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On Haar uniqueness properties for vector-valued random processes

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We consider a filtered space (f.s.) $\bigl(\Omega, \mathbb{F} = (\mathscr{F}_k)_{k=0}^N, \mathscr{F}\bigr)$ with N finite, all the σ-algebras $\mathscr{F}_k$ finite, $\mathscr{F}_0 = \{\Omega, \varnothing\}$, $\mathscr{F} = \mathscr{F}_N$, and $\mathscr{F}_k \neq \mathscr{F}_{k+1}$ for every k. Denote by $\mathscr{P}$ the set of probability measures on $(\Omega, \mathscr{F})$ that are non-zero on every atom in $\mathscr{F}$. Let $Z = \bigl(Z_k, \mathscr{F}_k\bigr)_{k=0}^N$ be an adapted vector-valued random process (v.r.p.) on $(\Omega, \mathscr{F})$, where $Z_k = (Z_k^{(1)}, \dots, Z_k^{(l)})$. As usual, $f|_A$ denotes the restriction of a random variable (r.v.) f to an event $A \in \mathscr{F}$. Suppose that

$$A = B_1 + \dots + B_m,$$

where A is an atom in $\mathscr{F}_k$, B_i is an atom in $\mathscr{F}_{k+1}$ $(i = 1, \dots, m)$, and $1 \leqslant m < \infty$ (m depends on A; we will call this number the splitting characteristic of the atom $A \in \mathscr{F}_k$). Denote by $\mathscr{P}(Z, \mathbb{F})$ (by $\mathscr{P}(Z^{(i)}, \mathbb{F})$) the set of measures P in $\mathscr{P}$ with respect to which the v.r.p. $Z = (Z_k, \mathscr{F}_k, \mathsf{P})_{k=0}^N$ $(Z^{(i)} = (Z_k^{(i)}, \mathscr{F}_k^{(i)}, \mathsf{P})_{k=0}^N$, respectively) is a martingale. We have $\mathscr{P}(Z, \mathbb{F}) = \bigcap_{i=1}^l \mathscr{P}(Z^{(i)}, \mathbb{F})$.

A filtration $\mathbb{H} = (\mathscr{H}_n)_{n=0}^L$ on $(\Omega, \mathscr{F})$ is called a Haar filtration if for each n $(0 \leqslant n \leqslant L)$ the σ-algebra $\mathscr{H}_n$ is generated by the partition of Ω into exactly $n+1$ atoms. A Haar filtration $\mathbb{H} = (\mathscr{H}_n)_{n=0}^L$ is called an interpolating Haar filtration (i.H.f.) for the filtration $\mathbb{F}$ if there exists a sequence of positive integers $0 = n_0 < n_1 < \dots < n_N = L$ such that $\mathscr{H}_{n_k} = \mathscr{F}_k$.

Suppose that $\mathscr{P}(Z, \mathbb{F}) \neq \varnothing$ and fix a measure $\mathsf{P} \in \mathscr{P}(Z, \mathbb{F})$. From the martingale $Z = (Z_k, \mathscr{F}_k, \mathsf{P})_{k=0}^N$, we construct a martingale Haar interpolation $Y = (Y_n, \mathscr{H}_n)_{n=0}^L$ using the formula $Y_n = \mathsf{E}^{\mathsf{P}}[Z_N \mid \mathscr{H}_n]$ (a probabilistic solution to the Dirichlet problem). For a fixed i.H.f. $\mathbb{H}$ of the filtration $\mathbb{F}$, the process Y is uniquely determined by the martingale Z.

Let $|M|$ denote the cardinality of a set M. A martingale measure $\mathsf{P} \in \mathscr{P}(Z, \mathbb{F})$ is said to satisfy the Haar uniqueness property ($\mathsf{P} \in \mathrm{HUP}(Z)$) if for the initial filtration $\mathbb{F}$ it is possible to construct an i.H.f. $\mathbb{H}$ such that $|\mathscr{P}(Y, \mathbb{H})| = 1$ for the corresponding martingale interpolation $Y = (Y_n, \mathscr{H}_n)_{n=0}^L$ (in other words, the process Y is a martingale only with respect to the initial measure P). The following definition is, however, more relevant. A martingale measure $\mathsf{P} \in \mathscr{P}(Z, \mathbb{F})$ is said to satisfy the universal Haar uniqueness property ($\mathsf{P} \in \mathrm{UHUP}(Z)$) if $|\mathscr{P}(Y, \mathbb{H})| = 1$ for each i.H.f. $\mathbb{H}$ of the initial filtration $\mathbb{F}$. Here $Y = (Y_n, \mathscr{H}_n)_{n=0}^L$ is the corresponding martingale interpolation of the process Z. The inclusions $\mathsf{P} \in \mathrm{HUP}(Z^{(i)})$ and $\mathsf{P} \in \mathrm{UHUP}(Z^{(i)})$ are defined in like manner. It is easily shown that $\mathscr{P}(Z, \mathbb{F}) \cap \mathrm{HUP}(Z^{(i)}) \subset \mathrm{HUP}(Z)$ and $\mathscr{P}(Z, \mathbb{F}) \cap \mathrm{UHUP}(Z^{(i)}) \subset \mathrm{UHUP}(Z)$ for each i $(0 \leqslant i \leqslant l)$. The results of [1] show that $\mathrm{HUP}(Z^{(i)}) = \mathscr{P}(Z^{(i)}, \mathbb{F})$ $(\mathrm{UHUP}(Z^{(i)}) = \mathscr{P}(Z^{(i)}, \mathbb{F})$, respectively) if $\mathrm{HUP}(Z^{(i)}) \neq \varnothing$ $(\mathrm{UHUP}(Z^{(i)}) \neq \varnothing$ and the splitting characteristic m is $\leqslant 3$ for each k $(0 \leqslant k < N)$ and any atom $A \in \mathscr{F}_k$, respectively). This implies the following lemma.

Lemma 1. *If* $\mathrm{HUP}(Z^{(i)}) \neq \varnothing$ $(\mathrm{UHUP}(Z^{(i)}) \neq \varnothing$ *and, for each* k $(0 \leqslant k < N)$ *and each atom* $A \in \mathscr{F}_k$, *the splitting characteristic* m *does not exceed* 3, *respectively*), *then* $\mathrm{HUP}(Z) = \mathscr{P}(Z, \mathbb{F})$ $(\mathrm{UHUP}(Z) = \mathscr{P}(Z, \mathbb{F})$, *respectively*).

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In fact, we have the following theorem.

Theorem 1. 1) *If* $\mathrm{HUP}(Z) \neq \varnothing$, *then, for each* k $(0 \leqslant k < N)$ *and every atom* $A \in \mathscr{F}_k$ *whose splitting characteristic* m *is strictly greater than* 1, *there exists an index* i *such that*

$$\min_{1 \leqslant j \leqslant m} Z^{(i)}_{k+1}|_{B_j} < Z^{(i)}_k|_A < \max_{1 \leqslant j \leqslant m} Z^{(i)}_{k+1}|_{B_j}. \tag{1}$$

2) $\mathrm{HUP}(Z) = \mathscr{P}(Z, \mathbb{F})$ *if for each* k $(0 \leqslant k < N)$ *and each atom* $A \in \mathscr{F}_k$ *with splitting characteristic* m *strictly greater than* 1 *there exists an index* i *such that* (1) *holds.*

This theorem extends the corresponding one-dimensional result in [2] to the case of vector-valued processes.

Corollary 1. *If* $\mathbb{F}$ *is a natural filtration of the process* Z, *then the condition* (1) *is satisfied. Therefore,* $\mathrm{HUP}(Z) = \mathscr{P}(Z, \mathbb{F})$.

Remark 1. Theorem 1 remains true if $l = \infty$ and also in the case when Ω is countable and $l < \infty$ (in the latter case, we refer the reader to [2] for the definition of interpolating Haar filtration and a discussion of the case $l = 1$).

Theorem 2. *Suppose that for each* k $(0 \leqslant k < N)$ *and each atom* $A \in \mathscr{F}_k$ *the splitting characteristic* m *does not exceed* 3 *and there exists an index* i *such that the numbers* $Z^{(i)}_k|_A, Z^{(i)}_{k+1}|_{B_1}, \dots, Z^{(i)}_{k+1}|_{B_m}$ *are distinct. Then* $\mathrm{UHUP}(Z) = \mathscr{P}(Z, \mathbb{F})$.

Remark 2. The converse of Theorem 2 is not true. Indeed, take $N = 1$ and $l = 2$, and consider a splitting of the set Ω into three atoms ω_1, ω_2, and ω_3. Suppose that $Z^{(1)}_0 = Z^{(2)}_0 = 2$, $Z^{(1)}_1(\omega_1) = 4$, $Z^{(1)}_1(\omega_2) = 2$, $Z^{(1)}_1(\omega_3) = 1$, and $Z^{(2)}_1(\omega_1) = 1$, $Z^{(2)}_1(\omega_2) = 4$, $Z^{(2)}_1(\omega_3) = 2$. In this case it is easy to deduce that $|\mathscr{P}(Z, \mathbb{F})| = 1$, that is, $\mathrm{UHUP}(Z) = \mathscr{P}(Z, \mathbb{F})$. However, the conditions of Theorem 2 fail to hold, both for $i = 1$ and $i = 2$. Thus, the fact that $\mathrm{UHUP}(Z) \neq \varnothing$ does not imply the existence of an index i such that $\mathrm{UHUP}(Z^{(i)}) \neq \varnothing$.

We now formulate a result which provides a construction of a martingale measure in $\mathrm{UHUP}(Z)$ in a case when the conditions of Theorem 2 are violated. Suppose that $\bigl(\Omega^{(i)}, (\mathscr{F}^{(i)}_k)^N_{k=0}, \mathscr{F}^{(i)}\bigr)$ is an f.s. $(i = 1, 2)$, where $N < \infty$, $\mathscr{F}^{(i)}_0 = \{\Omega^{(i)}, \varnothing\}$, and $\mathscr{F}^{(i)} = \mathscr{F}^{(i)}_N$ are finite σ-algebras. Denote by $\mathbb{F}^{(i)}$ the filtration $(\mathscr{F}^{(i)}_k)^N_{k=0}$. Let $\Omega = \Omega^{(1)} \times \Omega^{(2)}$, $\mathscr{F}_k = \mathscr{F}^{(1)}_k \otimes \mathscr{F}^{(2)}_k$, $\mathscr{F} = \mathscr{F}^{(1)} \otimes \mathscr{F}^{(2)}$, and $\mathbb{F} = (\mathscr{F}^{(1)}_k \otimes \mathscr{F}^{(2)}_k)^N_{k=0}$.

Theorem 3. *Suppose that a process* $(Z^{(i)}_k, \mathscr{F}^{(i)}_k)^N_{k=0}$ *admits a unique martingale measure* $\mathsf{P}^{(i)}$ *which is non-zero on any atom of the* σ*-algebra* $\mathscr{F}^{(i)}$ $(i = 1, 2)$. *Then the measure* $\mathsf{P} = \mathsf{P}^{(1)} \otimes \mathsf{P}^{(2)}$ *belongs to the set* $\mathrm{UHUP}(Z)$.

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