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SOME REMARKS ON THE OPERATOR OF DIAGONAL MAP AND ON MULTIFUNCTIONAL ANALYTIC SPACES

This article has nature of announcement of new results.

We consider the action of the operator of diagonalization and we assert the boundedness of this operator from one holomorphic space to another one. Assertions in the bidisc extend previously known onedimensional estimates. Also new sharp theorems on diagonal map are given. These results essentially compliment previously known subtle results on diagonal map in polydisc. Some estimates for multifunctional analytic spaces in higher dimension are also given. These inequalities were previously known in the case of unit disc.

Ключевые слова: *holomorphic spaces, diagonal map*

Introduction

The problem of study of diagonal map and its applications for the first time was suggested by W. Rudin in [1]. Later several papers appeared where complete solutions were given for classical holomorphic spaces such as Hardy, Bergman and Nevanlinna classes (see [2 — 4]). Recently complete answer was given for so-called mixed norm spaces in [4]. For many other holomorphic spaces such as analytic Lizorkin—Triebel spaces the answer is unknown. Partially the aim of this note is to fill this gap. We will give not only precise answers in some cases but also we will calculate sizes of diagonal for the large scale of classes of analytic functions in the unit polydisc. Different multilinear operators in $\mathbb{R}^n$ were also under intensive investigation recently (see [5]). It seems interesting and natural to consider similar problems in complex analysis. For the first time some investigation in this direction appeared in [6]. Our intention is to develop these ideas further, establishing some new criteria, theorems for multifunctional analytic spaces or on multilinear type operators acting on them. Let us note that some of our results here are not new for $n = 1$, and were obtained in unit disc case by other methods (see for example [3; 6]).

1. Definitions and notations

Let $\mathbb{D} = \{|z| < 1\}$ be the unit disc on the complex plane $\mathbb{C}$, $\mathbb{D}^n = \{z = (z_1, \dots, z_n) : |z_j| < 1\}$ be the unit polydisc in $\mathbb{C}^n$,

$$\Gamma_t(\xi) = \{z \in \mathbb{D} : |1 - \overline{z}\xi| < t(1 - |z|)\}, \qquad t > 1, \qquad \xi \in T;$$

$$[T_{n,\alpha}(f)](\omega) = \{C(n,\alpha)\} \int\limits_{\mathbb{D}} \frac{f(z)(1-|z|)^{\alpha}}{(1-\overline{z}\omega_1)^{\frac{\alpha+2}{n}} \dots (1-\overline{z}\omega_n)^{\frac{\alpha+2}{n}}} dm_2(z),$$

where $n \in \mathbb{N}$, $\alpha > -1$, $C(n,\alpha)$ is the Bergman projection constant. Note $[T_{1,\alpha}(f)](\omega) = f(w)$, $T_{n,\alpha}$-operator appears in problems connected with diagonal map (see [2 — 4]) and is known as anoperator of diagonalization.

Let further D^γ be fractional derivative

$$(D^\gamma f)(z) = \sum_{k\ge 0}(k+1)^\gamma a_k z^k, \quad \gamma \in \mathbb{R}, \quad f \in H(\mathbb{D}),$$

where $H(\mathbb{D})$ is a space of all holomorphic functions, $A^p_\beta(\mathbb{D})$ and $A^p_{\vec{\beta}}(\mathbb{D}^n)$ are known Bergman spaces in $\mathbb{D}$ and $\mathbb{D}^n$, $H^p(\mathbb{D})$ and $H^p(\mathbb{D}^n)$ are Hardy spaces in $\mathbb{D}$ and $\mathbb{D}^n$ (see [1; 3]). Denote by S_n a unit sphere, by $\mathbb{B}^n$ a unit ball in $\mathbb{C}^n$ (see [7]). Let further

$$A_\infty H^p(\mathbb{D}^n) = \left\{ f \in H(\mathbb{D}^n) : \int\limits_{T^n} \sup_{z\in\Gamma_\alpha(\xi)} |f(z)|^p dm_n(\xi) < \infty \right\},$$

where $H(\mathbb{D}^n)$ is a space of all holomorphic functions in $\mathbb{D}^n$, $T^n = \{z \in \mathbb{D}^n : |z_j| = 1; j = 1,\dots,n\}$; $\Gamma_\alpha(\xi) = \Gamma_{\alpha_1}(\xi_1)\times\ldots\times\Gamma_{\alpha_n}(\xi_n)$; by $m_n(\xi)$ we denote Lebesgue measure on T^n and by dm_{2n} normalized Lebesgue measure on $\mathbb{D}^n$, $d\sigma(\xi)$ is a Lebesgue measure on S_n.

Let us introduce subspace $M^p_\alpha(\mathbb{D}^n)$ of $H(\mathbb{D}^n)$ space. It is such set of functions $f \in H(\mathbb{D}^n)$ that

$$\int\limits_T \int\limits_{\Gamma_t(\xi)} |f(z)|^p (1-|z_1|)^{\alpha_1} \dots (1-|z_n|)^{\alpha_n} dm_{2n}(z) d\xi < \infty,$$

where $\alpha_j > -1$, $j = 1,\dots,n$, $p \in (0,\infty)$. Besides, $K^{\alpha,\beta}_q(\mathbb{D}^n)$ consists of the functions $f \in H(\mathbb{D}^n)$ such that

$$\int\limits_0^1 \left(\int\limits_{|z_1|<r} \dots \int\limits_{|z_n|<r} |f(z)| \prod_{j=1}^n (1-|z_j|)^{\alpha_j} dm_{2n}(z) \right)^q (1-r)^\beta dr < \infty,$$

$\beta \ge -1$, $\alpha_j > -1$, $j = 1,\dots,n$, $q \in (0,\infty)$, and $T^{\alpha,q}_{\beta,n}$ consists of the functions $f \in H(\mathbb{D}^n)$ such that

$$\int\limits_0^1 \left(\int\limits_{|z|<r} |f(z)|(1-|z|)^{|\alpha|+2n-2} dm_2(z) \right)^q (1-r)^\beta dr < \infty,$$

where $\beta > -1$, $n \in \mathbb{N}$, $|\alpha| = \sum\limits_{j=1}^n \alpha_j$, $q \in (0,\infty)$.

2. Formulation of Main Results

The following results on $T_{2,\alpha}$ operators generalize some known assertions from the unit disc to bidisc. Proofs of these results partially are based on some maximal theorems from [8].

Theorem 1

1) *Let* $p > 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $t \in (-2, -1)$, $\alpha > \max\{t + \frac{2}{q}, 0\}$. *If*

$$\int\limits_T \left(\sup_{z \in \Gamma_\gamma(\xi)} |f(\omega)|(1 - |\omega|)^{\frac{\alpha}{2}} \right)^p dm(\xi) < \infty,$$

then $\sup\limits_{z_1, z_2 \in \mathbb{D}} |T_{2,\alpha}(f)|(1 - |z_2|)^{\frac{t+2}{2}}(1 - |z_1|)^{\frac{\alpha - t}{2} - \frac{1}{q}} < \infty.$

2) *Let* $p > 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $t \in (-2, -1)$, $\alpha > \max\{t + \frac{2}{q}, 0\}$, $p(\frac{t}{2} + 1) > 1$. *If*

$$\sup_{\widetilde{\omega} \in \mathbb{D}} \int\limits_{\mathbb{D}} \frac{(1 - |\widetilde{\omega}|)^N |f(\omega)|^2 (1 - |\omega|)^{\alpha - 1}}{|1 - \widetilde{\omega}\overline{\omega}|^{N+1}} dm_2(\omega), \quad N \in \mathbb{N},$$

then $\sup\limits_{z_1, z_2 \in \mathbb{D}} |T_{2,\alpha}(z_1, z_2)|(1 - |z_2|)^{\frac{t+2}{2} - \frac{1}{p}}(1 - |z_1|)^{\frac{\alpha - t}{2} - \frac{1}{q}} < \infty.$

3) *Let* $p > 1$, $\tau \geq 0$, $\gamma \geq 0$, $\tau - \gamma p > -\frac{1}{2}$, $\alpha > 0$. *If*

$$\int\limits_{\mathbb{D}} |f(z)|^p (1 - |z|)^{2\tau - 2\gamma p} dm_2(z) < \infty, \quad \textit{then}$$

$$\int\limits_{T^2} \sup_{\Gamma_t(\xi)} |D^\gamma T_{2,\alpha}(f)(\omega_1, \omega_2)|^p (1 - |\omega_1|)^\tau (1 - |\omega_2|)^\tau dm_2(\xi_1, \xi_2) < \infty,$$

where $\Gamma_t(\xi) = \Gamma_t(\xi_1) \times \Gamma_t(\xi_2)$, $\xi_i \in T$, $i = 1, 2$.

Remark 1. As we noted in introduction $T_{1,\alpha} f = f$. Putting $n = 1$, $\alpha = 0$ and letting $t \to -2$ in the first statement, we'll get known assertion on Hardy classes (see [3, Chapt.1]).

Putting $n = 1$, and letting $t \to -\frac{2}{q}$ we'll get known assertion for BMOA classes (see[7]) in the unit disc in 2).

Putting $\tau = 0$, $\gamma = 0$ in 3), $n = 1$, $0 < p < \infty$, we'll get well known assertions (see [3]).

Theorem 2. *Let* $n \in \mathbb{N}$, $\alpha_j > -2$, $j = 1, \ldots, n$, $|\alpha| = \sum\limits_{j=1}^n |\alpha_j|$, $p \leq 1$, *then* $T_{n,\widetilde{\alpha}}(f)$ *maps* $A^p_\beta(\mathbb{D}^n)$ *to* $S^p_{\overrightarrow{\alpha}}(\mathbb{D}^n)$, *where* $\beta = \sum\limits_{j=1}^n |\alpha_j| + 2n - 1$, $S^p_{\overrightarrow{\alpha}}(\mathbb{D}^n)$ *is the set of functions* $f \in H(\mathbb{D}^n)$ *such that*

$$\int\limits_T \int\limits_{\Gamma_t(\xi)} \ldots \int\limits_{\Gamma_t(\xi)} |f(\omega)|^p \prod_{j=1}^n (1 - |\omega_j|)^{\alpha_j} dm_2(\omega_1) \ldots dm_2(\omega_n) d\xi < \infty.$$

Remark 2. Note that $S^p_{\overrightarrow{\alpha}}(\mathbb{D}^n)$ spaces are different from classical Bergman spaces in the polydisc. It is easy to see that for $n = 1$ Theorem 2 is an obvious consequence of Bergman integral representation (see [3]).

Theorem 3. *Let $\beta \in (0, \frac{1}{2})$, $\alpha > \beta$, then*

$$\int_T \left(\sup_{z_1 \in \Gamma_\gamma(\xi)} \sup_{z_2 \in \Gamma_\gamma(\xi)} |D^\alpha_{z_2} T_{2,0}(f)(z_1, z_2)|(1 - |z_2|)^{\alpha-\beta}(1 - |z_1|)^\beta \right)^2 dm(\xi) \lesssim$$

$$\lesssim C\|f\|_{H^2(\mathbb{D}^n)}.$$

Remark 3. Put in Theorem 3 $n = 1$ and let $\alpha \to 0$, $\beta \to 0$, then from the inequality of Theorem 3 we have the following well-known estimate (maximal theorem)

$$\int_T \left(\sup_{z \in \Gamma_\gamma(\xi)} |\phi(z)|^2 \right) dm(\xi) \leq C\|\phi\|_{H^2(\mathbb{D}^n)}.$$

Let's formulate some sharp results on diagonal map. Let

$$\wedge^{\alpha,1}(\mathbb{D}^2) = \left\{ f \,\text{measurable in}\, \mathbb{D}^2 : \sup_{z \in \mathbb{D}^2} |f(z)|(1 - |z|)^\alpha < \infty \right\},$$

$$\widetilde{\wedge}^\alpha(\mathbb{D}) = \left\{ \phi \in H(\mathbb{D}) : \sup_{z \in \mathbb{D}} |\phi(z)|(1 - |z|)^{\alpha+1} < \infty \right\}.$$

Let X be a quazinormed subspace of $H(\mathbb{D}^n)$, Y be a subspace of $H(\mathbb{D})$. We will say and write $\text{Diag}X = Y$ if for every $f \in X$ we have $f(z, ..., z) \in Y$, and for every $g \in Y$ there exist a function $f \in X$, such that $\text{Diag}f = f(z, ..., z) = g(z)$.

Theorem 4

1) *Let $n \in \mathbb{N}$, $a_j > -1$, $j = 1, \ldots, n$, $|\alpha| = \sum_{i=1}^n \alpha_i$, $p \leq 1$, then* $\text{Diag}M^p_\alpha(\mathbb{D}^n) = A^p_{|\alpha|+2n-1}(\mathbb{D})$.

2) *Let $\alpha \in (\frac{1}{2}, 1)$, then* $\text{Diag}\, \wedge^{\alpha,1} (\mathbb{D}^2) = \widetilde{\wedge}^\alpha(\mathbb{D})$.

3) *Let $\beta > -1$, $n \in \mathbb{N}$, $\alpha_j > -1$, $j = 1, \ldots, n$, $q \in (0, \infty)$, then*

$$\text{Diag}K^{\alpha,\beta}_q(\mathbb{D}^n) = T^{\alpha,q}_{\beta,n}(\mathbb{D}).$$

Remark 4. All assertions of Theorem 4 are easy to check for $n = 1$ (see [3; 6]).

In the following theorem we collect some estimates for multifunctional analytic spaces.

Theorem 5

1) *Let $\sum_{i=1}^n \frac{q_i}{p_i} = 1$, and let $\mu_1, \ldots \mu_n$ be positive Borel measures on $\mathbb{D}$, then*

$$\int_{\mathbb{D}^n} \prod_{k=1}^n |f_k(z_1, \ldots, z_n)|^{q_k} d\mu_1(z_1) \ldots d\mu_n(z_n) \leq$$

$$\leq C\|f_1\|^{q_1}_{A_\infty H^{p_1}(\mathbb{D}^n)} \cdots \|f_n\|^{q_n}_{A_\infty H^{p_n}(\mathbb{D}^n)},$$

iff

$$\sup_{a\in\mathbb{D}^n}\int\limits_{\mathbb{D}^n}\frac{\prod\limits_{k=1}^{n}(1-|a_k|)d\mu_1(z_1)\dots d\mu_n(z_n)}{\prod\limits_{k=1}^{n}|1-(\overline{a_k},z_k)|^2}<\infty.$$

2) *Let* $n\in\mathbb{N}$, $q_i<p_i$, $i=1,\dots,n$, $\sum\limits_{i=1}^{n}\frac{q_i}{p_i}=\lambda<1$, *then*

$$\int\limits_{\mathbb{D}^n}|f_1(z_1,\dots,z_n)|^{q_1}\dots|f_n(z_1,\dots,z_n)|^{q_n}d\mu_1(z_1)\dots d\mu_n(z_n)\lesssim$$

$$\lesssim\prod_{k=1}^{n}\|f_k\|^{q_k}_{A_\infty H^{p_k}(\mathbb{D}^n)}\int\limits_{T^n}\left(\prod_{k=1}^{n}\int\limits_{\Gamma_{\alpha(\xi_k)}}\frac{d\mu_k(z)}{1-|z|}\right)^{\frac{1}{1-\lambda}}dm_n(\xi).$$

3) *Let* $n\in\mathbb{N}$, $f_i\in L^{p_i}(S_n,d\sigma)$, $1<p_i<q<\infty$, $i=1,\dots,n$,

$$1-\frac{1}{S_i}=\frac{1}{p_i}-\frac{1}{q},\quad b_i=\frac{S_in}{S_i-1},\quad C_i=\frac{p_in}{p_i-1},$$

$$T(f_1,\dots,f_n)(z)=\int\limits_{S^n}\frac{f_1(\xi)\dots f_n(\xi)}{|1-(z,\xi)|^n}d\sigma(\xi),\quad \widetilde{f}=f_1\cdot\ldots\cdot f_n,$$

then

$$\int\limits_{S^n}\left|T(\widetilde{f})\right|^{qn^2}d\sigma(\xi)\lesssim C\left[\prod_{k=1}^{n}\left(\int\limits_{S^n}|\widetilde{f}|^{p_i}d\sigma\right)^{\frac{qn^2}{b_i}}\right]\times$$

$$\times\left[\int\limits_{S^n}\dots\int\limits_{S^n}|\widetilde{f}|^{n\sum\limits_{i=1}^{n}p_i}d\sigma(\xi_1)\dots d\sigma(\xi_n)\right](1-|z|)^{qn^3\sum\limits_{i=1}^{n}\frac{1-S_i}{C_i}-n^2\sum\limits_{j=1}^{n}S_j+n}.$$

Remark 5. All assertions of the last theorem for $n=1$ are known (see [6; 7]). Assertions 1) and 2) in Theorem 5 for $n=1$ were proved in [6] by other purely one-dimentional methods.
Remark 6. The proofs of assertions 1) and 2) in Theorem 5 are based on estimates for tent spaces (see for example [9]) and their complete analogues are true in the unit ball.

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