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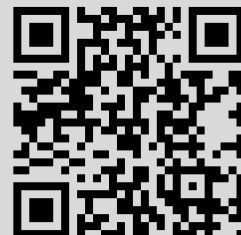
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Mathematical Analysis of a Generalized Chiral Quark Soliton Model

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Abstract. A generalized version of the so-called chiral quark soliton model (CQSM) in nuclear physics is introduced. The Hamiltonian of the generalized CQSM is given by a Dirac type operator with a mass term being an operator-valued function. Some mathematically rigorous results on the model are reported. The subjects included are: (i) supersymmetric structure; (ii) spectral properties; (iii) symmetry reduction; (iv) a unitarily equivalent model.

Key words: chiral quark soliton model; Dirac operator; supersymmetry; ground state; symmetry reduction

2000 Mathematics Subject Classification: 81Q10; 81Q05; 81Q60; 47N50

1 Introduction

The chiral quark soliton model (CQSM) [5] is a model describing a low-energy effective theory of the quantum chromodynamics, which was developed in 1980's (for physical aspects of the model, see, e.g., [5] and references therein). The Hamiltonian of the CQSM is given by a Dirac type operator with iso-spin, which differs from the usual Dirac type operator in that the mass term is a matrix-valued function with an effect of an interaction between quarks and the pion field. It is an interesting object from the purely operator-theoretical point of view too. But there are few mathematically rigorous analyses for such Dirac type operators (e.g., [2], where the problem on essential self-adjointness of a Dirac operator with a variable mass term given by a scalar function is discussed).

In the previous paper [1] we studied some fundamental aspects of the CQSM in a mathematically rigorous way. In this paper we present a slightly general form of the CQSM, which we call a *generalized CQSM*, and report that results similar to those in [1] hold on this model too, at least, as far as some general aspects are concerned.

2 A Generalized CQSM

The Hilbert space of a Dirac particle with mass $M > 0$ and iso-spin $1/2$ is taken to be $L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathbb{C}^2$. For a generalization, we replace the iso-spin space \mathbb{C}^2 by an arbitrary complex Hilbert space \mathcal{K} . Thus the Hilbert space \mathcal{H} in which we work in the present paper is given by

$$\mathcal{H} := L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathcal{K}.$$

We denote by $\mathbf{B}(\mathcal{K})$ the Banach space of all bounded linear operators on \mathcal{K} with domain \mathcal{K} . Let $T : \mathbb{R}^3 \rightarrow \mathbf{B}(\mathcal{K})$; $\mathbb{R}^3 \ni \mathbf{x} = (x_1, x_2, x_3) \mapsto T(\mathbf{x}) \in \mathbf{B}(\mathcal{K})$ be a Borel measurable mapping

such that, for all $\mathbf{x} \in \mathbb{R}^3$, $T(\mathbf{x})$ is a non-zero bounded self-adjoint operator on \mathcal{K} such that $\|T\|_\infty := \sup_{\mathbf{x} \in \mathbb{R}^3} \|T(\mathbf{x})\| < \infty$, where $\|T(\mathbf{x})\|$ denotes the operator norm of $T(\mathbf{x})$.

Example 1. In the original CQSM, $\mathcal{K} = \mathbb{C}^2$ and $T(\mathbf{x}) = \boldsymbol{\tau} \cdot \mathbf{n}(\mathbf{x})$, where $\mathbf{n} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a measurable vector field with $|\mathbf{n}(\mathbf{x})| = 1$, a.e. (almost everywhere) $\mathbf{x} \in \mathbb{R}^3$ and $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ is the set of the Pauli matrices.

We denote by $\{\alpha_1, \alpha_2, \alpha_3, \beta\}$ the Dirac matrices, i.e., 4×4 -Hermitian matrices satisfying

$$\{\alpha_j, \alpha_k\} = 2\delta_{jk}, \quad \{\alpha_j, \beta\} = 0, \quad \beta^2 = 1, \quad j, k = 1, 2, 3,$$

where $\{A, B\} := AB + BA$.

Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ be measurable, a.e., finite and

$$U_F := (\cos F) \otimes I + i(\sin F)\gamma_5 \otimes T,$$

where I denotes identity and $\gamma_5 := -i\alpha_1\alpha_2\alpha_3$. We set $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \alpha_3)$ and $\nabla := (D_1, D_2, D_3)$ with D_j being the generalized partial differential operator in the variable x_j . Then the one particle Hamiltonian of a generalized CQSM is defined by

$$H := -i\boldsymbol{\alpha} \cdot \nabla \otimes I + M(\beta \otimes I)U_F$$

acting in the Hilbert space \mathcal{H} . For a linear operator L , we denote its domain by $D(L)$. It is well-known that $-i\boldsymbol{\alpha} \cdot \nabla$ is self-adjoint with $D(-i\boldsymbol{\alpha} \cdot \nabla) = \cap_{j=1}^3 D(D_j)$. Since the operator $M(\beta \otimes I)U_F$ is bounded and self-adjoint, it follows that H is self-adjoint with domain $D(H) = \cap_{j=1}^3 D(D_j \otimes I) = H^1(\mathbb{R}^3; \mathbb{C}^4 \otimes \mathcal{K})$, the Sobolev space of order 1 consisting of $\mathbb{C}^4 \otimes \mathcal{K}$ -valued measurable functions on \mathbb{R}^3 . In the context of the CQSM, the function F is called a profile function. In what follows we sometimes omit the symbol of tensor product \otimes in writing equations down.

Example 2. Usually profile functions are assumed to be rotation invariant with boundary conditions

$$F(0) = -\pi, \quad \lim_{|\mathbf{x}| \rightarrow \infty} F(\mathbf{x}) = 0.$$

The following are concrete examples [6]:

- (I) $F(\mathbf{x}) = -\pi \exp(-|\mathbf{x}|/R)$, $R = 0.55 \times 10^{-15}$ m;
- (II) $F(\mathbf{x}) = -\pi \{a_1 \exp(-|\mathbf{x}|/R_1) + a_2 \exp(-|\mathbf{x}|^2/R_2^2)\}$,
 $a_1 = 0.65$, $R_1 = 0.58 \times 10^{-15}$ m, $a_2 = 0.35$, $R_2 = \sqrt{0.3} \times 10^{-15}$ m;
- (III) $F(\mathbf{x}) = -\pi \left(1 - \frac{|\mathbf{x}|}{\sqrt{\lambda^2 + |\mathbf{x}|^2}}\right)$, $\lambda = \sqrt{0.4} \times 10^{-15}$ m.

We say that a self-adjoint operator A on \mathcal{H} has chiral symmetry if $\gamma_5 A \subset A \gamma_5$.

Proposition 1. *The Hamiltonian H has no chiral symmetry.*

Proof. It is easy to check that, for all $\psi \in D(H)$, $\gamma_5 \psi \in D(H)$ and $[\gamma_5, H]\psi = 2M\gamma_5\beta U_F \psi$. Note that $U_F \neq 0$. Hence, $[\gamma_5, H] \neq 0$ on $D(H)$. \blacksquare

We note that, if F and T are differentiable on \mathbb{R}^3 with $\sup_{\mathbf{x} \in \mathbb{R}^3} |\partial_j F(\mathbf{x})| < \infty$ and $\sup_{\mathbf{x} \in \mathbb{R}^3} \|\partial_j T(\mathbf{x})\| < \infty$ ($j = 1, 2, 3$), then the square of H takes the form

$$H^2 = (-\Delta + M^2) \otimes I - iM\beta\boldsymbol{\alpha} \cdot (\nabla U_F) + M^2 \sin^2 F \otimes (T^2 - I).$$

This is a Schrödinger operator with an operator-valued potential.

3 Operator matrix representation

For more detailed analyses of the model, it is convenient to work with a suitable representation of the Dirac matrices. Here we take the following representation of α_j and β (the Weyl representation):

$$\alpha_j = \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where σ_1, σ_2 and σ_3 are the Pauli matrices. Let $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$ and

$$\Phi_F := (\cos F) \otimes I + i(\sin F) \otimes T.$$

Then we have the following operator matrix representation for H :

$$H = \begin{pmatrix} -i\boldsymbol{\sigma} \cdot \nabla & M\Phi_F^* \\ M\Phi_F & i\boldsymbol{\sigma} \cdot \nabla \end{pmatrix}.$$

4 Supersymmetric aspects

Let $\xi : \mathbb{R}^3 \rightarrow \mathcal{B}(\mathcal{K})$ be measurable such that, for all $\mathbf{x} \in \mathbb{R}^3$, $\xi(\mathbf{x})$ is a bounded self-adjoint operator on \mathcal{K} and $\xi(\mathbf{x})^2 = I, \forall \mathbf{x} \in \mathbb{R}^3$. Let

$$\Gamma(\mathbf{x}) := i\gamma_5\beta \otimes \xi(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3.$$

We define an operator $\hat{\Gamma}$ on \mathcal{H} by

$$(\hat{\Gamma}\psi)(\mathbf{x}) := \Gamma(\mathbf{x})\psi(\mathbf{x}), \quad \psi \in \mathcal{H}, \quad \text{a.e. } \mathbf{x} \in \mathbb{R}^3.$$

The following fact is easily proven:

Lemma 1. *The operator $\hat{\Gamma}$ is self-adjoint and unitary, i.e., it is a grading operator on \mathcal{H} : $\hat{\Gamma}^* = \hat{\Gamma}, \hat{\Gamma}^2 = I$.*

Theorem 1. *Suppose that ξ is strongly differentiable with $\sup_{\mathbf{x} \in \mathbb{R}^3} \|\partial_j \xi(\mathbf{x})\| < \infty$ ($j = 1, 2, 3$) and*

$$\sum_{j=1}^3 \alpha_j \otimes D_j \xi(\mathbf{x}) = M\gamma_5\beta \{\xi(\mathbf{x}), T(\mathbf{x})\} \sin F(\mathbf{x}). \quad (1)$$

Then $\hat{\Gamma}D(H) \subset D(H)$ and $\{\hat{\Gamma}, H\}\psi = 0, \forall \psi \in D(H)$.

Proof. For all $\psi \in D_0 := C_0^\infty(\mathbb{R}^3) \otimes_{\text{alg}} (\mathbb{C}^4 \otimes \mathcal{K})$ (\otimes_{alg} denotes algebraic tensor product), we have

$$D_j \hat{\Gamma}\psi = i\gamma_5\beta \otimes (D_j \xi)\psi + i\gamma_5\beta \otimes \xi(D_j \psi). \quad (2)$$

By a limiting argument using the fact that D_0 is a core of $D_j \otimes I$, we can show that, for all $\psi \in D(D_j)$, $\hat{\Gamma}\psi$ is in $D(D_j)$ and (2) holds. Hence, for all $\psi \in D(H)$, $\hat{\Gamma}\psi \in D(H)$ and (2) holds. Thus we have for all $\psi \in D(H)$ $\{\hat{\Gamma}, H\}\psi = C_1\psi + C_2\psi$ with $C_1 := \sum_{j=1}^3 \{\gamma_5\beta \otimes \xi, \alpha_j D_j\}$ and $C_2 := iM\{\gamma_5\beta \otimes \xi, \beta U_F\}$. Using the fact that $\{\gamma_5, \beta\} = 0$ and $[\gamma_5, \alpha_j] = 0$ ($j = 1, 2, 3$), we obtain $C_1\psi = -\gamma_5\beta \left(\sum_{j=1}^3 \alpha_j D_j \xi \right) \psi$. Similarly direct computations yield $(C_2\psi)(\mathbf{x}) = -M \sin F(\mathbf{x}) \otimes \{\xi(\mathbf{x}), T(\mathbf{x})\}\psi(\mathbf{x})$. Thus (1) implies $\{\hat{\Gamma}, H\}\psi = 0$. ■

Theorem 1 means that, under its assumption, H may be interpreted as a generator of a supersymmetry with respect to $\hat{\Gamma}$.

Example 3. Consider the case $\mathcal{K} = \mathbb{C}^2$. Let $f, g : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a continuously differentiable function such that

$$(1 + C^2)f(\mathbf{x})^2 + g(\mathbf{x})^2 = 1.$$

with a real constant $C \neq 0$ and $\mathbf{n}(\mathbf{x}) := (f(\mathbf{x}), Cf(\mathbf{x}), g(\mathbf{x}))$. Then $|\mathbf{n}(\mathbf{x})| = 1, \forall \mathbf{x} \in \mathbb{R}^3$. Let

$$\xi := \frac{C}{\sqrt{1 + C^2}}\tau_1 - \frac{1}{\sqrt{1 + C^2}}\tau_2, \quad T(\mathbf{x}) := \boldsymbol{\tau} \cdot \mathbf{n}(\mathbf{x}).$$

Then $\xi^2 = I$ and (ξ, T) satisfies (1).

To state spectral properties of H , we recall some definitions. For a self-adjoint operator S , we denote by $\sigma(S)$ the spectrum of S . The point spectrum of S , i.e., the set of all the eigenvalues of S is denoted $\sigma_p(S)$. An isolated eigenvalue of S with finite multiplicity is called a discrete eigenvalue of S . We denote by $\sigma_d(S)$ the set of all the discrete eigenvalues of S . The set $\sigma_{\text{ess}}(S) := \sigma(S) \setminus \sigma_d(S)$ is called the essential spectrum of S .

Theorem 2. *Under the same assumption as in Theorem 1, the following holds:*

- (i) $\sigma(H)$ is symmetric with respect to the origin of \mathbb{R} , i.e., if $\lambda \in \sigma(H)$, then $-\lambda \in \sigma(H)$.
- (ii) $\sigma_{\#}(H)$ ($\# = p, d$) is symmetric with respect to the origin of \mathbb{R} with

$$\dim \ker(H - \lambda) = \dim \ker(H - (-\lambda))$$

for all $\lambda \in \sigma_{\#}(H)$.

- (iii) $\sigma_{\text{ess}}(H)$ is symmetric with respect to the origin of \mathbb{R} .

Proof. Theorem 1 implies a unitary equivalence of H and $-H$ ($\hat{\Gamma}H\hat{\Gamma}^{-1} = -H$). Thus the desired results follow. \blacksquare

Remark 1. Suppose that the assumption of Theorem 1 holds. In view of supersymmetry breaking, it is interesting to compute $\dim \ker H$. This is related to the index problem: Let

$$\mathcal{H}_+ := \ker(\hat{\Gamma} - 1), \quad \mathcal{H}_- := \ker(\hat{\Gamma} + 1)$$

and

$$H_{\pm} := H|_{\mathcal{H}_{\pm}}.$$

Then H_+ (resp. H_-) is a densely defined closed linear operator from \mathcal{H}_+ (resp. \mathcal{H}_-) to \mathcal{H}_- (resp. \mathcal{H}_+) with $D(H_+) = D(H) \cap \mathcal{H}_+$ (resp. $D(H_-) = D(H) \cap D(H_-)$). Obviously

$$\ker H = \ker H_+ \oplus \ker H_-.$$

The analytical index of H_+ is defined by

$$\text{index}(H_+) := \dim \ker H_+ - \dim \ker H_+^*,$$

provided that at least one of $\dim \ker H_+$ and $\dim \ker H_+^*$ is finite. We conjecture that, for a class of F and T , $\text{index}(H_+) = 0$.

5 The essential spectrum and finiteness of the discrete spectrum of H

5.1 Structure of the spectrum of H

Theorem 3. *Suppose that $\dim \mathcal{K} < \infty$ and*

$$\lim_{|\mathbf{x}| \rightarrow \infty} F(\mathbf{x}) = 0. \quad (3)$$

Then

$$\sigma_{\text{ess}}(H) = (-\infty, -M] \cup [M, \infty), \quad (4)$$

$$\sigma_{\text{d}}(H) \subset (-M, M). \quad (5)$$

Proof. We can rewrite H as $H = H_0 \otimes I + V$ with $H_0 := -i\boldsymbol{\alpha} \cdot \nabla + M\beta$ and $V := M(\beta \otimes I)(U_F - I)$. We denote by χ_R ($R > 0$) the characteristic function of the set $\{\mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}| < R\}$. It is well-known that, for all $z \in \mathbb{C} \setminus \mathbb{R}$, $(H_0 - z)^{-1}\chi_R$ is compact [7, Lemma 4.6]. Since \mathcal{K} is finite dimensional, it follows that $(H_0 \otimes I - z)^{-1}\chi_R \otimes I$ is compact. We have

$$\|V(\mathbf{x})\| \leq M(|\cos F(\mathbf{x}) - 1| + |\sin F(\mathbf{x})| \|T\|_{\infty}) \leq M \left(\frac{|F(\mathbf{x})|^2}{2} + |F(\mathbf{x})| \|T\|_{\infty} \right).$$

Hence, by (3), we have $\lim_{R \rightarrow \infty} \sup_{|\mathbf{x}| > R} \|V(\mathbf{x})\| = 0$. Then, in the same way as in the method

described on [7, pp. 115–117], we can show that, for all $z \in \mathbb{C} \setminus \mathbb{R}$, $(H - z)^{-1} - (H_0 \otimes I - z)^{-1}$ is compact. Hence, by a general theorem (e.g., [7, Theorem 4.5]), $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0 \otimes I)$. Since $\sigma_{\text{ess}}(H_0) = (-\infty, -M] \cup [M, \infty)$ ([7, Theorem 1.1]), we obtain (4). Relation (5) follows from (4) and $\sigma_{\text{d}}(H) = \sigma(H) \setminus \sigma_{\text{ess}}(H)$. \blacksquare

5.2 Bound for the number of discrete eigenvalues of H

Suppose that $\dim \mathcal{K} < \infty$ and (3) holds. Then, by Theorem 3, we can define the number of discrete eigenvalues of H counting multiplicities:

$$N_H := \dim \text{Ran } E_H((-M, M)), \quad (6)$$

where E_H is the spectral measure of H .

To estimate an upper bound for N_H , we introduce a hypothesis for F and T :

Hypothesis (A).

- (i) $T(\mathbf{x})^2 = I$, $\forall \mathbf{x} \in \mathbb{R}^3$ and T is strongly differentiable with $\sum_{j=1}^3 (D_j T(\mathbf{x}))^2$ being a multiplication operator by a scalar function on \mathbb{R}^3 .
- (ii) $F \in C^1(\mathbb{R}^3)$.
- (iii) $\sup_{\mathbf{x} \in \mathbb{R}^3} |D_j F(\mathbf{x})| < \infty$, $\sup_{\mathbf{x} \in \mathbb{R}^3} \|D_j T(\mathbf{x})\| < \infty$ ($j = 1, 2, 3$).

Under this assumption, we can define

$$V_F(\mathbf{x}) := \sqrt{|\nabla F(\mathbf{x})|^2 + \sum_{j=1}^3 (D_j T(\mathbf{x}))^2 \sin^2 F(\mathbf{x})}.$$

Theorem 4. *Let $\dim \mathcal{K} < \infty$. Assume (3) and Hypothesis (A). Suppose that*

$$C_F := \int_{\mathbb{R}^6} \frac{V_F(\mathbf{x})V_F(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} d\mathbf{x}d\mathbf{y} < \infty.$$

Then N_H is finite with

$$N_H \leq \frac{(\dim \mathcal{K})M^2 C_F}{4\pi^2}.$$

A basic idea for the proof of Theorem 4 is as follows. Let

$$L(F) := H^2 - M^2.$$

Then we have

$$L(F) = -\Delta + M \begin{pmatrix} 0 & W_F^* \\ W_F & 0 \end{pmatrix}$$

with $W_F := i\boldsymbol{\sigma} \cdot \nabla \Phi_F$. Note that

$$W_F^* W_F = W_F W_F^* = V_F^2.$$

Let

$$L_0(F) := -\Delta - MV_F.$$

For a self-adjoint operator S , we introduce a set

$$N_-(S) := \text{the number of negative eigenvalues of } S \text{ counting multiplicities.}$$

The following is a key lemma:

Lemma 2.

$$N_H \leq N_-(L(F)) \leq N_-(L_0(F)). \tag{7}$$

Proof. For each $\lambda \in \sigma_d(H) \cap (-M, M)$, we have $\ker(H - \lambda) \subset \ker(L(F) - E_\lambda)$ with $E_\lambda = \lambda^2 - M^2 < 0$. Hence the first inequality of (7) follows. The second inequality of (7) can be proven in the same manner as in the proof of [1, Lemma 3.3], which uses the min-max principle. ■

On the other hand, one has

$$N_-(L_0(F)) \leq \frac{(\dim \mathcal{K})M^2 C_F}{4\pi^2}$$

(the Birman–Schwinger bound [4, Theorem XIII.10]). In this way we can prove Theorem 4.

As a direct consequence of Theorem 4, we have the following fact on the absence of discrete eigenvalues of H :

Corollary 1. *Assume (3) and Hypothesis (A). Let $(\dim \mathcal{K})M^2 C_F < 4\pi^2$. Then $\sigma_d(H) = \emptyset$, i.e., H has no discrete eigenvalues.*

6 Existence of discrete ground states

Let A be a self-adjoint operator on a Hilbert space and bounded from below. Then

$$E_0(A) := \inf \sigma(A)$$

is finite. We say that A has a *ground state* if $E_0(A) \in \sigma_p(A)$. In this case, a non-zero vector in $\ker(A - E_0(A))$ is called a *ground state of A* . Also we say that A has a discrete ground state if $E_0(A) \in \sigma_d(A)$.

Definition 1. Let

$$E_0^+(H) := \inf [\sigma(H) \cap [0, \infty)], \quad E_0^-(H) := \sup [\sigma(H) \cap (-\infty, 0]].$$

- (i) If $E_0^+(H)$ is an eigenvalue of H , then we say that H has a *positive energy ground state* and we call a non-zero vector in $\ker(H - E_0^+(H))$ a *positive energy ground state* of H .
- (ii) If $E_0^-(H)$ is an eigenvalue of H , then we say that H has a *negative energy ground state* and we call a non-zero vector in $\ker(H - E_0^-(H))$ a *negative energy ground state* of H .
- (iii) If $E_0^+(H)$ (resp. $E_0^-(H)$) is a discrete eigenvalue of H , then we say that H has a *discrete positive* (resp. *negative*) *energy ground state*.

Remark 2. If the spectrum of H is symmetric with respect to the origin of \mathbb{R} as in Theorem 2, then $E_0^+(H) = -E_0^-(H)$, and H has a positive energy ground state if and only if it has a negative energy ground state.

Assume Hypothesis (A). Then the operators

$$S_{\pm}(F) := -\Delta \pm M(D_3 \cos F)$$

are self-adjoint with $D(S_{\pm}(F)) = D(\Delta)$ and bounded from below.

As for existence of discrete ground states of the Dirac operator H , we have the following theorem:

Theorem 5. *Let $\dim \mathcal{K} < \infty$. Assume Hypothesis (A) and (3). Suppose that $E_0(S_+(F)) < 0$ or $E_0(S_-(F)) < 0$. Then H has a discrete positive energy ground state or a discrete negative ground state.*

Proof. We describe only an outline of proof. We have

$$\sigma_{\text{ess}}(L(F)) = [0, \infty), \quad \sigma_d(L(F)) \subset [-M^2, 0).$$

Hence, if $L(F)$ has a discrete eigenvalue, then H has a discrete eigenvalue in $(-M, M)$. By the min-max principle, we need to find a unit vector Ψ such that $\langle \Psi, L(F)\Psi \rangle < 0$. Indeed, for each $f \in D(\Delta)$, we can find vectors $\Psi_f^{\pm} \in D(L(F))$, such that $\langle \Psi_f^{\pm}, L(F)\Psi_f^{\pm} \rangle = \langle f, S_{\pm}f \rangle$. By the present assumption, there exists a non-zero vector $f_0 \in D(\Delta)$ such that $\langle f_0, S_+(F)f_0 \rangle < 0$ or $\langle f_0, S_-(F)f_0 \rangle < 0$. Thus the desired results follow. \blacksquare

To find a class of F such that $E_0(S_+(F)) < 0$ or $E_0(S_-(F)) < 0$, we proceed as follows. For a constant $\varepsilon > 0$ and a function f on \mathbb{R}^d , we define a function f_{ε} on \mathbb{R}^d by

$$f_{\varepsilon}(x) := f(\varepsilon x), \quad x \in \mathbb{R}^d.$$

The following are key Lemmas.

Lemma 3. *Let $V : \mathbb{R}^d \rightarrow \mathbb{R}$ be in $L^2_{\text{loc}}(\mathbb{R}^d)$ and*

$$S_\varepsilon := -\Delta + V_\varepsilon.$$

Suppose that:

- (i) *For all $\varepsilon > 0$, S_ε is self-adjoint, bounded below and $\sigma_{\text{ess}}(S_\varepsilon) \subset [0, \infty)$.*
- (ii) *There exists a nonempty open set $\Omega \subset \{x \in \mathbb{R}^d | V(x) < 0\}$.*

Then there exists a constant $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, S_ε has a discrete ground state.

Proof. A basic idea for the proof of this lemma is to use the min-max principle (see [1, Lemma 4.3]). ■

Lemma 4. *$V : \mathbb{R}^d \rightarrow \mathbb{R}$ be continuous with $V(x) \rightarrow 0(|x| \rightarrow \infty)$. Suppose that $\{x \in \mathbb{R}^d | V(x) < 0\} \neq \emptyset$. Then:*

- (i) *$-\Delta + V$ is self-adjoint and bounded below.*
- (ii) *$\sigma_{\text{ess}}(-\Delta + V) = [0, \infty)$.*
- (iii) *S_ε has a discrete ground state for all $\varepsilon \in (0, \varepsilon_0)$ with some $\varepsilon_0 > 0$.*

Proof. The facts (i) and (ii) follow from the standard theory of Schrödinger operators. Part (iii) follows from a simple application of Lemma 3 (for more details, see the proof of [1, Lemma 4.4]). ■

We now consider a one-parameter family of Dirac operators:

$$H_\varepsilon := (-i)\boldsymbol{\alpha} \cdot \nabla + \frac{1}{\varepsilon}M(\beta \otimes I)U_{F_\varepsilon}.$$

Theorem 6. *Let $\dim \mathcal{K} < \infty$. Assume Hypothesis (A) and (3). Suppose that $D_3 \cos F$ is not identically zero. Then there exists a constant $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, H_ε has a discrete positive energy ground state or a discrete negative ground state.*

Proof. This follows from Theorem 5 and Lemma 4 (for more details, see the proof of [1, Theorem 4.5]). ■

7 Symmetry reduction of H

Let T_1, T_2 and T_3 be bounded self-adjoint operators on \mathcal{K} satisfying

$$\begin{aligned} T_j^2 &= I, & j &= 1, 2, 3, \\ T_1 T_2 &= iT_3, & T_2 T_3 &= iT_1, & T_3 T_1 &= iT_2. \end{aligned}$$

Then it is easy to see that the anticommutation relations

$$\{T_j, T_k\} = 2\delta_{jk}I, \quad j, k = 1, 2, 3$$

hold. Since each T_j is a unitary self-adjoint operator with $T_j \neq \pm I$, it follows that

$$\sigma(T_j) = \sigma_{\text{p}}(T_j) = \{\pm 1\}.$$

We set $\mathbf{T} = (T_1, T_2, T_3)$.

In this section we consider the case where $T(\mathbf{x})$ is of the following form:

$$T(\mathbf{x}) = \mathbf{n}(\mathbf{x}) \cdot \mathbf{T},$$

where $\mathbf{n}(\mathbf{x})$ is the vector field in Example 1. We use the cylindrical coordinates for points $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$:

$$x_1 = r \cos \theta, \quad x_2 = r \sin \theta, \quad x_3 = z,$$

where $\theta \in [0, 2\pi)$, $r > 0$. We assume the following:

Hypothesis (B). There exists a continuously differentiable function $G : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ such that

- (i) $F(\mathbf{x}) = G(r, z)$, $\mathbf{x} \in \mathbb{R}^3 \setminus \{0\}$;
- (ii) $\lim_{r+|z| \rightarrow \infty} G(r, z) = 0$;
- (iii) $\sup_{r>0, z \in \mathbb{R}} (|\partial G(r, z)/\partial r| + |\partial G(r, z)/\partial z|) < \infty$.

We take the vector field $\mathbf{n} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ to be of the form

$$\mathbf{n}(\mathbf{x}) := (\sin \Theta(r, z) \cos(m\theta), \sin \Theta(r, z) \sin(m\theta), \cos \Theta(r, z)),$$

where $\Theta : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and m is a natural number.

Let L_3 be the third component of the angular momentum acting in $L^2(\mathbb{R}^3)$ and

$$K_3 := L_3 \otimes I + \frac{1}{2} \Sigma_3 \otimes I + \frac{m}{2} I \otimes T_3 \quad (8)$$

with $\Sigma_3 := \sigma_3 \oplus \sigma_3$. It is easy to see that K_3 is a self-adjoint operator acting in \mathcal{H} .

Lemma 5. *Assume that*

$$\Theta(\varepsilon r, \varepsilon z) = \Theta(r, z), \quad (r, z) \in (0, \infty) \times \mathbb{R}, \quad \varepsilon > 0. \quad (9)$$

Then, for all $t \in \mathbb{R}$ and $\varepsilon > 0$, the operator equality

$$e^{itK_3} H_\varepsilon e^{-itK_3} = H_\varepsilon \quad (10)$$

holds.

Proof. Similar to the proof of [1, Lemma 5.2]. We remark that, in the calculation of

$$e^{itK_3} T(\mathbf{x}) e^{-itK_3} = \sum_{j=1}^3 e^{itL_3} n_j(\mathbf{x}) e^{-itL_3} e^{itmT_3} T_j e^{-itmT_3},$$

the following formulas are used:

$$(T_1 \cos mt - T_2 \sin mt) e^{itmT_3} = T_1, \quad (T_1 \sin mt + T_2 \cos mt) e^{itmT_3} = T_2. \quad \blacksquare$$

Definition 2. We say that two self-adjoint operators on a Hilbert space strongly commute if their spectral measures commute.

Lemma 6. *Assume (9). Then, for all $\varepsilon > 0$, H_ε and K_3 strongly commute.*

Proof. By (10) and the functional calculus, we have for all $s, t \in \mathbb{R}$ $e^{itK_3} e^{isH_\varepsilon} e^{-itK_3} = e^{isH_\varepsilon}$, which is equivalent to $e^{itK_3} e^{isH_\varepsilon} = e^{isH_\varepsilon} e^{itK_3}$, $s, t \in \mathbb{R}$. By a general theorem (e.g., [3, Theorem VIII.13]), this implies the strong commutativity of K_3 and H_ε . \blacksquare

Lemma 6 implies that H_ε is reduced by eigenspaces of K_3 . Note that

$$\sigma(K_3) = \sigma_p(K_3) = \left\{ \ell + \frac{s}{2} + \frac{mt}{2} \mid \ell \in \mathbb{Z}, s = \pm 1, t = \pm 1 \right\}.$$

The eigenspace of K_3 with eigenvalue $\ell + (s/2) + (mt/2)$ is given by

$$\mathcal{M}_{\ell,s,t} := \mathcal{M}_\ell \otimes \mathcal{C}_s \otimes \mathcal{T}_t$$

with $\mathcal{C}_s := \ker(\Sigma_3 - s)$ and $\mathcal{T}_t := \ker(T_3 - t)$. Then \mathcal{H} has the orthogonal decomposition

$$\mathcal{H} = \bigoplus_{\ell \in \mathbb{Z}, s, t \in \{\pm 1\}} \mathcal{M}_{\ell,s,t}.$$

Thus we have:

Lemma 7. *Assume (9). Then, for all $\varepsilon > 0$, H_ε is reduced by each $\mathcal{M}_{\ell,s,t}$.*

We denote by $H_\varepsilon(\ell, s, t)$ by the reduced part of H_ε to $\mathcal{M}_{\ell,s,t}$ and set

$$H(\ell, s, t) := H_1(\ell, s, t).$$

For $s = \pm 1$ and $\ell \in \mathbb{Z}$, we define

$$L_s(G, \ell) := -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\ell^2}{r^2} + \frac{\partial^2}{\partial z^2} + sMD_z \cos G$$

acting in $L^2((0, \infty) \times \mathbb{R}, r dr dz)$ with domain

$$D(L_s(G, \ell)) := C_0^\infty((0, \infty) \times \mathbb{R})$$

and set

$$\mathcal{E}_0(L_s(G, \ell)) := \inf_{f \in C_0^\infty((0, \infty) \times \mathbb{R}), \|f\|_{L^2((0, \infty) \times \mathbb{R}, r dr dz)} = 1} \langle f, L_s(G, \ell) f \rangle.$$

The following theorem is concerned with the existence of discrete ground states of $H(\ell, s, t)$.

Theorem 7. *Assume Hypothesis (B) and (9). Fix an $\ell \in \mathbb{Z}$ arbitrarily, $s = \pm 1$ and $t = \pm 1$. Suppose that $\dim \mathcal{T}_t < \infty$ and*

$$\mathcal{E}_0(L_s(G, \ell)) < 0.$$

Then $H(\ell, s, t)$ has a discrete positive energy ground state or a discrete negative ground state.

Proof. Similar to the proof of Theorem 5 (for more details, see the proof of [1, Theorem 5.5]). ■

Theorem 8. *Assume Hypothesis (B) and (9). Suppose that $\dim \mathcal{T}_t < \infty$ and that $D_z \cos G$ is not identically zero. Then, for each $\ell \in \mathbb{Z}$, there exists a constant $\varepsilon_\ell > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\ell)$, each $H_\varepsilon(\ell, s, t)$ has a discrete positive energy ground state or a discrete negative ground state.*

Proof. Similar to the proof of Theorem 6 (for more details, see the proof of [1, Theorem 5.6]). ■

Theorem 8 immediately yields the following result:

Corollary 2. *Assume Hypothesis (B) and (9). Suppose that $\dim \mathcal{T}_t < \infty$ and that $D_z \cos G$ is not identically zero. Let ε_ℓ be as in Theorem 8 and, for each $n \in \mathbb{N}$ and $k > n$ ($k, n \in \mathbb{Z}$), $\nu_{k,n} := \min_{n+1 \leq \ell \leq k} \varepsilon_\ell$. Then, for each $\varepsilon \in (0, \nu_{k,n})$, H_ε has at least $(k - n)$ discrete eigenvalues counting multiplicities.*

Proof. Note that $\sigma_p(H_\varepsilon) = \bigcup_{\ell \in \mathbb{Z}, s, t = \pm 1} \sigma_p(H_\varepsilon(\ell, s, t))$. ■

8 A unitary transformation

We go back again to the generalized CQSM defined in Section 2. It is easy to see that the operator

$$X_F := \frac{1 + \gamma_5}{2} \exp\left(iF \otimes \frac{T}{2}\right) + \frac{1 - \gamma_5}{2} \exp\left(-iF \otimes \frac{T}{2}\right)$$

is unitary. Under Hypothesis (A), we can define the following operator-valued functions:

$$B_j(\mathbf{x}) := \frac{1}{2} D_j[F(\mathbf{x})T(\mathbf{x})], \quad \mathbf{x} \in \mathbb{R}^3, \quad j = 1, 2, 3.$$

We set

$$\mathbf{B} := (B_1, B_2, B_3)$$

and introduce

$$H(\mathbf{B}) := (-i)\boldsymbol{\alpha} \cdot \nabla + M\beta - \boldsymbol{\sigma} \cdot \mathbf{B}$$

acting in \mathcal{H} . Since $\boldsymbol{\sigma} \cdot \mathbf{B}$ is a bounded self-adjoint operator, $H(\mathbf{B})$ is self-adjoint with $D(H(\mathbf{B})) = \cap_{j=1}^3 D(D_j \otimes I)$.

Proposition 2. *Assume Hypothesis (A) and that $T(\mathbf{x})$ is independent of \mathbf{x} . Then*

$$X_F H X_F^{-1} = H(\mathbf{B}).$$

Proof. Similar to the proof of [1, Proposition 6.1]. ■

Using this proposition, we can prove the following theorem:

Theorem 9. *Let $\dim \mathcal{K} < \infty$. Assume Hypothesis (A) and that $T(\mathbf{x})$ is independent of \mathbf{x} . Suppose that*

$$\lim_{|\mathbf{x}| \rightarrow \infty} |\nabla F(\mathbf{x})| = 0.$$

Then

$$\sigma_{\text{ess}}(H) = (-\infty, -M] \cup [M, \infty). \quad (11)$$

Proof. By Proposition 2, we have $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H(\mathbf{B}))$. By the present assumption, $B_j(\mathbf{x}) = D_j F(\mathbf{x})T(\mathbf{0})/2$. Hence

$$\sup_{|\mathbf{x}| > R} \|\boldsymbol{\sigma} \cdot \mathbf{B}(\mathbf{x})\| \leq \sum_{j=1}^3 (\|T(\mathbf{0})\|/2) \sup_{|\mathbf{x}| > R} |D_j F(\mathbf{x})| \rightarrow 0 \quad (R \rightarrow \infty).$$

Therefore, as in the proof of Theorem 3, we conclude that $\sigma_{\text{ess}}(H(\mathbf{B})) = (-\infty, -M] \cup [M, \infty)$. Thus (11) follows. ■

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