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Second-order arithmetic and the consistency of first-order theories

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It is known that the proposition $\mathrm{con}_{\mathrm{PA}}$ expressing the consistency of the Peano first-order arithmetic PA cannot be proved within PA, but can be proved in the second-order arithmetic A_2 [1], [2].

In the present note we show that an analogue of Gödel's second incompleteness theorem holds for theories given by the set of non-logical axioms of any level of the Kleene-Mostowski arithmetical hierarchy [2]. Then the propositions expressing the consistency of the corresponding theories are provable in A_2.

Let L be the language of the first-order arithmetic PA [2], [3]. A fixed Gödel numbering of L is assumed, and the symbol $\ulcorner\varphi\urcorner$ denotes the numeral corresponding to the number of the formula φ. The terms imp and Sub have the usual meaning: for any formulae φ, ψ and any $n \in \omega$ the formulae $\mathrm{imp}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) = \ulcorner\varphi \to \psi\urcorner$ and $\mathrm{Sub}(\ulcorner\varphi(x)\urcorner, \bar{n}) = \ulcorner\varphi(\bar{n})\urcorner$ are provable in PA. Following [3], [4], we write $\ulcorner\varphi(\dot{x})\urcorner$ instead of $\mathrm{Sub}(\ulcorner\varphi(x)\urcorner, x)$.

For any $n > 0$, as non-logical axioms of the theory $T(n)$ we take the set of Π^0_n-propositions of the language L that are true in the standard interpretation $\langle\omega, +, \cdot\rangle$ and the axioms of PA [3], [4]. As is known [4], [1], there exists a Π^0_n-formula $\mathrm{Tr}_n(x)$ such that for any Π^0_n-formula $\varphi(x_1, \ldots, x_k)$

$$(1) \qquad \mathrm{PA} \vdash \varphi(x_1, \ldots, x_k) \leftrightarrow \mathrm{Tr}_n(\ulcorner\varphi(\dot{x}_1, \ldots, \dot{x}_k)\urcorner).$$

The predicate of provability for $T(n)$ can be given as follows:

$$\mathrm{Prf}_{T(n)}(x, y) \rightleftharpoons \mathrm{Seg}(x) \,\&\, \forall u \leqslant \mathrm{lh}(x)\,(LAx_{PA}((x)_u) \lor$$
$$\lor Ax_{PA}((x)_u) \lor \mathrm{Tr}_n((x)_u) \lor \exists z, v < u\,((x)_z =$$
$$= \mathrm{imp}((x)_v, (x)_u) \lor (x)_u = \mathrm{Gen}((x)_z)) \,\&\, \mathrm{Form}((x)_u) \,\&\, y = (x)_{\mathrm{lh}(x)},$$
$$\mathrm{Pr}_{T(n)}(y) \rightleftharpoons \exists x\, \mathrm{Prf}_{T(n)}(x, y), \quad \mathrm{con}_{T(n)} \rightleftharpoons \neg \mathrm{Pr}_{T(n)}(\ulcorner\bar{0} = \bar{1}\urcorner),$$

where Seg, LAx_{PA}, Ax_{PA}, Form, imp, Gen, lh, and $(x)_u$ are the corresponding primitive recursive formulae and terms [3], [4].

Theorem 1. *For any formulae* φ, ψ *the following relations hold*:

1) $$T(n) \vdash \varphi \Rightarrow T(n) \vdash \mathrm{Pr}_{T(n)}(\ulcorner\varphi\urcorner),$$
2) $$\mathrm{PA} \vdash \mathrm{Pr}_{T(n)}(\ulcorner\varphi\urcorner) \,\&\, \mathrm{Pr}_{T(n)}(\ulcorner\varphi \to \psi\urcorner) \to \mathrm{Pr}_{T(n)}(\ulcorner\psi\urcorner),$$
3) $$\mathrm{PA} \vdash \mathrm{Pr}_{T(n)}(\ulcorner\varphi\urcorner) \to \mathrm{Pr}_{T(n)}(\ulcorner\mathrm{Pr}_{T(n)}(\ulcorner\varphi\urcorner)\urcorner),$$
4) $$T(n) \nvdash \mathrm{con}_{T(n)}.$$

Proof. 1) If $T \vdash \varphi$, then $\langle\omega, +, \cdot\rangle \models \mathrm{Pr}_{T(n)}(\ulcorner\varphi\urcorner)$ by the definition of $\mathrm{Pr}_{T(n)}(y)$. Since $\mathrm{Pr}_{T(n)}(\ulcorner\varphi\urcorner)$ is equivalent in PA to *a* $\sum_{n+1}$-proposition, it follows that $\mathrm{Pr}_{T(n)}(\ulcorner\varphi\urcorner)$ is provable in $T(n)$.

2) This follows from Theorem 4.6 of [3].

3) Using (1), for the Π^0_n-formula $\varphi(y)$ we have

$$\mathrm{PA} \vdash \varphi(y) \to \mathrm{Pr}_{T(n)}(\ulcorner\varphi(\dot{y})\urcorner).$$

Hence

$$\mathrm{PA} \vdash \exists y \varphi(y) \to \mathrm{Pr}_{T(n)}(\ulcorner\exists y \varphi(y)\urcorner).$$

4) This is proved as in the classical case, using 1)-3).

In the language of second-order arithmetic A_2 there is a formula $\mathrm{Tr}(x)$ (a true formula of PA) such that for any formula φ of L

$$A_2 \vdash \varphi(x_1, \ldots, x_k) \leftrightarrow \mathrm{Tr}(\ulcorner\varphi(\dot{x}_1, \ldots, \dot{x}_k)\urcorner).$$

Taking into account the fact that in A_2 the formula

$$\mathrm{Tr}_n(x) \to \mathrm{Tr}\,(x)$$

is provable, by analogy with [1] we can prove the following lemma.

Lemma.

$$A_2 \vdash \mathrm{Pr}_{T(n)}(x) \to \mathrm{Tr}\,(x).$$

Using this Lemma, we obtain

$$A_2 \vdash \neg\, \mathrm{Tr}\,(\ulcorner \bar{0} = \bar{1} \urcorner) \to \neg\, \mathrm{Pr}_{T(n)}(\ulcorner \bar{0} = \bar{1} \urcorner).$$

Hence, $A_2 \vdash \mathrm{con}_{T(n)}$.

Thus, the following result is valid.

***Theorem* 2.** *For any $n > 0$*

$$A_2 \vdash \mathrm{con}_{T(n)}.$$

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