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# Combined Stochastic Control and Optimal Stopping, and Application to Numerical Approximation of Combined Stochastic and Impulse Control[1]

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This paper is twofold. The first aim is to study a combined stochastic control and optimal stopping problem: we prove a verification theorem and give a characterization of the value function as a unique viscosity solution to the associated Hamilton–Jacobi–Bellman variational inequality (HJBVI). Although these results have independent interest, they are also motivated by the fact that they are the main ingredients in solving a combined stochastic control and impulse control problem. Indeed, this problem can be reduced to an iterative sequence of combined stochastic control and optimal stopping problems. This method is implemented to solve numerically the quasi-variational inequality (QVI) associated with the problem of portfolio optimization with both fixed and proportional transaction costs. Numerical results are provided.

## 1. INTRODUCTION

The first part of this paper, Section 2, is dedicated to combined stochastic control and optimal stopping. This problem is illustrated on a financial application of optimal consumption and optimal stopping (Subsection 2.1). Such combined problems appear in many applications (see, e.g., [4, 6, 8]). The results of these papers do not apply, however, to the situation that we will consider because there it is assumed that the control does not influence the state and that the profit rate is a bounded function of the control. These conditions are not satisfied in our case.

In this paper, we establish a verification theorem for a combined stochastic control and optimal stopping problem (Subsection 2.2). We then show in Subsection 2.3 that the value function of the problem is characterized as a unique viscosity solution to the associated Hamilton–Jacobi–Bellman variational inequality (HJBVI).

We point out that it has been proved by Pham [10] that the value function of a certain class of combined stochastic control and optimal stopping problems is indeed a unique viscosity solution to the corresponding HJBVI. However, his conditions are not satisfied in our situation.

In the second part of the article, Section 3, we use the results of the first part in order to solve a combined stochastic control and impulse control problem.

We show how a combined stochastic control and *impulse control* problem can be reduced to an iterative sequence of combined stochastic control and *optimal stopping* problems. Our method can be regarded as a nonlinear version of the iterative method developed in [2] for reducing quasi-

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variational inequalities to a sequence of variational inequalities. This method is applied to a portfolio optimization problem with proportional and fixed transaction costs studied in [9]. A similar type of reduction result but in a different setting, with partial observations, discretized time and space, and with a different proof, was obtained in [11].

Subsection 3.1 is devoted to the description of the portfolio optimization problem. In Subsection 3.2, we show that the value function, which is a solution to a quasi-variational inequality, can be obtained as the limit of an iterative procedure, where each step is a combined stochastic control and optimal stopping problem of the type described in Subsection 2.1. Finally, in Subsection 3.3, we provide a numerical method based on this iterative method and the Howard algorithm and give numerical results for the optimal investment strategy.

Note that our aim in this paper is not to prove results in the most general case, but restrict the attention to a simple, yet typical, case in order to make the proofs simpler. It will be clear from the proofs that the results obtained apply to many other cases as well, when suitably modified.

## 2. COMBINED STOCHASTIC CONTROL AND OPTIMAL STOPPING

**2.1. Optimal consumption and stopping.** We consider a market with two possible investment types: a safe investment (e.g., a bond or a bank account) and a risky investment (e.g., a stock). We assume that, if there is no consumption (and no transactions), the wealth $X(t)$ invested in the bank grows exponentially with fixed relative growth rate $r > 0$. If the investor consumes at the rate $c(t) \geq 0$ at time $t \geq 0$ and the consumption is taken from the bank account, the dynamics of $X(t)$ gets the form

$$dX_x^{(c)}(t) = dX(t) = (rX(t) - c(t))dt, \qquad X(0) = x \geq 0. \tag{2.1}$$

We assume that the wealth $Y(t)$ invested in the stock grows like a geometric Brownian motion:

$$dY_y(t) = dY(t) = \alpha Y(t)dt + \sigma Y(t)dW(t), \qquad Y(0) = y \geq 0. \tag{2.2}$$

Here, $\alpha$ and $\sigma$ are constants; $W(t) = W(t,\omega)$, $t \geq 0$, $\omega \in \Omega$, is a Brownian motion (Wiener process) on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$; and the stochastic differential equation (2.2) is interpreted in the Itô sense.

We allow $c(t) = c(t,\omega)$ to depend on $\omega$, but only in an *adapted* way; i.e., we assume that

$$c(t,\cdot) \ \text{ is } \mathcal{F}_t\text{-measurable for all } \ t \geq 0. \tag{2.3}$$

If, in addition, the corresponding process $X^c(t,\omega)$ remains nonnegative for all $t \geq 0$ a.s., then we call $c$ an *admissible* consumption rate and write $c \in \mathcal{A} = \mathcal{A}(x,y)$.

If our investor consumes at the rate $c(t)$ up to a certain $\mathcal{F}_t$-stopping time $\tau(\omega) \leq \infty$, then we assume that the total expected discounted utility obtained from this consumption is given by

$$E\left[\int_0^\tau e^{-\delta t}\frac{c^\gamma(t)}{\gamma}dt\right], \tag{2.4}$$

where $\delta > 0$ and $\gamma \in (0,1)$ are constants ($1-\gamma$ is the relative risk aversion coefficient) and $E$ denotes expectation with respect to $P$. Assume that our investor also has the option of closing down and selling out his business. If he decides to do this at time $t$ and the wealths invested in the bank and the stock at that moment are $\xi \geq 0$ and $\eta \geq 0$, respectively, we assume that the reward/payoff utility he obtains is given by a reward function $g$ of the form

$$g(t,\xi,\eta) = e^{-\delta t}h(\xi,\eta), \tag{2.5}$$

where $h\colon \mathcal{S}_+ \to \mathbb{R}$ is a given continuous, lower bounded function. Here, $\mathcal{S}_+$ is the *solvency region*, defined by

$$\mathcal{S}_+ := [0,\infty) \times [0,\infty), \tag{2.6}$$

corresponding to not allowing any borrowing or shortselling.

Combining (2.4) and (2.5), we see that the total expected discounted utility obtained from an admissible consumption rate $c(t,\omega) \geq 0$ and a sellout time $\tau(\omega)$ is given by

$$J^{c,\tau}(x,y) = E^{x,y}\left[\int\limits_0^\tau e^{-\delta t}\frac{c^\gamma(t)}{\gamma}dt + e^{-\delta\tau}h(X(\tau),Y(\tau))\right], \tag{2.7}$$

where $E^{x,y}$ denotes expectation when $X(0) = x \geq 0$ and $Y(0) = y \geq 0$ and we interpret $h(X(\tau),Y(\tau))$ as 0 if $\tau = \infty$.

We seek the *value function*

$$V(x,y) = \sup_{c,\tau} J^{c,\tau}(x,y), \tag{2.8}$$

where the supremum is taken over all $c,\tau$ with $c \in \mathcal{A}$, and we seek the corresponding optimal combined control $(c^*,\tau^*)$ such that

$$V(x,y) = J^{c^*,\tau^*}(x,y). \tag{2.9}$$

Problem (2.8), (2.9) is an example of a *combined stochastic control and optimal stopping problem.*

**2.2. A verification theorem for the combined control problem.** In this subsection, we state a simple but useful verification theorem for our combined control problem. Basically, the result says that, *if* a given function $v$ is smooth enough and satisfies the appropriate HJBVI, then necessarily $v = V$. Several versions of this result have been applied in the literature. The result given here is an adaptation of a corresponding result for combined stochastic control and *impulse* control given in [3].

Suppose (without loss of generality) that $c(t) = c(t, X(t), Y(t))$ is Markovian. Then, the corresponding time–space process

$$Z^c(t) = (s+t, X^c(t), Y^c(t))$$

is a Markov process with the generator

$$A^c(\varphi(s,x,y) = \frac{\partial\varphi}{\partial s} + (rx - c(s,x,y))\frac{\partial\varphi}{\partial x} + \alpha y\frac{\partial\varphi}{\partial y} + \frac{1}{2}\sigma^2y^2\frac{\partial^2\varphi}{\partial y^2} \tag{2.10}$$

for any $\varphi\colon [0,\infty) \times \mathcal{S}_+ \to \mathbb{R}$ such that the partial derivatives exist. In particular, if $\varphi(s,x,y) = e^{-\delta s}\psi(x,y)$, then

$$(A^c\varphi)(s,x,y) = e^{-\delta s}L^c\psi(x,y),$$

where

$$L^cg(x,y) = -\delta g + (rx - c)\frac{\partial g}{\partial x} + \alpha y\frac{\partial g}{\partial y} + \frac{1}{2}\sigma^2y^2\frac{\partial^2 g}{\partial y^2}. \tag{2.11}$$

Following [3], we call a locally bounded function $h\colon [0,\infty) \times \mathcal{S}_+ \to \mathbb{R}^+$ *stochastically* $C^2$ with respect to $Z^c$ if $(A^ch)(z)$ exists for a.a. $z = (s,x,y)$ with respect to the *Green measure* (expected occupation time measure) $G(z_0,\cdot)$, and the generalized Dynkin formula holds for $h$,

$$E^{z_0}[h(Z^c(\tau))] = h(z_0) + E^{z_0}\left[\int\limits_0^\tau (A^ch)(Z^c(t))dt\right]$$

for all stopping times $\tau$ such that

$$\tau \leq T_R := R \wedge \inf\{t > 0;\ |Z^c(t)| \geq R\} \qquad \text{for some } R < \infty. \tag{2.12}$$

Recall that the Green measure $G(z_0, \cdot)$ of the process $Z^c$ is defined by

$$G(z_0, H) = E^{z_0}\left[\int_0^\infty \mathcal{X}_H(Z^c(t))dt\right] \qquad \text{for all Borel sets } H \subset \widetilde{\mathcal{S}}_+,$$

where $\mathcal{X}_H(z) = 1$ if $z \in H$, $\mathcal{X}_H(z) = 0$ if $z \notin H$, and

$$\widetilde{\mathcal{S}}_+ = [0,\infty) \times \mathcal{S}_+ = [0,\infty) \times [0,\infty) \times [0,\infty). \tag{2.13}$$

If $h\colon \mathcal{S}_+ \to \mathbb{R}$ is a function, we define

$$\mathcal{L}h(x,y) = \sup_{c\ge 0}\left\{L^c h(x,y) + \frac{c^\gamma}{\gamma}\right\}, \qquad (x,y) \in \mathcal{S}_+, \tag{2.14}$$

for all points $(x, y)$ where the partial derivatives of $h$ involved in $L^c h$ exist.

**Theorem 2.1.** (a) *Suppose that we can find a function* $v\colon \mathcal{S}_+ \to [0,\infty)$ *such that* $v \in C^1(\mathcal{S}_+)$,

$$u(s,x,y) := e^{-\delta s} v(x,y) \quad \textit{is stochastically } C^2 \textit{ with respect to } Z^c(t) \\ \textit{for all Markov controls } c = c(s,x,y), \tag{2.15}$$

$$\mathcal{L}v(x,y) \le 0 \qquad \textit{a.e. with respect to } G(z_0,\cdot) \textit{ on } \mathcal{S}_+, \textit{ for all } (x,y) \in \mathcal{S}_+, \tag{2.16}$$

*and*

$$v(x,y) \ge h(x,y) \qquad \textit{for all } (x,y) \in \mathcal{S}_+. \tag{2.17}$$

*Then,*

$$v(x,y) \ge V(x,y) \qquad \textit{for all } (x,y) \in \mathcal{S}_+. \tag{2.18}$$

(b) *Define*

$$D = \{(x,y) \in \mathcal{S}_+;\ v(x,y) > h(x,y)\}. \tag{2.19}$$

*Suppose that*

$$\mathcal{L}v(x,y) = 0 \qquad \textit{in } D. \tag{2.20}$$

*Define*

$$c^*(x,y) = \left(\frac{\partial v}{\partial x}\right)^{\frac{1}{\gamma-1}}.$$

*Suppose that* $c^*$ *is admissible,*

$$e^{-\delta t} v(X^{c^*}(t), Y^{c^*}(t)) \to 0 \qquad \textit{as } t \to \infty, \tag{2.21}$$

*and the family*

$$\left\{e^{-\delta\tau} v(X^{c^*}(\tau), Y^{c^*}(\tau));\ \tau \textit{ stopping time}\right\} \tag{2.22}$$

*is uniformly integrable.*

*Moreover, assume that*

$$\tau_D = \inf\{t > 0;\ (X^{c^*}(t), Y^{c^*}(t)) \notin D\} < \infty \qquad \textit{a.s.} \tag{2.23}$$

*Then,*

$$c^*, \tau_D \quad \textit{is an optimal combined control} \tag{2.24}$$

*and*

$$v(x,y) = V(x,y) \qquad \textit{for all } (x,y) \in \mathcal{S}_+. \tag{2.25}$$

**Proof.** The proof is an adaptation of the proof of Theorem 3.1 in [3] and is omitted.

**Corollary 2.2.** *Suppose that there exists $C < \infty$ such that*

$$|h(x,y)| \le C(x+y)^{\gamma} \qquad \text{for all } (x,y) \in \mathcal{S}_+. \tag{2.26}$$

*Suppose also that*

$$\delta > \gamma\alpha. \tag{2.27}$$

*Then, there exists $K < \infty$ such that*

$$V(x,y) \le K(x+y)^{\gamma}. \tag{2.28}$$

**Proof.** Choose $K$ and define $v(x,y) = K(x+y)^{\gamma}$. We verify the requirements of Theorem 2.1(a): First, since $v$ is $C^2$, it is automatically stochastically $C^2$. Next, we compute

$$\mathcal{L}v(x,y) = (x+y)^{\gamma-2}\Big[\Big(\frac{1-\gamma}{\gamma}(K\gamma)^{\frac{\gamma}{\gamma-1}} - \delta K\Big)(x+y)^2 + K\gamma(rx+\alpha y)(x+y) - \frac{1}{2}\sigma^2 K\gamma(1-\gamma)y^2\Big].$$

Thus, we see that $\mathcal{L}v(x,y) \le 0$ for all $(x,y) \in \mathcal{S}_+$ if and only if

$$\Big[\frac{1-\gamma}{\gamma}(K\gamma)^{\frac{1}{\gamma-1}} - \delta K + K\gamma\alpha\Big](x+y)^2 \le \frac{1}{2}\sigma^2 K\gamma(1-\gamma)y^2$$

for all $(x,y) \in \mathcal{S}_+$, which holds iff

$$\delta > \gamma\alpha + (1-\gamma)(K\gamma)^{\frac{1}{\gamma-1}}. \tag{2.29}$$

We conclude that if (2.27) holds, then (2.29) holds for $K$ large enough. Hence, the conclusion follows from Theorem 2.1(a). □

We proceed to establish some other useful properties of $V$. Recall that a function $F\colon \mathcal{S}_+ \to \mathbb{R}$ is called *Hölder continuous* with exponent $\gamma$ if there exists a constant $C < \infty$ such that

$$|F(x,y) - F(x',y')| \le C(|x-x'|^{\gamma} + |y-y'|^{\gamma})$$

for all $(x,y) \in \mathcal{S}_+$, $(x',y') \in \mathcal{S}_+$.

$F$ is called *locally* Hölder continuous if $C$ is allowed to depend on $(x,y)$ in a locally bounded way.

**Lemma 2.3.** (i) *If $h(x,y)$ is increasing with respect to both $x$ and $y$, then so is $V(x,y)$.*

(ii) *Assume that* (2.27) *holds and that $h(x,y)$ is Hölder continuous with exponent $\gamma$ on $\mathcal{S}_+$. Then, $V(x,y)$ is locally Hölder continuous with exponent $\gamma$ on $\mathcal{S}_+$.*

(iii) *If $h(x,y)$ is concave, then $V(x,y)$ is concave and, hence, continuous.*

(iv) *$V(x,y)$ is always lower semicontinuous.*

**Proof.** (i) If $x \le x'$ and $y \le y'$, then $\mathcal{A}(x,y) \subseteq \mathcal{A}(x',y')$ (see (2.3) and below). Moreover, $X_x^{(c)}(t) \le X_{x'}^{(c)}(t)$ and $Y_y^{(c)}(t) \le Y_{y'}^{(c)}(t)$ for all $t \ge 0$ if $c \in \mathcal{A}(x,y)$. Hence, $V(x,y) \le V(x',y')$.

(ii) Suppose that there exists $C < \infty$ such that

$$|h(x,y) - h(x',y')| \le C(|x-x'|^{\gamma} + |y-y'|^{\gamma})$$

for all $(x,y), (x',y') \in \mathcal{S}_+$.

a) Suppose that $(x,y)\in\mathcal{S}_+$ and $(x',y')\in\mathcal{S}_+$ with $0\le x\le x'$ and $0\le y\le y'$. If $c\in\mathcal{A}(x,y)$, then $c\in\mathcal{A}(x,y')$. For $\varepsilon>0$, choose $c\in\mathcal{A}(x,y)$ and $\tau$ to be $\varepsilon$-optimal for $V(x,y)$. Then,

$$\begin{aligned}
V(x,y) &\le E^{x,y}\left[\int_0^\tau e^{-\delta t}\frac{c^\gamma(t)}{\gamma}dt+e^{-\delta\tau}h(X(\tau),Y(\tau))\right]+\varepsilon\\
&\le V(x',y')+E\Big[e^{-\delta\tau}\Big\{h(X_x(\tau),Y_y(\tau))-h(X_{x'}(\tau),Y_{y'}(\tau))\Big\}\Big]+\varepsilon\\
&\le V(x',y')+C\Big(E\Big[e^{-\delta\tau}|X_x(\tau)-X_{x'}(\tau)|^\gamma\Big]+E\Big[e^{-\delta\tau}|Y_y(\tau)-Y_{y'}(\tau)|^\gamma\Big]\Big)+\varepsilon\\
&\le V(x',y')+C\bigg(E\Big[e^{-\delta\tau}|x'-x|^\gamma e^{r\gamma\tau}\Big]+E\bigg[e^{-\delta\tau}|y'-y|^\gamma\exp\Big(\gamma\sigma W(\tau)+\gamma\Big(\alpha-\frac{1}{2}\sigma^2\Big)\tau\Big)\bigg]\bigg)+\varepsilon\\
&\le V(x',y')+C(|x-x'|^\gamma+|y-y'|^\gamma)+\varepsilon
\end{aligned}$$

by (2.27) and our assumption that $0<r<\alpha$. Since $\varepsilon>0$ was arbitrary, we conclude that

$$V(x,y)\le V(x',y')+C(|x-x'|^\gamma+|y-y'|^\gamma)$$

for all $(x,y)\in\mathcal{S}_+,(x',y')\in\mathcal{S}_+$ with $0\le x\le x'$, $0\le y\le y'$. We proceed to prove the opposite inequality.

b) First, consider $(x,y)\in\mathcal{S}_+$ and $(x,y')\in\mathcal{S}_+$ with $y<y'$. If $c\in\mathcal{A}(x,y)$, then $c\in\mathcal{A}(x,y')$, and vice versa. For all $\tau$, we have

$$\begin{aligned}
V(x,y) &\ge E^{x,y}\left[\int_0^\tau e^{-\delta t}\frac{c^\gamma(t)}{\gamma}dt+e^{-\delta\tau}h(X(\tau),Y(\tau))\right]\\
&= J^{c,\tau}(x,y')+E\Big[e^{-\delta\tau}\Big\{h(X_x(\tau),Y_y(\tau))-h(X_x(\tau),Y_{y'}(\tau))\Big\}\Big]\\
&\ge J^{c,\tau}(x,y')-CE\Big[e^{-\delta\tau}(Y_{y'}(\tau)-Y_y(\tau))^\gamma\Big]\\
&\ge J^{c,\tau}(x,y')-CE\bigg[e^{-\delta\tau}(y'-y)^\gamma\exp\Big(\gamma\sigma W(\tau)+\gamma\Big(\alpha-\frac{1}{2}\sigma^2\Big)\tau\Big)\bigg]\\
&\ge J^{c,\tau}(x,y')-C(y'-y)^\gamma
\end{aligned}$$

by (2.27). Since $c$ and $\tau$ were arbitrary, we conclude that

$$V(x,y)\ge V(x,y')-C(y'-y)^\gamma.$$

c) Next, suppose that $(x,y)\in\mathcal{S}_+,(x',y)\in\mathcal{S}_+$ with $0<x<x'$. Choose $\theta$ such that $x'=xe^{r\theta}$, and choose $c(t)$ such that $c(t)=0$ for $0\le t\le\theta$. Then, by the dynamic programming principle [5, Theorem 9, p. 134], we get

$$V(x,y)\ge E[e^{-r\theta}V(X_x(\theta),Y_y(\theta))]=e^{-r\theta}E[V(x',Y_y(\theta))].$$

Moreover, by parts a) and b) above,

$$|E[V(x',Y_y(\theta))-V(x',y)]|\le CE[|Y_y(\theta)-y|^\gamma]=Cy^\gamma\rho(\theta),$$

where

$$\begin{aligned}
\rho(\theta) &= E\bigg[\bigg(\exp\Big\{\Big(\alpha-\frac{1}{2}\sigma^2\Big)\theta+\sigma W(\theta)\Big\}-1\bigg)^\gamma\bigg]\\
&\le E\bigg[\exp\Big\{\Big(\alpha-\frac{1}{2}\sigma^2\Big)\theta+\sigma W(\theta)\Big\}-1\bigg]^\gamma=(\exp(\alpha\theta)-1)^\gamma\\
&\le C_1\theta^\gamma=C_2\left(\ln\frac{x'}{x}\right)^\gamma\le C_3|x'-x|^\gamma \qquad \text{for } x<x'<x+\beta
\end{aligned}$$

for some constant $C_3$ depending on $x$ and $\beta$ only.

Hence,

$$\begin{aligned}V(x',y)-V(x,y)&\le V(x',y)-E[V(x',Y_y(\theta))]+E[V(x',Y_y(\theta))]-V(x,y)\\&\le CC_3y^\gamma|x'-x|^\gamma+E[V(x',Y_y(\theta))]-e^{r\theta}V(x,y)+e^{r\theta}V(x,y)-V(x,y)\\&\le CC_3y^\gamma|x'-x|^\gamma+e^{r\theta}V(x,y)-V(x,y)\\&=CC_3y^\gamma|x'-x|^\gamma+\frac{V(x,y)}{x}(x'-x).\end{aligned}$$

d) Finally, we prove that $V(x,y)-V(0,y)\le Cx^\gamma$ for all $x,y\ge 0$. To this end, note that

$$X_x^{(c)}(t)=xe^{rt}-e^{rt}\int_0^t e^{-rs}c(s)ds\ge 0$$

and, therefore,

$$\int_0^\tau e^{-\delta t}c(t)^\gamma dt\le\left[\int_0^\tau e^{-rt}c(t)dt\right]^\gamma\left[\int_0^\tau e^{\frac{r\gamma-\delta}{1-\gamma}t}dt\right]^{1-\gamma}\le Cx^\gamma\qquad\text{for all }\tau$$

since $r\gamma-\delta\le 0$. Therefore, since $V(0,y)=\sup_{\tau\ge 0}E^y[h(0,Y(\tau))]$,

$$\begin{aligned}V(x,y)&\le C_1x^\gamma+\sup_\tau E\Big[e^{-\delta\tau}h(X_x(\tau),Y_y(\tau))\Big]\\&\le C_1x^\gamma+\sup_\tau E\Big[e^{-\delta\tau}\{C_2X_x^\gamma(\tau)+h(0,Y_y(\tau))\}\Big]\\&\le C_1x^\gamma+C_2\sup_\tau E[e^{-\delta\tau}X_x^\gamma(\tau)]+V(0,y)\\&\le C_1x^\gamma+C_2x^\gamma\sup_\tau E[\exp((\gamma r-\delta)\tau)]+V(0,y)\\&\le C_3x^\gamma+V(0,y)\end{aligned}$$

since $\gamma r-\delta\le 0$. This completes the proof of (ii).

(iii) To prove that $V$ is concave if $h$ is, choose $(x,y)\in\mathcal{S}_+$ and $(x',y')\in\mathcal{S}_+$ and consider

$$(\overline{x},\overline{y})=\lambda(x,y)+(1-\lambda)(x',y')\qquad\text{for }\lambda\in(0,1).$$

Choose $c\in\mathcal{A}(x,y)$ and $c'\in\mathcal{A}(x',y')$ and define $\overline{c}(t)=\lambda c(t)+(1-\lambda)c'(t)$. Then, it follows from (2.1) and (2.2) that

$$X_{\overline{x}}^{(\overline{c})}(t)=\lambda X_x^{(c)}(t)+(1-\lambda)X_{x'}^{(c')}(t)$$

and

$$Y_{\overline{y}}(t)=\lambda Y_y(t)+(1-\lambda)Y_{y'}(t)\qquad\text{for all }t.$$

Hence, $\overline{c}\in\mathcal{A}(\overline{x},\overline{y})$ and

$$V(\overline{x},\overline{y})\ge\sup_{c,c',\tau}E\left[\int_0^\tau e^{-\delta t}\,\frac{(\overline{c}(t))^\gamma}{\gamma}dt+e^{-\delta\tau}h\Big(X_{\overline{x}}^{(\overline{c})}(\tau),Y_{\overline{y}}^{(\overline{c})}(\tau)\Big)\right]\ge\lambda V(x,y)+(1-\lambda)V(x',y')$$

by the concavity of $c\to c^\gamma$ and $(x,y)\to h(x,y)$. This proves that $V$ is a concave function of two variables and, therefore, is continuous.

(iv) For $m = 1, 2, \dots$, define

$$\mathcal{R}_m = \Big\{F\colon \mathcal{S}_+ \to \mathbb{R};\ |F(x,y)| \le m(1+|x+y|)^m\Big\}$$

and let $\mathcal{A}_m(x,y)$ be the set of Markov controls $c(t) = \widetilde{c}(X(t),Y(t)) \in \mathcal{A}(x,y)$ with $\widetilde{c} \in \mathcal{R}_m$. Define

$$V_m(x,y) = \sup_{\tau, c \in \mathcal{A}_m(x,y)} E^{x,y}\left[\int_0^\tau e^{-\delta t}\,\frac{c^\gamma(t)}{\gamma}dt + h_m(X(\tau),Y(\tau))\right],$$

where

$$h_m(x,y) = \begin{cases} h(x,y) & \text{if } h(x,y) < m, \\ m & \text{if } h(x,y) \ge m. \end{cases}$$

Then, $V_m$ is continuous for all $m$ by [5, Theorem 5, p. 132]. By the monotone convergence, we have

$$V_m(x,y) \uparrow V(x,y) \qquad \text{as } m \to \infty.$$

Hence, $V$ is the limit of an increasing sequence of continuous functions and, therefore, is lower semicontinuous. □

**2.3. Viscosity solutions of variational HJB inequalities.** It is natural to ask if some kind of converse of Theorem 2.1 can be proved: Does $V(x,y)$ always satisfy the HJBVI's of that theorem? One problem is that $V$ need not be smooth enough for $\mathcal{L}V$ to make sense. However, we will prove that, when interpreted in the weak sense of *viscosity*, the value function is indeed a solution to these variational inequalities.

Combining inequalities (2.16), (2.17), and (2.20) and taking the boundary conditions into account, one obtains

$$F(D^2v(\zeta), Dv(\zeta), v(\zeta), h, \zeta) = 0 \qquad \text{for all } \zeta = (x,y) \in \mathcal{S}_+, \tag{2.30}$$

where

$$F(D^2v(\zeta), Dv(\zeta), v(\zeta), h, \zeta) = \begin{cases} \max\{\mathcal{L}v(\zeta), h(\zeta) - v(\zeta)\}, & \zeta \in \mathcal{S}_+ \setminus \ell, \\ \max\{\mathcal{L}_0 v(\zeta), h(\zeta) - v(\zeta)\}, & \zeta \in \ell, \end{cases} \tag{2.31}$$

where $\ell = \{(0,y);\ y \ge 0\}$,

$$\mathcal{L}v(\zeta) = \sup_{c \ge 0}\left\{-\delta v(\zeta) + (rx - c)\frac{\partial v}{\partial x} + \alpha y \frac{\partial v}{\partial y} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 v}{\partial y^2} + \frac{c^\gamma}{\gamma}\right\}, \tag{2.32}$$

and

$$\mathcal{L}_0 v(\zeta) = -\delta v(\zeta) + \alpha y \frac{\partial v}{\partial y}(\zeta) + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 v}{\partial y^2}(\zeta), \qquad \zeta = (x,y). \tag{2.33}$$

Our approach in the following is based on the presentation by Barles [1].

Recall that, if $u$ is a function on $\mathcal{S}_+$, then the upper semicontinuous (usc) envelope $\overline{u}$ and the lower semicontinuous (lsc) envelope $\underline{u}$ are defined, respectively, by

$$\overline{u}(\zeta) = \limsup_{\substack{\eta \to \zeta \\ \eta \in \mathcal{S}_+}} u(\eta) \qquad \text{and} \qquad \underline{u}(\zeta) = \liminf_{\substack{\eta \to \zeta \\ \eta \in \mathcal{S}_+}} u(\eta).$$

Let USC and LSC denote the set of usc and lsc functions on $\mathcal{S}_+$, respectively.

Note that, in general, we have

$$\underline{u} \le u \le \overline{u}$$

and that $u$ is usc iff $u = \overline{u}$ and $u$ is lsc iff $u = \underline{u}$. In particular, $u$ is continuous iff $\underline{u} = u = \overline{u}$.

We now define viscosity solutions to equations of type (2.30):

**Definition 2.4** [1]**.** a) A function $u \in \mathrm{USC}$ is a *viscosity subsolution* to

$$F(D^2u(\zeta), Du(\zeta), u(\zeta), h, \zeta) = 0 \qquad \text{for all } \zeta \in \mathcal{S}_+ \tag{2.34}$$

if, for every function $\varphi$ that is $C^2$ in a neighborhood of $\mathcal{S}_+$ and every $\zeta_0 \in \mathcal{S}_+$ such that $\varphi \ge u$ on $\mathcal{S}_+$ and $\varphi(\zeta_0) = u(\zeta_0)$, we have

$$\overline{F}(D^2\varphi(\zeta_0), D\varphi(\zeta_0), \varphi(\zeta_0), h, \zeta_0) \ge 0. \tag{2.35}$$

b) A function $u \in \mathrm{LSC}$ is a *viscosity supersolution* to (2.34) if, for every function $\varphi$ that is $C^2$ in a neighborhood of $\mathcal{S}_+$ and every $\zeta_0 \in \mathcal{S}_+$ such that $\varphi \le u$ on $\mathcal{S}_+$ and $\varphi(\zeta_0) = u(\zeta_0)$, we have

$$\underline{F}(D^2\varphi(\zeta_0), D\varphi(\zeta_0), \varphi(\zeta_0), h, \zeta_0) \le 0. \tag{2.36}$$

c) We say that a locally bounded function $u$ on $\mathcal{S}_+$ is a *viscosity solution* to (2.34) if $\overline{u}$ is a viscosity subsolution and $\underline{u}$ is a viscosity supersolution to (2.34).

Note that we always have

$$V(\zeta) \ge h(\zeta). \tag{2.37}$$

This is because we can obtain the expected payoff $h(\zeta)$ by stopping immediately.

**Theorem 2.5.** (a) *Suppose that* (2.26) *and* (2.27) *hold. Then, the value function $V$ is lsc on $\mathcal{S}_+$, continuous on $\mathcal{S}_+^0$ (the interior of $\mathcal{S}_+$), and is a unique viscosity solution to* (2.30) *with the property that there exists $C < \infty$ such that*

$$|V(x, y)| \le C(x + y)^\gamma \qquad \textit{for all } (x, y) \in \mathcal{S}_+. \tag{2.38}$$

(b) *If (in addition to* (2.26) *and* (2.27)*) $h$ is Hölder continuous on $\mathcal{S}_+$ with exponent $\gamma$, then $V$ is locally Hölder continuous on $\mathcal{S}_+$ with exponent $\gamma$.*

(c) *If (in addition to* (2.26) *and* (2.27)*) $h$ is concave on $\mathcal{S}_+$, then $V$ is concave on $\mathcal{S}_+$ and, hence, continuous on $\mathcal{S}_+$.*

**Proof.** The proof of (a) follows the proof of Theorem 3.10 in [9]. The proof is simpler here than in [9] because our obstacle function is given. We omit the details. Parts (b) and (c) are (ii) and (iii) of Lemma 2.3. □

## 3. NUMERICAL APPROXIMATION OF COMBINED STOCHASTIC AND IMPULSE CONTROL

**3.1. Portfolio optimization with proportional and fixed transaction costs.** We consider the problem studied in [9]. In this application, the setup is the same as in Subsection 2.1, but now the investor can decide at any time to transfer money from the bank account to the stock and conversely. We assume that a purchase or sale of stocks of amount $\xi$ incurs a transaction cost consisting of a sum of a fixed cost $k > 0$ (independent of the size of the transaction) plus a cost $\lambda|\xi|$ proportional to the transaction ($\lambda \ge 0$). These costs are drawn from the bank account. In this case, there is no final stopping time to consider, only the combination of the regular stochastic control $c(t)$ and an *impulse control* $v = (\tau_1, \tau_2, \dots; \xi_1, \xi_2, \dots)$. Here, $0 \le \tau_1 < \tau_2 < \dots$ are the

$\mathcal{F}_t$-stopping times when the investor decides to change his portfolio, and $\{\xi_j \in \mathbb{R},\ j = 1, 2, \dots\}$ are $\mathcal{F}_{\tau_j}$-measurable random variables giving the sizes of the transactions at these times. We assume that

$$\lim_{j\to\infty} \tau_j = \infty \quad \text{a.s.} \qquad (\text{possibly, } \tau_n = \infty \text{ a.s. for some } n < \infty). \tag{3.1}$$

If such a combined control $w = (c, v)$ is applied to the system $(X(t), Y(t))$, it gets the form $(X^w(t), Y^w(t))$ given by

$$dX^w(t) = (rX^w(t) - c(t))dt, \qquad \tau_i \le t < \tau_{i+1}, \tag{3.2}$$

$$dY^w(t) = \alpha Y^w(t)dt + \sigma Y^w(t)dW(t), \qquad \tau_i \le t < \tau_{i+1}, \tag{3.3}$$

$$X^w(\tau_{i+1}) = X^w(\tau_{i+1}^-) - k - \xi_{i+1} - \lambda|\xi_{i+1}|, \tag{3.4}$$

$$Y^w(\tau_{i+1}) = Y^w(\tau_{i+1}^-) + \xi_{i+1}. \tag{3.5}$$

Thus, $\xi_{i+1} > 0$ means a *purchase* of stocks, while $\xi_{i+1} < 0$ means a *sale* of stocks.

We call the control $w = (c, v)$ *admissible* if, in addition to (2.3) and (3.1) above, $w$ does not cause $(X(t), Y(t)) = (X^w(t), Y^w(t))$ to exit from the solvency region $\mathcal{S}_+$ defined in (2.6). We now define the performance $J^w(x, y)$ by

$$J^w(x, y) = E^{x,y}\left[\int_0^\infty e^{-\delta t}\,\frac{c^\gamma(t)}{\gamma}dt\right] \qquad (\delta > 0 \text{ and } \gamma \in (0, 1) \text{ constants}) \tag{3.6}$$

and seek the *value function* $V(x, y)$ and an *optimal combined control* $w^* = (c^*, v^*) \in \mathcal{W}$ such that

$$V(x, y) = \sup_{w\in\mathcal{W}} J^w(x, y) = J^{w^*}(x, y), \tag{3.7}$$

where $\mathcal{W}$ denotes the set of *admissible* controls, i.e., the set of controls $w = (c, v)$ that do not take $(X(t), Y(t))$ out of $\mathcal{S}_+$.

This problem is studied in [9] using quasi-variational inequalities and their viscosity solutions. (The case where the solvency region has the form

$$\mathcal{S} = \Big\{(x, y) \in \mathbb{R}^2;\ \max\{x + y - \lambda|y| - k, \min\{x, y\}\} \ge 0\Big\}, \tag{3.8}$$

which corresponds to requiring that the *net wealth* of the investor is nonnegative, is also considered in [9].)

**3.2. An iterative method to solve the combined stochastic control/impulse control problem.** From now on, we assume that (2.27) holds.

For $n = 0, 1, 2, \dots$, let $\mathcal{W}_n$ denote the set of all combined controls $w = (c, v) \in \mathcal{W}$, where $v = (\tau_1, \tau_2, \dots, \tau_n, \tau_{n+1}; \xi_1, \xi_2, \dots, \xi_n)$ with $\tau_{n+1} = \infty$ a.s. In other words, $\mathcal{W}_n$ consists of all admissible combined controls *with at most $n$ transactions.*

With $J^w(x, y)$ as in (3.6), we define, analogously to (3.7),

$$V_n(x, y) = \sup_{w\in\mathcal{W}_n} J^w(x, y), \qquad n = 0, 1, 2, \dots. \tag{3.9}$$

Then, $V_n(x, y) \le V_{n+1}(x, y) \le V(x, y)$ because $\mathcal{W}_n \subseteq \mathcal{W}_{n+1} \subseteq \mathcal{W}$ for all $n$.

Moreover, we have

**Lemma 3.1.** $\lim_{n\to\infty} V_n(x, y) = V(x, y)$ *for all* $(x, y) \in \mathcal{S}_+$.

**Proof.** We have already seen that

$$\lim_{n\to\infty} V_n(x,y) \le V(x,y). \tag{3.10}$$

To get the opposite inequality, choose $\varepsilon > 0$ and $w = (c,v) \in \mathcal{W}$ with $v = (\tau_1, \tau_2, \dots; \xi_1, \xi_2, \dots)$ such that

$$E^{x,y}\left[\int_0^\infty e^{-\delta t}\,\frac{c^\gamma(t)}{\gamma}dt\right] = J^w(x,y) \ge V(x,y) - \varepsilon. \tag{3.11}$$

Next, choose $n$ such that

$$E^{x,y}\left[\int_0^{\tau_n} e^{-\delta t}\,\frac{c^\gamma(t)}{\gamma}dt\right] \ge E^{x,y}\left[\int_0^\infty e^{-\delta t}\,\frac{c^\gamma(t)}{\gamma}dt\right] - \varepsilon. \tag{3.12}$$

This is possible because of (3.1).

Then, define $w_n := (c, v_n)$, where $v_n = (\tau_1, \dots, \tau_{n+1}; \xi_1, \dots, \xi_n)$ with $\tau_{n+1} = \infty$ a.s., i.e., $w_n$ is obtained by truncating the $v$ sequence after $n$ steps. Then, $w_n \in \mathcal{W}_n$ and

$$J^{w_n}(x,y) \ge E^{x,y}\left[\int_0^{\tau_n} e^{-\delta t}\,\frac{c^\gamma(t)}{\gamma}dt\right]. \tag{3.13}$$

Combining (3.13) with (3.12) and (3.11), we get the result. □

A crucial concept in connection with impulse control is the following:

**Definition 3.2.** Let $h\colon \mathcal{S}_+ \to \mathbb{R}$ be a measurable function. Then, we define the *intervention operator* $\mathcal{M}$ by

$$\mathcal{M}h(x,y) = \sup\Big\{h(x',y');\ \xi \in \mathbb{R}\setminus\{0\},\ (x',y') \in \mathcal{S}_+\Big\}, \tag{3.14}$$

where

$$x' = x'(\xi) = x - k - \xi - \lambda|\xi| \tag{3.15}$$

and

$$y' = y'(\xi) = y + \xi \tag{3.16}$$

are the new values of the bank wealth $x$ and the stock wealth $y$ after a transaction of size $\xi$.

If $(x',y') \notin \mathcal{S}_+$ for all $\xi \in \mathbb{R}\setminus\{0\}$, we put $\mathcal{M}h(x,y) = 0$.

If, for all $(x,y) \in \mathcal{S}_+$, there exists $(x',y') = (x'(\xi), y'(\xi)) \in \mathcal{S}_+$ such that

$$\mathcal{M}h(x,y) = h(x',y'),$$

then we put

$$\widehat{\xi}(x,y) = \widehat{\xi}_h(x,y) = (x',y'). \tag{3.17}$$

(More precisely, let $\widehat{\xi}(x,y)$ denote a measurable selection of the map $(x,y) \to (x',y')$.)

If $V$ is the value function of problem (3.7), then we can interpret $\mathcal{M}V(x,y)$ as the maximal value we can obtain for $V$ if we make an admissible transaction at $(x,y)$.

We state some useful properties of the operator $\mathcal{M}$.

**Lemma 3.3.** (i) *Suppose that $h\colon \mathcal{S}_+ \to \mathbb{R}$ is continuous. Then, $\mathcal{M}h\colon \mathcal{S}_+ \to \mathbb{R}$ is continuous.*

(ii) *Suppose that $h$ is increasing with respect to both $x$ and $y$. Then, the same holds for $\mathcal{M}h$.*

**Proof.** (i) For $(x,y)\in\mathcal{S}_+$, let

$$\ell(x,y)=\{(x'(\xi),y'(\xi))\in\mathcal{S}_+;\ \xi\in\mathbb{R}\}.$$

Then, $\ell(x,y)$ is a union of two compact line segments; therefore, for each $(x,y)\in\mathcal{S}_+$, there exists $(x^*,y^*)\in\ell(x,y)$ such that

$$\mathcal{M}h(x,y)=h(x^*,y^*).$$

Fix $(x_0,y_0)\in\mathcal{S}_+$. Let $\{(x_n,y_n)\}_{n=1}^{\infty}$ be a sequence in $\mathcal{S}_+$ such that $(x_n,y_n)\to(x_0,y_0)\in\mathcal{S}_+$ as $n\to\infty$. Let $(\overline{x},\overline{y})$ be a cluster point of $\{(x_n^*,y_n^*)\}_{n=1}^{\infty}$. Then, by taking a subsequence, we get

$$\mathcal{M}h(x_0,y_0)\ge h(\overline{x},\overline{y})=\lim_{n\to\infty}h(x_n^*,y_n^*)=\lim_{n\to\infty}\mathcal{M}h(x_n,y_n).$$

On the other hand, if $h(x_0^*,y_0^*)=\mathcal{M}h(x_0,y_0)>\limsup_{n\to\infty}\mathcal{M}h(x_n,y_n)+\varepsilon=\lim_{n\to\infty}h(x_n^*,y_n^*)+\varepsilon$ for some $\varepsilon>0$, then there exists a neighborhood $U$ of $(x_0^*,y_0^*)$ such that the same holds for all $(x',y')\in U$. But if $n$ is large enough, we have $\ell(x_n,y_n)\cap U\ne\varnothing$; however, this contradicts the fact that $(x_n^*,y_n^*)$ is a maximum point for $h$ on $\ell(x_n,y_n)$.

(ii) is clear. □

The iterative procedure is the following.

Define

$$v_0(x,y)=\sup_{c\ge0}E^{x,y}\left[\int_0^{\infty}e^{-\delta t}\,\frac{c^{\gamma}(t)}{\gamma}dt\right]\tag{3.18}$$

and, inductively, for $k=1,2,\dots,n$,

$$v_k(x,y)=\sup_{c,\tau}E^{x,y}\left[\int_0^{\tau}e^{-\delta t}\,\frac{c^{\gamma}(t)}{\gamma}dt+e^{-\delta\tau}\mathcal{M}v_{k-1}(X(\tau),Y(\tau))\right].\tag{3.19}$$

The main result of this section is the following.

**Theorem 3.4.**

$$v_n(x,y)=V_n(x,y)\qquad\textit{for all }(x,y)\in\mathcal{S}_+.\tag{3.20}$$

To prove this, we first introduce some simplifying notation:

For $z=(s,x,y)\in\widetilde{\mathcal{S}}_+:=\mathbb{R}^+\times\mathcal{S}_+$, put

$$Z(t)=(s+t,X(t),Y(t)),\qquad t\ge0.\tag{3.21}$$

Assuming that $c(t)$ is a Markov control, we can write

$$e^{-\delta(s+t)}\,\frac{c^{\gamma}(t)}{\gamma}=f(Z(t))\qquad\text{for a certain function } f.\tag{3.22}$$

Let $g\colon\widetilde{\mathcal{S}}_+\to\mathbb{R}^+$ be a continuous function such that, for $(s,x,y)\in\widetilde{\mathcal{S}}_+$, we have

$$g(s,x,y)\le K[1+(x+y)^m]\qquad\text{for some constant } K<\infty.\tag{3.23}$$

Then, we have the following *dynamic programming principle* (due to Krylov [5, Theorem 9 and Theorem 11, p. 134]).

**Lemma 3.5.** *Define*

$$\varphi(z) = \sup_{c,\tau} E^z \left[ \int_0^\tau f(Z(s))ds + g(Z(\tau)) \right], \qquad z \in \widetilde{\mathcal{S}}_+. \tag{3.24}$$

(a) *Then, for all stopping times* $\beta$, *we have*

$$\varphi(z) = \sup_{c,\tau} E^z \left[ \int_0^{\tau\wedge\beta} f(Z(t))dt + g(Z(\tau))\mathcal{X}_{\tau\le\beta} + \varphi(Z(\beta))\mathcal{X}_{\tau>\beta} \right]. \tag{3.25}$$

(b) *For* $\varepsilon > 0$, *define*

$$D^{(\varepsilon)} = \{z \in \widetilde{\mathcal{S}}_+;\ \varphi(z) > g(z) + \varepsilon\}$$

*and put*

$$\tau^{(\varepsilon)} = \inf\{t > 0;\ Z(t) \notin D^{(\varepsilon)}\}.$$

*Then, if* $\beta$ *is a stopping time such that* $\beta \le \tau^{(\varepsilon)}$ *for some* $\varepsilon > 0$, *we have*

$$\varphi(z) = \sup_{c} E^z \left[ \int_0^\beta f(Z(t))dt + \varphi(Z(\beta)) \right]. \tag{3.26}$$

**Remark 3.6.** In [5], this result is proved under the assumption that $f$ and $g$ have at most polynomial growth, i.e., that there exist constants $C$ and $m$ such that

$$|f(s,x,y)| \le C(1 + |x+y|^m) \tag{3.27}$$

and

$$|g(s,x,y)| \le C(1 + |x+y|^m) \qquad \text{for all } (s,x,y). \tag{3.28}$$

To verify (3.28) in our case ($g = v_k$, $k = 0, 1, \dots, n$), we use a recursive procedure.

First, note that, by Corollary 2.2, we have

$$v_0(x,y) \le V(x,y) \le K_0(x+y)^\gamma \qquad \text{for some constant } K_0 < \infty.$$

Then we see from (3.14)–(3.16) that

$$\mathcal{M}v_0(x,y) \le K(x+y)^\gamma.$$

Hence, again by Corollary 2.2,

$$v_1(x,y) \le K_1(x+y)^\gamma \qquad \text{for some constant } K_1 < \infty.$$

Repeating this $k$ times, we obtain

$$v_k(x,y) \le K_k(x+y)^\gamma \qquad \text{for some constant } K_k < \infty,$$

as requested.

Next, we discuss condition (3.27).

In our situation, this means that the (feedback) controls $c = c(s,x,y)$ should satisfy such a growth condition that we have not assumed. However, Lemma 3.5 still holds. To see this, define

$$\varphi_m(z) = \sup_{\tau, c\in\mathcal{R}_m} E^z\left[\int_0^\tau f(Z(s))ds + g(Z(\tau))\right],$$

where

$$\mathcal{R}_m = \Big\{c = c(x,y);\ \text{there exists } m \text{ with } c(x,y) \le m(1+|x+y|^m)\Big\},$$

$m = 1,2,\dots$ . If $c(x,y) \ge 0$ is admissible, then $c_m(x,y) := c(x,y)\wedge m \in \mathcal{R}_m$. Moreover, if $Z^{(c)}(t)$ and $Z^{(c_m)}(t)$ are the corresponding controlled processes, then

$$Z^{(c_m)}(t) \to Z^{(c)}(t) \qquad \text{as } m\to\infty \quad \text{for a.a. } (t,\omega).$$

Therefore, we see that

$$\varphi_m(z) \uparrow \varphi(z) \qquad \text{as } m\to\infty.$$

Using this, we can deduce that, since (3.25) holds for each $\varphi_m$ (assuming $c\in\mathcal{R}_m$), it will hold for $\varphi$ (with no bound on $c$). Similarly, we see that (3.26) holds without bounds on $c$.

**Corollary 3.7.** (a) *For every given $c$ and $\tau$ with $c\in\mathcal{A}$, the process*

$$U(t) := \int_0^{t\wedge\tau} f(Z(r))dr + \varphi(Z(t\wedge\tau)), \qquad t\ge 0,$$

*is a supermartingale. In particular, if $\tau_1 < \tau_2$ are stopping times, then*

$$E^z[\varphi(Z(\tau_1))] \ge E^z\left[\int_{\tau_1}^{\tau_2} f(Z(t))dt + \varphi(Z(\tau_2))\right]. \tag{3.29}$$

(b) *For $\varepsilon > 0$, let $\tau^{(\varepsilon)}$ be as in Lemma* 3.5(b) *and let $\beta_1 \le \beta_2 \le \tau^{(\varepsilon)}$ be stopping times. Then,*

$$E^z[\varphi(Z(\beta_1))] = \sup_{\substack{c(t),\\ \beta_1\le t\le\beta_2}} E^z\left[\int_{\beta_1}^{\beta_2} f(Z(t))dt + \varphi(Z(\beta_2))\right]. \tag{3.30}$$

**Proof.** (a) Define

$$R(t) = \left(Z(t\wedge\tau), \int_0^{t\wedge\tau} f(Z(r))dr\right) \in \mathbb{R}^4$$

and put

$$H(z,u) = \varphi(z) + u, \qquad z\in\mathbb{R}\times\mathcal{S}_+, \quad u\in\mathbb{R}.$$

Then, if $t > s$, by the Markov property we have

$$E^z[U(t)\,|\,\mathcal{F}_s] = E^{z,0}[H(R(t))\,|\,\mathcal{F}_s] = E^{R(s)}[H(R(t-s))].$$

Put $R(s) = (z,u)$ and $(t-s)\wedge\tau = \beta$. Then, by (3.25),

$$E^{R(s)}[H(R(t-s))] = E^{z,u}[H(R(t-s))] = E^z\left[u + \int_0^\beta f(Z(r))dr + \varphi(Z(\beta))\right]$$
$$\le u + \sup_{\tau\ge\beta} E^z\left[\int_0^{\tau\wedge\beta} f(Z(t))dt + \varphi(Z(\tau\wedge\beta))\right]$$
$$\le u + \varphi(z) = H(R(s)) = U(s).$$

Combined with the above, this yields

$$E^z[U(t)\,|\,\mathcal{F}_s] \le U(s) \qquad \text{for } t > s.$$

Hence, $U(t)$ is a supermartingale. The second statement (3.29) now follows from Doob's optional sampling theorem. This proves (a).

(b) By Lemma 3.5(b), we have

$$\varphi(z) = \sup_c E^z\left[\int_0^{\beta_2} f(Z(t))dt + \varphi(Z(\beta_2))\right]$$
$$= \sup_c \left\{E^z\left[\int_0^{\beta_1} f(Z(t))dt + \varphi(Z(\beta_1))\right] + E^z\left[\int_{\beta_1}^{\beta_2} f(Z(t))dt + \varphi(Z(\beta_2)) - \varphi(Z(\beta_1))\right]\right\}$$
$$= \sup_{\substack{c(t),\\ t\le\beta_1}} \left\{E^z\left[\int_0^{\beta_1} f(Z(t))dt + \varphi(Z(\beta_1))\right]\right\} + \sup_{\substack{c(t),\\ \beta_1\le t\le\beta_2}} \left\{E^z\left[\int_{\beta_1}^{\beta_2} f(Z(t))dt + \varphi(Z(\beta_2)) - \varphi(Z(\beta_1))\right]\right\}$$
$$= \varphi(z) + \sup_{\substack{c(t),\\ \beta_1\le t\le\beta_2}} \left\{E\left[\int_{\beta_1}^{\beta_2} f(Z(t))dt + \varphi(Z(\beta_2))\right]\right\} - E^z[\varphi(Z(\beta_1))],$$

which gives (3.30). □

In the following, it is notationally convenient to introduce

$$u_k(s,x,y) = e^{-\delta s} v_k(x,y), \qquad k = 0,1,\dots,n. \tag{3.31}$$

**Proof of Theorem 3.4.** a) We first prove that

$$v_n(x,y) \ge V_n(x,y) \qquad \text{for all } (x,y)\in\mathcal{S}_+.$$

To this end, choose $w = (c,v) \in \mathcal{W}_n$, where $v = (\tau_1,\tau_2,\dots,\tau_{n+1};\xi_1,\dots,\xi_n)$ with $\tau_{n+1} = \infty$. Let $Z(t) = Z^w(t)$. Then, by Corollary 3.7(a), we have

$$E^z[u_{n-j}(Z(\tau_j))] \ge E^z\left[\int_{\tau_j}^{\tau_{j+1}} f(Z(t))dt + u_{n-j}(Z(\tau_{j+1}^-))\right]. \tag{3.32}$$

By (3.19), we see that

$$u_{n-j} \ge \mathcal{M}u_{n-j-1} \qquad \text{if} \quad n-j\ge 1. \tag{3.33}$$

Moreover, from the definition of $\mathcal{M}$, it is clear that

$$\mathcal{M}u_{n-j}(Z(\tau_{j+1}^-)) \geq u_{n-j}(Z(\tau_{j+1})). \tag{3.34}$$

Combining (3.32)–(3.34), we get

$$E^z[u_{n-j}(Z(\tau_j))] \geq E^z\left[\int_{\tau_j}^{\tau_{j+1}} f(Z(t))dt + u_{n-j-1}(Z(\tau_{j+1}))\right] \tag{3.35}$$

for $j = 0, 1, \ldots, n-1$, where we have put $\tau_0 = 0$. Summing (3.35) from $j = 0$ to $j = n-1$, we get

$$\sum_{j=0}^{n-1} E^z\Big[u_{n-j}(Z(\tau_j)) - u_{n-j-1}(Z(\tau_{j+1}))\Big] \geq E^z\left[\int_0^{\tau_n} f(Z(t))dt\right],$$

or

$$E^z\Big[u_n(Z(0)) - u_0(Z(\tau_n))\Big] \geq E^z\left[\int_0^{\tau_n} f(Z(t))dt\right]. \tag{3.36}$$

Now,

$$E^z[u_n(Z(0))] = u_n(z), \tag{3.37}$$

and, by Corollary 3.7(a), we have

$$E^z[u_0(Z(\tau_n))] \geq E^z\left[\int_{\tau_n}^{\tau_{n+1}} f(Z(t))dt + u_0(Z(\tau_{n+1}))\right]. \tag{3.38}$$

Combining (3.36)–(3.38) and keeping in mind that $\tau_{n+1} = \infty$ and $u_0(Z(\infty)) = 0$, we get

$$u_n(z) \geq E^z\left[\int_0^{\infty} f(Z(t))dt\right] = e^{-\delta s} J^w(x, y). \tag{3.39}$$

Since $w \in \mathcal{W}_n$ was arbitrary, we conclude that

$$v_n(x, y) = u_n(0, x, y) \geq V_n(x, y). \tag{3.40}$$

b) To get the opposite inequality, choose $\varepsilon > 0$ and define an increasing sequence of stopping times $0 = \widehat{\tau}_0 < \widehat{\tau}_1 < \ldots < \widehat{\tau}_n$ as follows: For $k = 1, 2, \ldots, n$, let

$$D_k = D_k^{(\varepsilon)} = \{z;\, u_k(z) > \mathcal{M}u_{k-1}(z) + \varepsilon\}. \tag{3.41}$$

Let $\alpha_n(t)$ be an $\varepsilon$-optimal control for $u_n$ in the sense that (see (3.26))

$$u_n(z) \leq E^z\left[\int_0^{\widehat{\tau}_1} f(Z^{(0)}(t))dt + u_n(Z^{(0)}(\widehat{\tau}_1))\right] + \varepsilon, \tag{3.42}$$

where $Z(t) = Z^{(0)}(t)$ is the corresponding $\alpha_n$-controlled process and

$$\widehat{\tau}_1 = \inf\{t > 0;\, Z^{(0)}(t) \notin D_n^{(\varepsilon)}\}. \tag{3.43}$$

Then, define

$$\widehat{c}_1(t) = \begin{cases} \alpha_n(t), & t \le \widehat{\tau}_1, \\ 0, & t > \widehat{\tau}_1. \end{cases}$$

Next, choose the first transaction $\widehat{\xi}_1 = \widehat{\xi}_1(x,y)$ to be $\varepsilon$-optimal for $u_{n-1}$ in the sense that

$$u_{n-1}(x'(\widehat{\xi}_1), y'(\widehat{\xi}_1)) \ge \mathcal{M}u_{n-1}(x,y) - \varepsilon. \tag{3.44}$$

Inductively, if $\widehat{\tau}_0, \dots, \widehat{\tau}_j, \widehat{\xi}_1, \dots \widehat{\xi}_j$ and $\widehat{c}_j(t)$ have been chosen, where $j \le n-1$, let $Z(t) = Z^{(j)}(t)$ be the process obtained by applying $\widehat{w}_j := (\widehat{c}_j, (\widehat{\tau}_1, \dots, \widehat{\tau}_j; \widehat{\xi}_1, \dots, \widehat{\xi}_j))$. Now, choose $\alpha_{n-j}(t)$ to be an $\varepsilon$-optimal control for $u_{n-j}$ in the sense that (see (3.30))

$$E^z[u_{n-j}(Z(\widehat{\tau}_j))] \le E\left[\int_{\widehat{\tau}_j}^{\widehat{\tau}_{j+1}} f(Z(t))dt + u_{n-j}(Z(\widehat{\tau}_{j+1}))\right] + \varepsilon \tag{3.45}$$

and define

$$\widehat{c}_{j+1}(t) = \begin{cases} \widehat{c}_j(t) & \text{for } t \le \widehat{\tau}_j, \\ \alpha_{n-j}(t) & \text{for } \widehat{\tau}_j < t \le \widehat{\tau}_{j+1}, \\ 0 & \text{for } \widehat{\tau}_{j+1} < t, \end{cases} \tag{3.46}$$

where

$$\widehat{\tau}_{j+1} = \inf\{t > 0;\ Z^{(j)}(t) \notin D^{(\varepsilon)}_{n-j}\}. \tag{3.47}$$

To compute the $j$th step, choose the next transaction $\widehat{\xi}_{j+1} = \widehat{\xi}_{j+1}(x,y)$ to be $\varepsilon$-optimal for $u_{n-j}$ in the sense that

$$u_{n-j}(x'(\widehat{\xi}_{j+1}), y'(\widehat{\xi}_{j+1})) \ge \mathcal{M}u_{n-j}(x,y) - \varepsilon. \tag{3.48}$$

Finally, put $\tau_{n+1} = \infty$ and let $\alpha_0(t)$ be an $\varepsilon$-optimal control for $v_0$ in the sense that (see (3.26))

$$E^z[u_0(Z(\widehat{\tau}_n))] \le E^z\left[\int_{\widehat{\tau}_n}^{\infty} f(Z(t))dt\right] + \varepsilon. \tag{3.49}$$

Put

$$\widehat{c}(t) = \begin{cases} \widehat{c}_n(t) & \text{for } t \le \widehat{\tau}_n, \\ \alpha_0(t) & \text{for } t > \widehat{\tau}_n \end{cases}$$

and define

$$\widehat{w} = (\widehat{c}, (\widehat{\tau}_1, \dots, \widehat{\tau}_n; \widehat{\xi}_1, \dots, \widehat{\xi}_n)) \in \mathcal{W}_n.$$

Now, apply the argument (3.32)–(3.40) to $\widehat{w}$.

By (3.45), we have, with $Z = Z^{\widehat{w}}$,

$$E^z[u_{n-j}(Z(\widehat{\tau}_j))] \le E^z\left[\int_{\widehat{\tau}_j}^{\widehat{\tau}_{j+1}} f(Z(t))dt + u_{n-j}(Z(\widehat{\tau}_{j+1}^-))\right] + \varepsilon. \tag{3.50}$$

Since $Z(\widehat{\tau}_{j+1}^-) \notin D_{n-j}^{(\varepsilon)}$, we have

$$u_{n-j}(Z(\widehat{\tau}_{j+1}^-)) \leq \mathcal{M}u_{n-j-1}(Z(\widehat{\tau}_{j+1}^-)) + \varepsilon \tag{3.51}$$

and, by (3.48), we have

$$\mathcal{M}u_{n-j-1}(Z(\widehat{\tau}_{j+1}^-)) \leq u_{n-j-1}(Z(\widehat{\tau}_{j+1})) + \varepsilon. \tag{3.52}$$

Combining (3.50)–(3.52), we get

$$E^z[u_{n-j}(Z(\widehat{\tau}_j))] \leq E^z\left[\int\limits_{\widehat{\tau}_j}^{\widehat{\tau}_{j+1}} f(Z(t))dt + u_{n-j-1}(Z(\widehat{\tau}_{j+1}))\right] + 3\varepsilon. \tag{3.53}$$

Summing (3.53) from $j = 0$ to $j = n-1$, we get, with $\widehat{\tau}_0 = 0$,

$$u_n(z) \leq E^z\left[\int\limits_0^{\widehat{\tau}_n} f(Z(t))dt + u_0(Z(\widehat{\tau}_n))\right] + 3n\varepsilon. \tag{3.54}$$

Combining (3.49) and (3.54), we get, since $\widehat{\tau}_{n+1} = \infty$,

$$u_n(z) \leq E^z\left[\int\limits_0^{\infty} f(Z(t))dt\right] + (3n+1)\varepsilon.$$

Since $\varepsilon$ was arbitrary, it follows that

$$v_n(x,y) = u_n(0,x,y) \leq V_n(x,y).$$

Combined with (3.39), this proves Theorem 3.4. $\square$

**Corollary 3.8.** *Suppose that $\widetilde{v}_0$ is a viscosity solution to the HJBI*

$$\begin{cases} \mathcal{L}\widetilde{v}_0(\zeta) = 0, & \zeta \in \mathcal{S}_+ \setminus \ell, \\ \mathcal{L}_0\widetilde{v}_0(\zeta) = 0, & \zeta \in \ell, \end{cases} \qquad \textit{for } k = 0 \tag{3.55}$$

*and, for $k = 1, \ldots, n$, suppose that $\widetilde{v}_k$ is a viscosity solution to the HJBVI*

$$\begin{cases} \max\{\mathcal{L}\widetilde{v}_k(\zeta), \mathcal{M}\widetilde{v}_{k-1}(\zeta) - \widetilde{v}_k(\zeta)\} = 0, & \zeta \in \mathcal{S}_+ \setminus \ell, \\ \max\{\mathcal{L}_0\widetilde{v}_k(\zeta), \mathcal{M}\widetilde{v}_{k-1}(\zeta) - \widetilde{v}_k(\zeta)\} = 0, & \zeta \in \ell, \end{cases} \tag{3.56}$$

*where $\mathcal{L}$ and $\mathcal{L}_0$ are as in* (2.32) *and* (2.33), *respectively. Then,*

$$\widetilde{v}_n(x,y) = V_n(x,y) \qquad \textit{for all } (x,y) \in \mathcal{S}_+.$$

**Proof.** By the uniqueness of the viscosity solution to (3.55), (3.56) (Theorem 2.5), we know that $\widetilde{v}_k = v_k$ for $k = 0, 1, \ldots, n$. Therefore, the result follows from Theorem 3.4. $\square$

**3.3. Numerical solution of the quasi-variational inequality.** The value function $V$ of the combined stochastic/impulse control, defined in (3.7), satisfies the quasi-variational inequality:

$$\mathcal{P}:\quad \max\left(\mathcal{L}v, \mathcal{M}v - v\right) = 0 \qquad \text{in } \mathcal{S}_+, \tag{3.57}$$

where

$$\mathcal{L}v = \max_{c\geq 0}\{-\delta v + \mathcal{A}_c v + c^\gamma/\gamma\}$$

and

$$\mathcal{A}_c v = rx\frac{\partial v}{\partial x} + \alpha y\frac{\partial v}{\partial y} + \frac{1}{2}\sigma^2 y^2\frac{\partial^2 v}{\partial y^2} - c\frac{\partial v}{\partial x}.$$

We want to solve numerically this problem. We assume that the consumption $c$ is bounded: $c \leq c_{\max}$. We first localize the problem on $D := [0, L] \times [0, L]$, assuming zero Neumann boundary conditions on the localized boundary:

$$\frac{\partial v}{\partial x}(L, y) = \frac{\partial v}{\partial y}(x, L) = 0 \qquad \text{for all } x, y \in [0, L). \tag{3.58}$$

The localized problem, $(\mathcal{P}_L)$, is solved by using the iterative method presented in Subsection 3.2:

$$\mathcal{L}v_0 = 0 \qquad \text{in } [0, L) \times [0, L),$$

$$\max\left(\mathcal{L}v_{n+1}, \mathcal{M}v_n - v_{n+1}\right) = 0 \qquad \text{for } n \geq 0$$

with the boundary conditions (3.58). The stopping criterion for this iteration is

$$\|v_{n+1} - v_n\|_\infty < \varepsilon_1, \tag{3.59}$$

where $\varepsilon_1$ is some fixed positive number.

3.3.1. *An approximation scheme for $v_0$.* Since the operator $\mathcal{L}$ is degenerate, a finite difference scheme with nonsymmetric first derivative approximation is used to approximate $v_0(x, y)$ on $D$.

Let $D_h = (ih, jh)_{i\in[0,N],j\in[0,N]}$ be a finite difference grid (we suppose here that $N = L/h$ is an integer). We approximate $\mathcal{A}_c$ by the finite difference operator $\mathcal{A}^h_c$ on the grid $D_h$ defined by

$$\mathcal{A}^h_c v = rx\partial^{h+}_x v + \alpha y\partial^{h+}_y v + \frac{1}{2}\sigma^2 y^2\partial^{2,h}_{yy} v - c\partial^{h-}_x v, \tag{3.60}$$

where

$$\partial^{h\pm}_x v(x, y) = \pm\frac{v(x \pm h, y) - v(x, y)}{h},$$

$$\partial^{h\pm}_y v(x, y) = \pm\frac{v(x, y \pm h) - v(x, y)}{h},$$

$$\partial^{2,h}_{yy} v(x, y) = \frac{v(x, y + h) - 2v(x, y) + v(x, y - h)}{h^2}.$$

We obtain the discrete problem

$$\max_{c\in(0,c_{\max})}\left\{-\delta v + \mathcal{A}^h_c v + \frac{c^\gamma}{\gamma}\right\} = 0. \tag{3.61}$$

Note that, in the previous expressions, the value of $v$ at some grid points situated outside $D_h$ are used, but they are eliminated by using discretized Neumann boundary conditions $(v(x+h,y)-v(x-h,y))/h=0$ and $(v(x,y+h)-v(x,y-h))/h=0$. On the axes, we have

$(0,jh)_{j\in[1,n]}$: The only admissible value for $c$ is zero, and the expression of $\mathcal{A}_c^h$ simplifies to $\mathcal{A}_c^h v = \alpha y \partial_y^{h+} v + \frac{1}{2}\sigma^2 y^2 \partial_{yy}^h v$.

$(ih,0)_{i\in[1,n]}$: The expression of $\mathcal{A}_c^h$ simplifies to $\mathcal{A}_c^h v = rx\partial_x^h v - c\partial_x^{h-} v$.

$(0,0)$: $-\delta v + \mathcal{A}_c^h v = 0$.

The discrete operator $\mathcal{A}_c^h$ has a square matrix representation. One can easily check that the nondiagonal elements of $\mathcal{A}_c^h$ take values in the set $\{0, rx/h, \alpha y/h, (\sigma y/h)^2/2, c/h\}$ and are, therefore, positive. Moreover, the diagonal elements take values in the set $\{0, -rx/h - \alpha y/h - (\sigma y/h)^2/2 - c/h\}$, and $\sum_j \mathcal{A}_{ij}^c = 0$. Therefore, $-\delta I + \mathcal{A}_c^h$ is a diagonal dominant operator and $\|(-\delta I + \mathcal{A}_c^h)^{-1}\|_\infty \le K$ (here, $I$ denotes the identity matrix and $K$ is a constant). This implies that the discrete maximum principle is satisfied, which guarantees the convergence of the Howard algorithm (see [7]).

Now, let $\Delta(x,y)$ be a given function only depending on the grid points and such that

$$0 < \Delta(x,y) \le (rx/h + \alpha y/h + (\sigma y/h)^2/2 + c_{\max}/h)^{-1}.$$

Then, the operator $\mathbf{M}_c^h \equiv \Delta(x,y)\mathcal{A}_c^h + I$ has a stochastic matrix representation, and the discrete problem (3.61) can be rewritten as

$$v = \frac{1}{1+\delta\Delta(x,y)} \max_{c\in(0,c_{\max})} \left[\mathbf{M}_c^h v + \Delta(x,y)\frac{c^\gamma}{\gamma}\right]. \tag{3.62}$$

This equation can be interpreted as the Bellman equation of a discrete control problem over an infinite horizon:

$$v(x,y) = \max_{0\le c\le c_{\max}} E^{x,y}\left[\sum_{n=0}^{\infty} \frac{1}{(1+\delta\Delta(X_n,Y_n))^n}\Delta(X_n,Y_n)\frac{c^\gamma}{\gamma}\right].$$

Here, $(X_n, Y_n)$ is a controlled Markov chain defined on the grid $D_h$, with the transition matrix $\mathbf{M}_c^h$ and the initial value $(x,y)$. The running function is bounded, and we can solve the problem numerically by using the Howard algorithm: we construct a sequence $(v^{(n)}, c^{(n)})$ converging to the value function and optimal consumption of the discrete problem.

• Given $v^{(n)}(x,y)$, we compute $c^{(n)}(x,y)$ such that

$$c^{(n)}(x,y) = \arg\max_{0\le c\le c_{\max}} \left[\mathbf{M}_c^h v^{(n)} + \Delta(x,y)\frac{c^\gamma}{\gamma}\right].$$

This leads to the explicit formula

$$c^{(n)}(x,y) = \min(c_{\max}, (dv_n(x,y))^{1/(\gamma-1)})\mathbb{I}_{\{dv^{(n)}(x,y)\ge 0\}} \tag{3.63}$$

with $dv^{(n)}(x,y) = (v^{(n)}(x,y) - v^{(n)}(x-h,y))/h$.

• We compute then $v^{(n+1)}$ as a solution to the linear equation

$$v = \frac{1}{1+\delta\Delta(x,y)}\left[\mathbf{M}_{c^{(n)}}^h v + \Delta(x,y)\frac{c^{(n)\gamma}}{\gamma}\right], \tag{3.64}$$

which is equivalent to the linear equation $\mathcal{L}_{c^{(n)}}^h v + c^{(n)\gamma}/\gamma = 0$. Since the operator $-\delta I + \mathcal{A}_c^h$ is diagonal dominant, the previous linear system is well posed and has a unique solution $v^{(n+1)}$.

3.3.2. *An approximation scheme for the solution $v_n$ of the iterative variational inequalities.* We present here an algorithm to solve, for each $n$, the variational inequality

$$\max(\mathcal{L}v_n, w_{n-1} - v_n) = 0 \qquad \text{in } [0,L) \times [0,L), \tag{3.65}$$
$$\mathcal{L}v_0 = 0 \qquad \text{in } [0,L) \times [0,L),$$

where $w_{n-1} = \mathcal{M}v_{n-1}$, with the boundary condition (3.58).

For fixed $n$, we will merely denote by $v$ the solution of equation (3.65) and write $w$ instead of $w_{n-1}$.

We thus want to solve the variational inequality

$$\max\left(\mathcal{L}v, w - v\right) = 0 \qquad \text{in } [0,L) \times [0,L) \tag{3.66}$$

with the boundary condition (3.58). Using the same finite difference approximation as above, the discrete version of the problem can be stated as follows:

$$\max\left(\max_{0 \le c \le c_{\max}} \left\{-\delta v + \frac{\mathbf{M}_c^h v - v}{\Delta(x,y)} + \frac{c^\gamma}{\gamma}\right\}, w - v\right) = 0. \tag{3.67}$$

In the previous equation, $w - v$ can be changed to $\frac{(1+\delta\Delta(x,y))}{\Delta(x,y)}(w - v)$, and, therefore, equation (3.67) can be rewritten as

$$v = \frac{1}{1+\delta\Delta(x,y)} \max\left(\max_{0 \le c \le c_{\max}} \left\{\mathbf{M}_c^h v + \Delta(x,y)\frac{c^\gamma}{\gamma}\right\}, (1+\delta\Delta(x,y))w\right).$$

Again, we use a Howard algorithm (which will be justified in the next item): the solution of (3.67) is approximated by a sequence $(v^{(k)})$, and the stopping criterion is

$$\|v^{(k+1)} - v^{(k)}\|_\infty < \varepsilon_2, \tag{3.68}$$

where $\varepsilon_2$ is some prescribed precision.

• Given $v^{(k)}(x,y)$, we compute $c^{(k)}(x,y)$ such that

$$c^{(k)}(x,y) = \operatorname*{arg\,max}_{0 \le c \le c_{\max}} \left[\mathbf{M}_c^h v^{(k)} + \Delta(x,y)\frac{c^\gamma}{\gamma}\right].$$

This leads again to the explicit formula (3.63). We compute a partition of $D_h$ such that

$$\mathcal{L}_{c^{(k)}} v^{(k)} + {c^{(k)}}^\gamma/\gamma \ge w - v^{(k)} \qquad \text{in } D_1^k,$$
$$\mathcal{L}_{c^{(k)}} v^{(k)} + {c^{(k)}}^\gamma/\gamma < w - v^{(k)} \qquad \text{in } D_2^k.$$

• The next step is to solve a linear system

$$\mathcal{L}_{c^{(k)}} v + {c^{(k)}}^\gamma/\gamma = 0 \qquad \text{in } D_1^k$$

with $v = w$ in $D_2^k$. This, again, is a well-posed problem that leads to $v^{(k+1)}$.

We thus obtain by this method the solution of the discrete problem, the optimal consumption and the boundary of the continuation set.

3.3.3. *A Howard algorithm for variational inequalities.* In this item, we explain how a variational inequality can be solved by a Howard algorithm by reducing the optimal stopping problem to a control problem. Let $(X_n,\, n \in \mathbb{N})$ be a controlled Markov chain with the transition matrix $M_c$ on a finite state space $\mathcal{E}$. We consider an augmented state space $\overline{\mathcal{E}} = \{\mathcal{E}, \Delta\}$ and build a new controlled transition matrix on $\overline{\mathcal{E}}$ with an additional control $u$ that can only take two values $u \in \{C, S\}$:

$$\overline{M}_{c,u=C} = \begin{pmatrix} M_c & \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \\ 0 & 1 \end{pmatrix}, \qquad \overline{M}_{c,u=S} = \begin{pmatrix} 0 & \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \\ 0 & 1 \end{pmatrix}.$$

In other words, as long as the control $u = C$ is used, the chain behaves as $X_n$, and, when the $S$ control is chosen, the chain jumps to the cemetery point $\Delta$. We want to show that the solution of the combined optimal control and optimal stopping problem

$$v(x) = \sup_{\tau,\,(c_k)_{0 \le k \le \tau}} E^x \left[ \sum_{k=0}^{\tau-1} \frac{1}{(1+\lambda)^{k+1}} \ell(c_k, X_k) + \frac{\psi(X_\tau)}{(1+\lambda)^{\tau+1}} \right] \tag{3.69}$$

coincides on $\mathcal{E}$ with the solution of the infinite horizon optimal control problem

$$v(x) = \max_{(U_k, c_k)_{k \ge 0}} E^x \left[ \sum_{k=0}^{\infty} \frac{1}{(1+\lambda)^{k+1}} \overline{\ell}_{U_k}(c_k, X_k) \right] \tag{3.70}$$

with $U_j \in \{C, S\}$ and

$$\overline{\ell}_u(c,x) \equiv \begin{cases} \psi(x) & \text{for } u = S \text{ and } x \in \mathcal{E}, \\ \ell(c,x) & \text{for } u = C \text{ and } x \in \mathcal{E}, \\ 0 & \text{for } x = \Delta. \end{cases}$$

The Bellman equation for problem (3.70) is

$$v(x) = \frac{1}{1+\lambda} \max_{u,c} \left\{ (\overline{M}_{c,u} v)(x) + \overline{\ell}_u(c,x) \right\} \qquad \text{for all } x \in \overline{\mathcal{E}}. \tag{3.71}$$

First, for $x = \Delta$, we have $v(\Delta) = 1/(1+\lambda) \max_{u,c} v(\Delta)$, which gives $v(\Delta) = 0$. Now, for $x \in \mathcal{E}$, considering the two possible actions $u$, we obtain

$$\begin{aligned} v(x) &= \frac{1}{1+\lambda} \max \left( \max_c M_c v(x) + \ell(c,x), v(\Delta) + \psi(x) \right) \\ &= \frac{1}{1+\lambda} \max \left( \max_c M_c v(x) + \ell(c,x), \psi(x) \right). \end{aligned} \tag{3.72}$$

We thus get the Bellman equation for the value function (3.69). We can apply the Howard algorithm to solve problem (3.69).

3.3.4. *Numerical simulations.* Numerical tests are performed with the following numerical constants: $\sigma = 0.3$, $\gamma = 0.3$, $\alpha = 0.11$, $r = 0.07$, $\delta = 0.1$, $c_{\max} = 100$, $L = 100$, $k = 0.05$, $\lambda = 0.1$, $\varepsilon_1 = 0.01$, and $\varepsilon_2 = 10^{-4}$. We take $N = \frac{L}{h} = 200$, where $h$ is the discretization step.

Note that, with $N = 200$, about two days are necessary to achieve the whole computation with a biprocessor Linux PC (Pentium II 450 MHz with 1 Gb RAM).

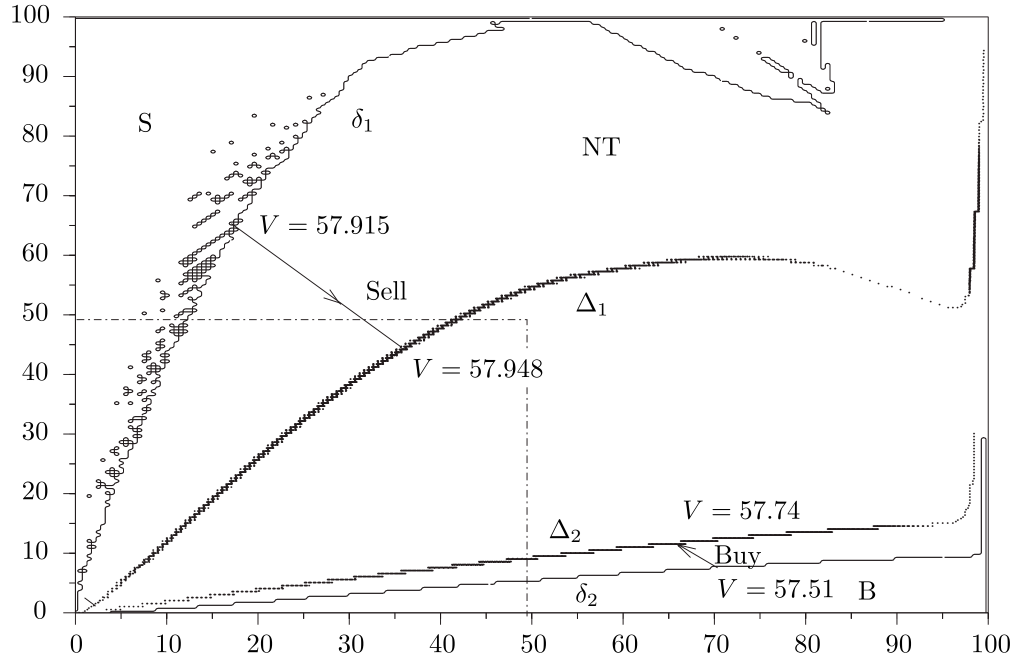


**Fig. 1.** Optimal transaction policy

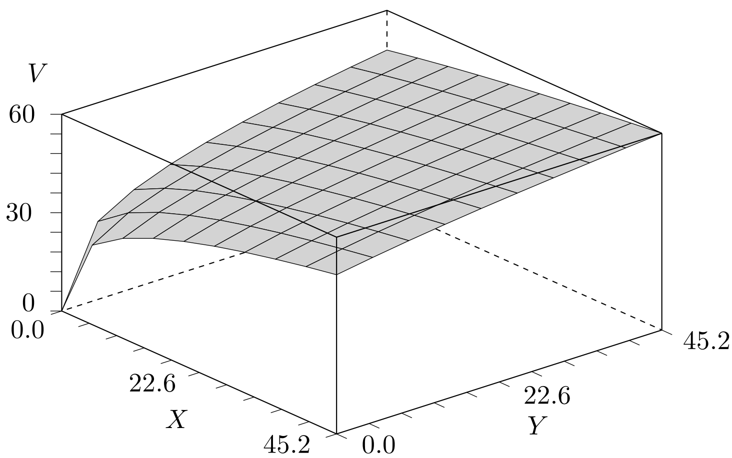


**Fig. 2.** The value function $V(x, y)$

The results are relevant only in a smaller domain, for example, $[0, 50] \times [0, 50]$, because of the side effects of the truncation and the artificial boundary conditions set for $x = 100$ and $y = 100$.

The partition of the domain is displayed in Fig. 1. It consists of three regions: Buy (B), Sell (S), and No Transaction region (NT).

The value function and the optimal consumption are displayed in Figs. 2 and 3.

The set of states $\Delta_1$ and $\Delta_2$ reached after a purchase or a sale of a stock are plotted. They are situated inside the continuation set NT. Unlike the case of no fixed costs, these lines do not coincide with the boundaries $\delta_1$ and $\delta_2$ of NT.

After a transaction, the position of the investor evolves as a pure diffusion process inside NT, until it reaches the boundary. Then, a jump occurs back to the closest of the two lines $\Delta_1$ or $\Delta_2$ in the transaction directions.

We are facing the following numerical difficulties: the slopes of the direction of the jumps are very similar (since $\lambda$ is small) and also very close to the contour lines of the function $V$. This

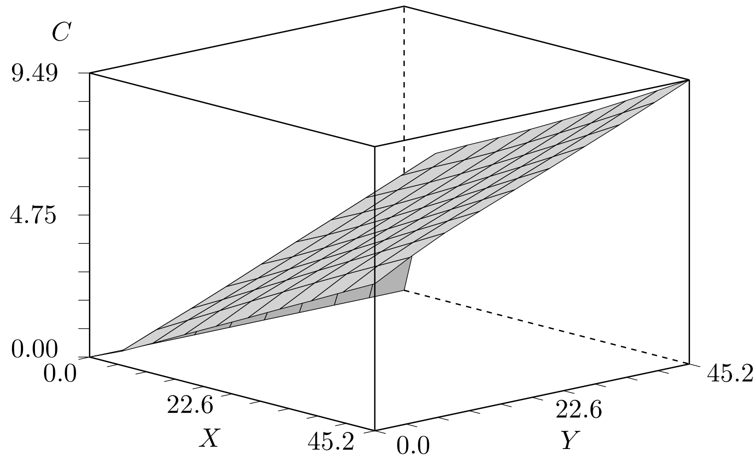


**Fig. 3.** Optimal consumption

means that the intervention operator $\mathcal{M}V$ is quite flat in the neighborhood of the maximum, which implies that the determination of the maximum might not be very accurate.

The effect of the fixed cost is more important when only small size of transaction is involved.

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