrac{2}{(x_0-1)^2} \ln^2 \dfrac{r}{\delta}$ and $\overline{\gamma}_2(\delta) = \dfrac{1}{2(x_0-1)^2} \ln^2 \dfrac{r}{\delta}$, and from (74),

$$\overline{\gamma}(\delta) \sim \ln^2 \delta \quad \text{with} \quad \delta \to 0.$$

From the theorem proved in [26] it follows that

$$G(\overline{\gamma}_2(\delta)) \leq G(\overline{\gamma}(\delta)) \leq G(\overline{\gamma}_1(\delta)), \tag{75}$$

where $G(\overline{\gamma}_1(\delta)) = \dfrac{4}{\sqrt{1 + \frac{4}{(x_0-1)^4} \cdot \ln^4 \delta}}$ and $G(\overline{\gamma}_2(\delta)) = \dfrac{1}{4 \cdot \sqrt{1 + \frac{1}{4(x_0-1)^4} \cdot \ln^4 \delta}}$.

From (66) we obtain that the estimate is exact in order, that is,

$$\sup\left\{ \|\hat{q}_\delta^{\overline{\gamma}(\delta)}(\sigma) - \hat{q}_0(\sigma)\| \ : \ \hat{q}_0(\sigma) \in \hat{M}_r, \|\hat{f}_\delta(\sigma) - \hat{f}_0(\sigma)\| \leq \delta \right\}$$

$$\geq \frac{r}{4 \cdot \sqrt{1 + \frac{1}{4(x_0-1)^4} \cdot \ln^4 \frac{r}{\delta}}}. \tag{76}$$

From Lemma 4.1, (72), (75) and (76) it follows that

Theorem 4.1. *For the method* $\{P_{\hat{\gamma}(\hat{f}_\delta, \delta)} \ : \ 0 < \delta \leq \delta_0\}$ *, the error estimate exact in an order of value exists:*

$$\frac{r}{4 \cdot \sqrt{1 + \frac{1}{4(x_0-1)^4} \cdot \ln^4 \delta}} \leq \Delta_\delta[P_{\hat{\gamma}(\hat{f}_\delta, \delta)}] \leq \frac{7 \cdot 4r}{\sqrt{1 + \frac{4}{(x_0-1)^4} \cdot \ln^4 \delta}}.$$

Applying to $\hat{q}_\delta(\sigma)$ the transform

$$q_\delta(t) = \begin{cases} Re[\overline{F}^{-1}[\hat{q}_\delta(\sigma)], & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{77}$$

where $\overline{F}^{-1}$ is the inverse Fourier transform, we obtain an approximate solution $q_\delta(t)$ to the inverse problem (1)–(3), (23)–(26).

It follows from (77) that, for an approximate solution $q_\delta(t)$ to problem (1)–(3), (23)–(26)), the following error estimation exact in an order of magnitude will be correct:

$$\|q_\delta(t) - q_0(t)\| \leq \frac{28r}{\sqrt{1 + \frac{4}{(x_0-1)^4} \cdot \ln^4 \delta}}. \tag{78}$$

5 Improvement of estimation (78)

Note that the nonlinear projective regularization method possesses an important property useful for obtaining more precise error estimates. The fact is that this method uses only and for approximate solution but, on the condition that the exact solution belongs to the correctness class M_r defined by the continuous and increasing function, the deviation of the approximate solution from the exact one is estimated as $7\omega(\delta, r)$. The similar property was mentioned earlier in [17], [27], and [28] for an iterative method with the stopping rule on the residual error for the correctness classes defined by power functions.

For more precise estimation, we need Sobolev spaces $W_2^\alpha[0,\infty)$, where $\alpha > 0$. These spaces are described in [29] on p.252.

$$W_2^\alpha[0,\infty) = \left\{ u(x) \ : \ u(x) \in L_2[0,\infty); \int_{-\infty}^{\infty} \sigma^{2\alpha} \cdot |\hat{u}(\sigma)|^2 d\sigma < \infty \right\}, \tag{79}$$

where $\hat{u}(\sigma) = F[u(x)]$ and F is the Fourier transform. The norm in this space is specified by the formula

$$\|u(x)\|^2_{W_2^\alpha} = \|u(x)\|^2_{L_2[0,\infty)} + \int_{-\infty}^{\infty} \sigma^{2\alpha} \cdot |\hat{u}(\sigma)|^2 d\sigma. \tag{80}$$

Let $q_0(t)$ denote the exact solution to the inverse boundary problem (1)–(3), (23)–(26), (55), and (56) while $q_0'(t)$ denotes the derivative of that solution.

Note that the function $q_0'(t)$ has no more than one point of discontinuity, with this point being the discontinuity point of the first kind disposed at the point $t = T$ and $\lim\limits_{t \to T-0} q_0'(t) = 0$.

It follows from the above mentioned that the function $q_0'(t)$ may be represented as a sum of two functions

$$q_0'(t) = \chi(t) + \psi(t), \quad -\infty < t < \infty, \tag{81}$$

where

$$\psi(t) \in H_0^1[0,\infty), \tag{82}$$

and

$$\chi(t) = \begin{cases} 0; \quad 0 < t \le T, \\ (t-T) - b_1; \quad T < t \le T_1, \\ -k_1(t-T_1) + b_2; \quad T_1 < t \le T_2, \\ 0; \quad t \ge T_2, \end{cases} \tag{83}$$

where $b_1 = -\lim_{t \to T+0} q_0'(t), \quad T_1 - T - b_1 > 0,$

$$-k_1(T_2 - T_1) + b_2 = 0, \quad (T_3 - T) = b_1 \quad \text{and} \tag{84}$$

$$\int_T^{T_3} [(t-T) - b_1]dt = -\Big\{ \int_{T_3}^{T_1} [(t-T) - b_1]dt + \int_{T_1}^{T_2} [-k_1(t-T_1) + b_2]dt \Big\}.$$

It follows from (83) and (84) that

$$\chi(t) \in \bigcap_{\varepsilon \in (0,1/4)} W_2^{\frac{1}{2}-\varepsilon}[0,\infty). \tag{85}$$

From (81), (82), and (85) we obtain that

$$q_0'(t) \in \bigcap_{\varepsilon \in (0,1/4)} W_2^{\frac{1}{2}-\varepsilon}[0,\infty). \tag{86}$$

Lemma 5.1. *If $q_0'(t)$ is defined by the formulas (81)–(84) then there is a number $c > 0$ such that, for any $\varepsilon \in (0, 1/4)$*

$$\int_{-\infty}^{\infty} |\sigma|^{1-2\varepsilon} \cdot |\hat{q}_0'(\sigma)|^2 d\sigma \le \frac{4c}{\varepsilon},$$

where $\hat{q}_0'(\sigma) = F[q_0'(t)]$.

It follows from (82) that

$$\int_{-\infty}^{\infty} (1 + |\sigma|^2)|\hat{\psi}(\sigma)|^2 d\sigma < \infty, \tag{87}$$

and from (87) it follows that, for any $\varepsilon \in (0, 1/4)$

$$\int_{-\infty}^{\infty} |\sigma|^{1-2\varepsilon} \cdot |\hat{\psi}(\sigma)|^2 d\sigma \le \int_{-\infty}^{\infty} (1 + |\sigma|^2) \cdot |\hat{\psi}(\sigma)|^2 d\sigma \le \frac{c}{\varepsilon}, \tag{88}$$

where $c \ge \int_{-\infty}^{\infty} (1 + |\sigma|^2) \cdot |\hat{\psi}(\sigma)|^2 d\sigma.$

Next, we turn to the function $\chi(t)$, defined by the formulas (83) and (84).

Having estimated the Fourier transform of the function $\chi(t)$, we obtain the existence of a number $c_1 > 0$ such that

$$|\hat{\chi}(\sigma)| \le \frac{c_1}{\sigma}, \tag{89}$$

and it follows from (89) that there exists a number $c \geq c_1$ such that, for any $\varepsilon \in (0, 1/4)$

$$\int_{-\infty}^{\infty} |\sigma|^{1-2\varepsilon} \cdot |\hat{\psi}(\sigma)|^2 d\sigma \leq \frac{c}{\varepsilon}, \tag{90}$$

while (89) and (90) imply that

$$\int_{-\infty}^{\infty} |\sigma|^{1-2\varepsilon} \cdot |\hat{q}_0'(\sigma)|^2 d\sigma \leq \frac{4c}{\varepsilon}.$$

That establishes the lemma. Let us introduce a notation

$$r^2(\varepsilon) = \frac{4c}{\varepsilon}, \tag{91}$$

as well as, by analogy with (62), the correctness classes

$$\hat{M}^{\varepsilon}_{r(\varepsilon)} = \left\{ \hat{q}(\sigma) \ : \ \hat{q}(\sigma) \in \overline{H}, \ \int_{-\infty}^{\infty} (1 + |\sigma|^{3-2\varepsilon}) |\hat{q}(\sigma)|^2 d\sigma \leq r^2(\varepsilon) \right\}.$$

Let $\omega_\varepsilon(\delta, r(\varepsilon))$ denote the module of continuity for the equation (60) on the correctness class $\hat{M}^{\varepsilon}_{r(\varepsilon)}$.

$$\omega_\varepsilon(\delta, r(\varepsilon)) = \sup\{\|\hat{q}(\sigma)\| \ : \ \hat{q}(\sigma) \in \hat{M}^{\varepsilon}_{r(\varepsilon)}, \quad \|A\hat{q}(\sigma)\| \leq \delta\}. \tag{92}$$

Since $q_0(t) \in \bigcap\limits_{\varepsilon \in (0,1/4)} W_2^{\frac{3}{2}-\varepsilon}[0, \infty)$, then, from (65),

$$\|\hat{q}_\delta(\sigma) - \hat{q}_0(\sigma)\| \leq 7\omega_\varepsilon(\delta, r(\varepsilon)). \tag{93}$$

Next, using the formula (93), we find a dependence $\varepsilon = \varepsilon(\delta)$ such that it implies the estimate $7\omega_{\varepsilon(\delta)}(\delta, r(\varepsilon(\delta)))$ to be an infinitely small value compared with the estimate (78).

To estimate the module of continuity $\omega_\varepsilon(\delta, r(\varepsilon))$, see (92), we invoke the technique described in (69) and (70), that is, solve the equation

$$r(\varepsilon) \cdot \gamma \cdot G_\varepsilon(\gamma) = \delta. \tag{94}$$

After that, we put the solution $\overline{\gamma}_\varepsilon(\delta)$ to this equation in the function $G_\varepsilon(\gamma)$ defined parametrically:

$$\overline{G}_\varepsilon(\beta) = (1 + |\beta|^{3-\varepsilon})^{-1/2}, \quad \gamma = e^{(x_0 - 1)\sqrt{\beta/2}}. \tag{95}$$

It follows from (94) and (95) that

$$\omega_\varepsilon(\delta, r(\varepsilon)) = r(\varepsilon) \cdot G_\varepsilon(\overline{\gamma}_\varepsilon(\delta)). \tag{96}$$

It follows from (95) that we need of applying (94) to determine the function $\overline{\gamma}_\varepsilon(\delta)$.

Next, to estimate the function $\overline{\gamma}_\varepsilon(\delta)$, taking into account (68), we consider two equations:

$$\frac{1}{4}e^{(1-x_0)\sqrt{\gamma/2}} = \frac{r(\varepsilon)}{\delta}, \quad 4e^{2(1-x_0)\sqrt{\gamma/2}} = \frac{r(\varepsilon)}{\delta}. \tag{97}$$

The solutions to the equations (97) we denote by $\gamma_\varepsilon^1(\delta)$ and $\gamma_\varepsilon^2(\delta)$, respectively.

From (94) and (97) it follows that, for sufficiently small values of δ, the following relations are correct for $\overline{\gamma}_\varepsilon(\delta)$:

$$\gamma_\varepsilon^2(\delta) \le \overline{\gamma}_\varepsilon(\delta) \le \gamma_\varepsilon^1(\delta),$$

where $\gamma_\varepsilon^1(\delta) = \dfrac{2}{(1-x_0)^2}\ln^2\left(\dfrac{4r(\varepsilon)}{\delta}\right)$ and $\gamma_\varepsilon^2(\delta) = \dfrac{1}{2(1-x_0)^2}\ln^2\left(\dfrac{r(\varepsilon)}{4\delta}\right)$. Then

$$\overline{\gamma}_\varepsilon(\delta) \sim \ln^2\left(\frac{r(\varepsilon)}{4\delta}\right) \quad \text{as} \quad \delta \to 0.$$

The theorem proved in [26] implies that

$$G_\varepsilon(\gamma_\varepsilon^2(\delta)) \le G_\varepsilon(\overline{\gamma}_\varepsilon(\delta)) \le G_\varepsilon(\gamma_\varepsilon^1(\delta)). \tag{98}$$

From (96) and (98) it follows that

$$\omega_\varepsilon(\delta, r(\varepsilon)) \le \frac{r(\varepsilon)(1-x_0)^3}{\ln^{3-\varepsilon}\left(\dfrac{r(\varepsilon)}{4\delta}\right)} \le \frac{r(\varepsilon)(1-x_0)^3}{\ln^{3-\varepsilon}\left(\dfrac{1}{\delta}\right)}.$$

Thus, for sufficiently small values of δ and any $\varepsilon \in (0, 1/4)$, the following estimate holds true:

$$\|\hat{q}_\delta(\sigma) - \hat{q}_0(\sigma)\| \le \frac{7r(\varepsilon)(1-x_0)^3}{\ln^{3-\varepsilon}\left(\dfrac{1}{\delta}\right)}.$$

Considering the above, we get

$$\|\hat{q}_\delta(\sigma) - \hat{q}_0(\sigma)\| \le 7 \min_{\varepsilon\in(0,1/4)} \frac{r(\varepsilon)(1-x_0)^3}{\ln^{3-\varepsilon}\left(\dfrac{1}{\delta}\right)}. \tag{99}$$

To avoid extending the length of the article by means of detailed calculations of constants, we limit ourselves to proving the fact that, for a new estimation, the rate of its tending to zero will be higher than in (78).

From (91) it follows that there exists $p_1 > 0$ such that

$$r(\varepsilon) \le \frac{p_1}{\sqrt{\varepsilon}}. \tag{100}$$

To estimate the right part of (99), we assume that

$$\varepsilon = \varepsilon(\delta) = \frac{1}{\ln\ln\frac{1}{\delta}}. \tag{101}$$

Thus, from (99)–(101) we obtain

$$\|\hat{q}_\delta(\sigma) - \hat{q}_0(\sigma)\| \le \frac{7p_1(1-x_0)^3\sqrt{\ln\ln\frac{1}{\delta}}}{\ln^{3-\frac{1}{\ln\ln\frac{1}{\delta}}}\left(\frac{1}{\delta}\right)}. \tag{102}$$

Let us consider the value of $\ln^{-3+\frac{1}{\ln\ln\frac{1}{\delta}}}\left(\frac{1}{\delta}\right)$ in the estimate (102)

$$\ln^{-3+\frac{1}{\ln\ln\frac{1}{\delta}}}\left(\frac{1}{\delta}\right) = \ln^{-3}\left(\frac{1}{\delta}\right)\cdot\ln^{\frac{1}{\ln\ln\frac{1}{\delta}}}\left(\frac{1}{\delta}\right), \tag{103}$$

$$\ln\ln^{\frac{1}{\ln\ln\frac{1}{\delta}}}\left(\frac{1}{\delta}\right) = \frac{1}{\ln\ln\frac{1}{\delta}}\cdot\ln\ln\frac{1}{\delta} = 1. \tag{104}$$

From (102)–(104) it follows that there exists a number $p_2 > 0$ such that, for sufficiently small values of δ , the following estimation holds true

$$\|\hat{q}_\delta(\sigma) - \hat{q}_0(\sigma)\| \le p_2\cdot\sqrt{\ln\ln\frac{1}{\delta}}\cdot\ln^{-3}\left(\frac{1}{\delta}\right). \tag{105}$$

Applying the transformation (77) to $\hat{q}(\sigma)$ and using (105), we get

$$\|q_\delta(t) - q_0(t)\| \le p_2\cdot\sqrt{\ln\ln\frac{1}{\delta}}\cdot\ln^{-3}\left(\frac{1}{\delta}\right), \tag{106}$$

for sufficiently small values of δ.

It follows from (106) that the improved estimation is an infinitely small value compared with the estimate (78).

6 Conclusion

All the estimation constants in (105) might be computed with higher accuracy but it would have increased the article length dramatically.

To estimate an error of the nonlinear projective regularization method in the equation $\|A\hat{q}_\delta^\gamma(\sigma) - \hat{f}_\delta(\sigma)\|^2 = 16\delta^2$ defining a regularization parameter $\gamma(\hat{f}_\delta, \delta)$, it was sufficient to choose $9\delta^2$ instead of $16\delta^2$, but, in that case, the reference [23] would not have been valid, requiring a new proof and, again, extending the paper.

Note that in this paper, as compared with the earlier ones of the similar subject matter, in the direct problem we present the statement and treatment of the combined problem describing the heating and self-cooling processes for a combustion chamber wall more adequately.

The main result of this paper is the use of the property of the nonlinear projective regularization method to improve the error estimate substantially. This is of great importance for heat-loaded engineering structures.

The rest of the paper results, including the statement of the inverse boundary value problem, are new but less meaningful.

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