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Sevastyanov Branching Processes with Non-homogeneous Poisson Immigration

Kosto V. Mitov[1], **Nickolay M. Yanev**[2]

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Sevastyanov age-dependent branching processes allowing an immigration component are considered in the case when the moments of immigration form a non-homogeneous Poisson process with intensity $r(t)$. The asymptotic behavior of the expectation and of the probability of non-extinction is investigated in the critical case depending on the asymptotic rate of $r(t)$. Corresponding limit theorems are also proved using different types of normalization. Among them we obtained limiting distributions similar to the classical ones of Yaglom (1947) and Sevastyanov (1957) and also discovered new phenomena due to the non-homogeneity.

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1. INTRODUCTION

From the mathematical point of view, the modern period of the theory of branching processes started in fact with the papers of Kolmogorov and Dmitriev [16], Kolmogorov and Sevastyanov [17], and Yaglom [35], where the term "*branching processes*" was also proposed.

Recall that the first model of branching processes with immigration was introduced and investigated by Sevastyanov [29] in the continuous-time Markov case when the moments of immigration form a homogeneous Poisson process. The class of age-dependent branching processes with dependence of the individual lifespan τ and the individual offspring ν was proposed and studied in [30–32]. Note that all these models and results are also completely presented in the monograph [33]. In fact the Sevastyanov process generalizes the age-dependent model of Bellman and Harris [3, 4], where the individual evolution is also described by the vector (τ, ν), but in this case the random variables τ and ν are independent. Bellman–Harris branching processes with finite offspring variance are well described in the books of Harris [8] and Athreya and Ney [2]. Bellman–Harris processes with infinite variance were studied by Vatutin [34].

The Sevastyanov immigration model was considered in the discrete-time case by Heathcote [9, 10]. The discrete-time model was further developed by many authors (see, for example, the books of Athreya and Ney [2], Jagers [13], and Asmussen and Hering [1]). Foster [6] and Pakes [24] considered a discrete-time branching process with immigration only in the state zero. Jagers [12] obtained a limit theorem for a subcritical Bellman–Harris branching process allowing immigration. Further investigation of this model was carried out by Pakes [25], Radcliffe [27], Pakes and Kaplan [26], and Kaplan and Pakes [14]. Limit theorems for Sevastyanov branching processes with homogeneous Poisson immigration were obtained in [39]. On the other hand, Yanev [40] considered Sevastyanov processes allowing immigration at the moments of time-homogeneous renewal process. Bellman–

[1]Faculty of Aviation, Vasil Levski National Military University, Pleven, Bulgaria.
E-mail: kmitov@yahoo.com

[2]Department of Probability and Statistics, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria.
E-mail: yanev@math.bas.bg

Harris branching processes with finite offspring variance and state-dependent immigration were investigated in [21, 22], and the case with infinite offspring variance, in [23].

Galton–Watson branching processes with decreasing state-dependent immigration component were studied in [20], and the similar continuous-time Markov case was considered in [19]. A more complicated case of branching processes with non-homogeneous migration was investigated by Yanev and Mitov [41]. Branching processes with time-decreasing immigration component are also considered in the monograph of Rahimov [28].

Age-dependent branching processes with immigration have been proposed to describe the temporal development of populations of differentiated cells *in vivo* [37]. For a comprehensive review of branching processes and their biological applications, the reader is referred to [13, 36, 15, 7]. Bellman–Harris branching processes with non-homogeneous Poisson immigration are considered by Hyrien and Yanev [11] as models of cell proliferation kinetics.

In this paper we introduce and investigate Sevastyanov age-dependent branching processes allowing immigration of new particles in the jump points of a time-inhomogeneous Poisson process with intensity $r(t)$. Note that these processes are non-Markov and non-homogeneous in time. The limiting results now are presented in the critical case under some conditions on the intensity $r(t)$.

The paper is organized as follows. The considered processes are described in Section 2. The asymptotic behavior of the probabilities of non-extinction is presented in Section 3, and the expectation is considered in Section 4. Corresponding limit theorems are proved in Section 5 using different types of normalization for different intensities $r(t)$.

2. DEFINITIONS AND NOTATIONS

2.1. Sevastyanov branching process. Consider a population of particles (individuals, objects) of a given type which evolve independently of each other. Each particle lives random time τ with a distribution function (d.f.) $G(t) = \Pr\{\tau \le t\}$, $t \ge 0$, and at the end of its life the particle produces n new particles with probability $p_n(u)$ provided that its age equals u. Denote by $f(u,s) = \sum_{n=0}^{\infty} p_n(u)$, $f(u,1) = 1$, the offspring probability generating function (p.g.f.).

Assume that the evolution started at time $t = 0$ with one new particle. The number of particles $\{Z(t),\, t \ge 0\}$ forms the stochastic process, known as the Sevastyanov branching process. It is well known that the p.g.f.

$$F(t,s) = \mathbf{E}\big[s^{Z(t)} \bigm| Z(0) = 1\big], \qquad t \ge 0, \quad s \in [0,1],$$

of the process $Z(t)$, $t \ge 0$, satisfies the following non-linear integral equation:

$$F(t,s) = s(1 - G(t)) + \int_0^t f(u, F(t-u,s))\, dG(u);$$

with the initial condition $F(0,s) = s$ and under mild regularity conditions, it is the only solution of this equation in the class of p.g.f.s.

Denote

$$a(u) = \frac{\partial f(u,s)}{\partial s}\bigg|_{s=1}, \qquad b(u) = \frac{\partial^2 f(u,s)}{\partial s^2}\bigg|_{s=1}, \qquad c(u) = \frac{\partial^3 f(u,s)}{\partial s^3}\bigg|_{s=1},$$

and let

$$a = \int_0^\infty a(u)\, dG(u), \qquad b = \int_0^\infty b(u)\, dG(u), \qquad c = \int_0^\infty c(u)\, dG(u).$$

Definition 2.1. A Sevastyanov branching process is said to be *subcritical*, *critical*, or *supercritical* if $a < 1$, $a = 1$, or $a > 1$, respectively.

2.2. Non-homogeneous Poisson immigration. Let us suppose that along with the Sevastyanov branching process $\{Z(t)\}$ there is a sequence of random vectors (S_k, I_k), $k = 0, 1, 2, \dots$, independent of $\{Z(t)\}$, where $0 = S_0 < S_1 < S_2 < S_3 < \dots$ are the jump points of a non-homogeneous Poisson process $\xi(t)$ independent of $Z(t)$ and $\{I_k\}$ are independent identically distributed non-negative integer-valued random variables.

Let $r(t)$ be the intensity of $\xi(t)$ and $R(t) = \int_0^t r(u)\,du$, which means that $\Pr\{\xi(t) = n\} = e^{-R(t)} R^n(t)/n!$, $n = 0, 1, 2, \dots$.

Assume that at every jump point S_n, a random number I_n of new particles immigrate into the process $\{Z(t)\}$, and they participate in the evolution as the other particles. Denote by $g(s) = \mathbf{E}[s^{I_n}] = \sum_{k=0}^{\infty} q_k s^k$ the p.g.f. of the immigrants.

Then the considered process $\{Y(t),\, t \ge 0\}$ is defined as follows:

$$Y(t) = \sum_{k=1}^{\xi(t)} Z^{I_k}(t - S_k) \quad \text{if } \xi(t) > 0, \qquad Y(t) = 0 \quad \text{if } \xi(t) = 0,$$

where $\{Z^{I_k}(t)\}$ are independent processes which have the same branching mechanism as $Z(t)$, but they started with a random number of ancestors I_k. Note that the Sevastyanov branching process $Z^{I_k}(t - S_k)$ starts with I_k ancestors but at the random moment S_k.

Definition 2.2. The process $\{Y(t),\, t \ge 0\}$ is called the *Sevastyanov branching process with non-homogeneous Poisson immigration.*

Introduce the p.g.f. $\Phi(t; s) = \mathbf{E}[s^{Y(t)} \mid Y(0) = 0]$, $\Phi(0; s) = 1$. The main tool for the investigation of the process $\{Y(t)\}$ is the equation

$$\Phi(t; s) = \exp\left\{ -\int_0^t r(t-u)\big(1 - g(F(u; s))\big)\,du \right\}, \tag{2.1}$$

which can be obtained following the method in the proof of Theorem 2 in [38].

Definition 2.3. The process $\{Y(t)\}$ is called *subcritical*, *critical*, or *supercritical* when the process $Z(t)$ is, i.e. when $a < 1$, $a = 1$, or $a > 1$, respectively.

Note that in the critical case $a = \int_0^\infty a(u)\,dG(u) = 1$, and therefore $G_a(t) = \int_0^t a(u)\,dG(u)$ is a proper distribution function on $[0, \infty)$.

Further on we will consider only the critical case assuming the following local characteristics are finite.

Basic conditions. (i) $a = 1$ and $0 < b < \infty$;

(ii) $m_I = g'(1-) \in (0, \infty)$ and $b_I = g''(1-) \in (0, \infty)$;

(iii) $M = \int_0^\infty u\,dG(u) \in (0, \infty)$ and $M_a = \int_0^\infty u\,dG_a(u) \in (0, \infty)$.

In the next sections we will investigate the asymptotic behavior of the critical processes depending on the intensity $r(t)$ of the Poisson immigration process.

3. PROBABILITY OF NON-EXTINCTION

Since the p.g.f. of the immigrants has finite first and second moments, it is not difficult to find that as $t \to \infty$

$$Q(t) := 1 - g(F(t, 0)) = \frac{m_I + \gamma(t)}{1 + \frac{b}{2M_a} t} \sim \frac{K}{t}, \qquad \text{where} \quad \gamma(t) \to 0 \quad \text{and} \quad K = \frac{2M_a m_I}{b}.$$

Denote $D(t) := \Pr\{Y(t) > 0\} = 1 - \Phi(t; 0) = 1 - e^{-J(t)}$, where

$$J(t) = \int_0^t r(t-u)Q(u)\,du.$$

Remark 3.1. We will use the mean value theorem (MVT) (see, e.g., [18]) for $J(t)$, where continuity is needed. But in our case $r(t)$ and $Q(t)$ are regularly varying functions (r.v.f.s) and it is well known (see, e.g., [5, Sect. 1.3.2]) that "any slowly varying function (s.v.f.) is asymptotic to another with much enhanced properties," especially continuity and differentiability. So, without any restrictions we can consider continuous versions of our r.v.f.s. Similar arguments go in the other cases where we apply the MVT.

Theorem 3.1. *Assume the Basic conditions and let $t \to \infty$.*

(a) *If $r(t) = L_r(t)t^\delta$, $-1 < \delta \le 0$, where in the case $\delta = 0$ the s.v.f. $L_r(t) \downarrow 0$ for large enough t, then $D(t) \sim Kr(t)\log t$.*

(b) *Assume one of the following conditions holds true*:

  (i) $r(t) \to r > 0$,

  (ii) $r(t) = L_r(t)t^\delta$, $\delta \ge 0$, *where in the case $\delta = 0$ the s.v.f. $L_r(t) \to \infty$.*

  Then $D(t) \to 1$.

(c) *If $r(t) = r/(t+1)$, $r > 0$, then $D(t) \sim 2Q(t)R(t) \sim 2Krt^{-1}\log t$.*

(d) *If $R = \int_0^\infty r(u)\,du < \infty$, $r(t)\downarrow$, and there exists a function $k(t)$ such that $k(t) \to \infty$, $k(t)/t \to 0$, and $r(k(t)) = o(1/(t\log t))$, $t \to \infty$, then $D(t) \sim RQ(t) \sim KR/t$.*

Proof. For any $\varepsilon \in (0,1)$ we get by the MVT that

$$J(t) = \int_0^t r(t-u)Q(u)\,du = \int_0^{t\varepsilon} + \int_{t\varepsilon}^t = J_1(t) + J_2(t),$$

$$J_1(t) = r(t_1^*)\int_0^{t\varepsilon} Q(u)\,du \qquad \text{for some} \quad t_1^* \in [t(1-\varepsilon), t],$$

$$J_2(t) = Q(t_2^*)\int_0^{t(1-\varepsilon)} r(u)\,du = Q(t_2^*)R(t(1-\varepsilon)) \qquad \text{for some} \quad t_2^* \in [t\varepsilon, t].$$

(a) For a fixed $\varepsilon \in (0,1)$ and large enough t one has $J_2(t) = O(r(t))$ and $r(t) \le r(t_1^*) \le r(t(1-\varepsilon))$. Therefore, $J_1(t) \sim r(t)K\log t$ and $J_2(t) = o(J_1(t))$. Then $D(t) \sim J(t) \sim Kr(t)\log t$.

(b) Note that in subcase (i) one has $J_1(t) \sim C\log t$ for some $C > 0$ and $J_2(t) \sim Q(t)R(t) = O(1)$. Therefore, $J(t) \to \infty$. The same goes for subcase (ii) because $J_1(t) \sim r(t)C\log t$ for some $C > 0$. Then $D(t) = 1 - e^{-J(t)} \to 1$, which completes the proof.

(c) In this case we have

$$\begin{aligned} J(t) &= r\int_0^t \frac{1}{t+1-u}Q(u)\,du \\ &= rm_I\left\{\int_0^t \frac{1}{t+1-u}\frac{1}{1+ub/(2M_a)}\,du + \int_0^t \frac{1}{t+1-u}\frac{\gamma(u)}{1+ub/(2M_a)}\,du\right\} \\ &= J_1(t) + J_2(t). \end{aligned}$$

Then we obtain

$$J_1(t) = \frac{rm_I}{1+(t+1)b/(2M_a)} \left\{ \int_0^t \frac{du}{u+1} + \int_0^t \frac{b/(2M_a)}{1+ub/(2M_a)}\, du \right\}$$

$$= \frac{rm_I}{1+(t+1)b/(2M_a)} \log\left((t+1)\left(1+\frac{tb}{2M_a}\right)\right) \sim \frac{2rK}{t}\log t.$$

Note that

$$J_2(t) = \int_0^t \frac{1}{t+1-u}\, \frac{\gamma(u)}{1+ub/(2M_a)}\, du = \int_0^{t/2} + \int_{t/2}^{t} = J_{21}(t) + J_{22}(t).$$

By the MVT we get for some $t_1 \in [t/2, t]$

$$J_{21}(t) = \frac{1}{t_1/2+1} \int_0^{t/2} \frac{\gamma(u)}{1+ub/(2M_a)}\, du = o(t^{-1}\log t).$$

Similarly for some $t_2 \in [t/2, t]$

$$J_{22}(t) = \frac{\gamma(t_2)}{1+bt_2/(4M_a)} \int_0^{t/2} \frac{du}{u+1} = o(t^{-1}\log t).$$

Therefore,

$$D(t) = 1 - e^{-J(t)} = J(t)(1+o(1)) = \frac{2rK\log t}{t}(1+o(1)),$$

which completes the proof.

(d) Consider

$$J(t) = \int_0^t r(t-u)Q(u)\, du = \int_0^{t-k(t)} + \int_{t-k(t)}^{t} = J_1(t) + J_2(t).$$

First we get

$$J_1(t) \le r(k(t)) \int_0^{t-k(t)} Q(u)\, du = O\big(r(k(t))\log t\big) = o\left(\frac{1}{t}\right).$$

For any fixed $\varepsilon > 0$ and t large enough we obtain

$$\frac{m_I-\varepsilon}{1+tb/(2M_a)} \int_0^{k(t)} r(u)\, du \le J_2(t) \le \frac{m_I+\varepsilon}{1+[1-k(t)/t]tb/(2M_a)} \int_0^{k(t)} r(u)\, du.$$

Therefore, $D(t) = 1 - e^{-J(t)} \sim J(t) \sim RQ(t)$.

4. MOMENTS

Denote by $M_1(t) = \mathbf{E}[Z(t)]$ and $M_2(t) = \mathbf{E}[Z(t)(Z(t)-1)]$ the moments of the Sevastyanov process $Z(t)$ and respectively for the process with immigration $A(t) = \mathbf{E}[Y(t)]$ and $B(t) = \mathbf{E}[Y(t)(Y(t)-1)]$. Further on we will use the following results for Sevastyanov branching processes.

Definition 4.1. The d.f. F is said to be of *absolutely continuous type* (a.c.t.) if for any k the k-fold convolution F^{k*} has absolutely continuous component.

Theorem 4.1 [33, § VIII.8, теорема 6]**.** *Let $a = 1$, $G(t)$ be non-lattice, $G_a(t)$ be a d.f. of a.c.t., $\int_0^\infty u^2\,dG(u) < \infty$, and $\int_0^\infty u^2\,dG_a(u) < \infty$. Then*

$$M_1(t) = \frac{M}{M_a} + o\left(\frac{1}{t}\right), \qquad t \to \infty. \tag{4.1}$$

Theorem 4.2 [33, § VIII.8, теорема 13]**.** *Let $a = 1$, $G(t)$ be non-lattice, $G_a(t)$ be a d.f. of a.c.t., $\int_0^\infty u^3\,dG(u) < \infty$, $\int_0^\infty u^3\,dG_a(u) < \infty$, $0 < b = \int_0^\infty b(u)\,dG(u) < \infty$, and $\int_0^\infty u^2 b(u)\,dG(u) < \infty$. Then*

$$M_2(t) = \frac{M^2}{M_a^3} bt + B_1 + o(1), \qquad t \to \infty, \tag{4.2}$$

where B_1 is a constant.

Differentiating (2.1) with respect to s and substituting $s = 1$, we get

$$A(t) = \mathbf{E}[Y(t)] = m_I \int_0^t r(t-u) M_1(u)\,du.$$

Theorem 4.3. *Let $M_1(t) = M/M_a + o(1/t)$, $t \to \infty$. Then*

$$A(t) = m_I \frac{M}{M_a} R(t)[1 + o(1)].$$

Proof. Note that $A(t) = m_I M R(t)/M_a + m_I V(t)$, where

$$V(t) = \int_0^t r(t-u)\varphi(u)\,du \qquad \text{and} \qquad \varphi(t) = o\left(\frac{1}{t}\right), \qquad t \to \infty.$$

Since $V(t) = \int_0^{t/2} + \int_{t/2}^t = V_1(t) + V_2(t)$, by the MVT one can obtain

$$V_1(t) = r(t^*) \int_0^{t/2} \varphi(u)\,du \qquad \text{for some} \quad t^* \in \left[\frac{t}{2}, t\right],$$

and therefore $V_1(t) = o(r(t)\log t)$. Similarly for large enough t one has

$$V_2(t) = \int_{t/2}^t r(t-u)\varphi(u)\,du = o\left(\int_{t/2}^t \frac{r(t-u)}{u}\,du\right) = o\left(\frac{R(t)}{t}\right),$$

because $t^{-1}R(t/2) \le \int_{t/2}^t (r(t-u)/u)\,du \le 2t^{-1}R(t/2)$, which completes the proof.

Corollary 4.1. *In fact, $A(t) = m_I M R(t)[1 + \psi(t)]/M_a$, where $\psi(t) = o(1/t)$ if $r(t) = r/(t+1)$ and $\psi(t) = o(t^{-1}\log t)$ if $R(t) \to R < \infty$ or $r(t) = L(t)t^\delta$, $\delta > -1$, and it is additionally assumed that $L(t) \equiv r > 0$ for $\delta = 0$.*

5. LIMIT THEOREMS

Further on we will use the following result proved by Sevastyanov.

Theorem 5.1 [33, § IX.2, теорема 1]. *Let* $a = 1$, $0 < b < \infty$, $c < \infty$, $\int_0^\infty u^3 dG(u) < \infty$, $\int_0^\infty u^3 c(u)\,dG(u) < \infty$, *and the factorial moments* $M_1(t)$ *and* $M_2(t)$ *satisfy* (4.1) *and* (4.2). *Then for* $s \in [0, 1)$ *and* $t \to \infty$

$$1 - F(t; s) = \frac{M_1(t)(1-s)}{1 + M_2(t)(1-s)/(2M_1(t))}\biggl(1 + O\biggl(\frac{1}{\log(t + 2/(1-s))}\biggr)\biggr).$$

Corollary 5.1. *Assume that the function* $h(t)$ *is positive and* $h(t) \to \infty$ *as* $t \to \infty$. *Let* $Q(u; s) := 1 - g(F(u; s))$, $\lambda > 0$ *be fixed, and* $u, t \to \infty$. *Then under the conditions of the theorem*

$$Q\bigl(u; e^{-\lambda/h(t)}\bigr) = \frac{m_I M\lambda/(M_a h(t))}{1 + Mb\lambda u/(2M_a^2 h(t))}\biggl(1 + O\biggl(\frac{1}{\log h(t)}\biggr)\biggr).$$

Since the first two moments of the immigration are finite, the proof follows from (4.1), (4.2) and the relation $1 - e^{-\lambda/h(t)} \sim \lambda/h(t)$, $t \to \infty$.

Remark 5.1. We will denote $D(t; s) := 1 - \Phi(t; s) = 1 - \exp(-J(t, s))$, where

$$J(t, s) = \int_0^t r(t-u)Q(u; s)\,du.$$

Remark 5.2. Further on we will always assume that the conditions of Theorem 5.1 hold true.

Theorem 5.2. *If* $R = \int_0^\infty r(x)\,dx < \infty$, $r(t) \downarrow 0$, *and there exists a function* $k(t)$ *such that* $k(t) \to \infty$, $k(t)/t \to 0$, *and* $r(k(t)) = o(1/(t\log t))$, $t \to \infty$, *then*

$$\lim_{t\to\infty} \Pr\bigl\{Y(t)D(t) \le x \bigm| Y(t) > 0\bigr\} = 1 - e^{-(m_I MR/M_a)x}, \qquad x \ge 0.$$

Proof. We have

$$J\bigl(t, e^{-\lambda D(t)}\bigr) = \left(\int_0^{t-k(t)} + \int_{t-k(t)}^t\right) r(t-u)Q\bigl(u; e^{-\lambda D(t)}\bigr)\,du = J_1(t, \lambda) + J_2(t, \lambda).$$

Since $Q(t; s)$ is non-increasing in s, we find from Theorem 3.1(d) that

$$J_1(t, \lambda) \le \int_0^{t-k(t)} r(t-u)Q(u)\,du = o\biggl(\frac{1}{t}\biggr), \qquad t \to \infty.$$

Let $\varepsilon > 0$ be fixed. From Corollary 5.1 with $h(t) = 1/D(t)$ it follows that there exists $T > 0$ such that for $t - k(t) > T$

$$\frac{m_I M\lambda D(t)(1-\varepsilon)/M_a}{1 + Mb\lambda t D(t)(1+\varepsilon)/(2M_a^2)} \int_0^{k(t)} r(u)\,du \le J_2(t, \lambda)$$

$$\le \frac{m_I M\lambda D(t)(1+\varepsilon)/M_a}{1 + Mb\lambda(t - k(t))D(t)(1-\varepsilon)/(2M_a^2)} \int_0^{k(t)} r(u)\,du.$$

Therefore,

$$D\big(t; e^{-\lambda D(t)}\big) \sim J\big(t, e^{-\lambda D(t)}\big) \sim D(t)\frac{Rm_I M\lambda/M_a}{1 + Rm_I M\lambda/M_a}, \qquad t \to \infty.$$

Since

$$\mathbf{E}\big[e^{-\lambda Y(t)D(t)} \mid Y(t) > 0\big] = 1 - \frac{D(t; e^{-\lambda D(t)})}{D(t)},$$

by the continuity theorem for Laplace transforms the theorem is proved.

Comment 5.1. The limiting distribution of the theorem corresponds to that obtained by Yaglom [35].

Theorem 5.3. *Let $r(t) = L_r(t)t^\delta$, $-1 < \delta \le 0$, where in the case $\delta = 0$ the s.v.f. $L_r(t) = o(1/\log t)$ as $t \to \infty$. Then*

$$\lim_{t\to\infty} \Pr\left\{ \frac{\log Y(t)}{\log t} \le x \;\Big|\; Y(t) > 0 \right\} = x, \qquad x \in [0, 1].$$

Proof. First we have for $0 < \varepsilon < 1$

$$J\big(t, e^{-\lambda t^{-x}}\big) = \int_0^t r(t-u)Q\big(u; e^{-\lambda t^{-x}}\big)\, du$$

$$= \int_0^{t\varepsilon} r(t-u)Q\big(u; e^{-\lambda t^{-x}}\big)\, du + \int_{t\varepsilon}^{t} r(t-u)Q\big(u; e^{-\lambda t^{-x}}\big)\, du = J_1(t,\lambda) + J_2(t,\lambda).$$

By the MVT we obtain for some $t_1 \in [t(1-\varepsilon), t]$ and $t_2 \in [t\varepsilon, t]$

$$J_1(t,\lambda) = r(t_1)\int_0^{t\varepsilon} Q\big(u; e^{-\lambda t^{-x}}\big)\, du,$$

$$J_2(t,\lambda) = Q\big(t_2; e^{-\lambda t^{-x}}\big) \int_0^{t(1-\varepsilon)} r(u)\, du = Q\big(t_2; e^{-\lambda t^{-x}}\big) R(t(1-\varepsilon)).$$

Since $Q(t; e^{-\lambda t^{-x}}) \sim 2M_a m_I/(bt)$ as $t \to \infty$ and $R(t) = tr(t)/(1+\delta)$, it then follows that $J_1(t,\lambda) = O(r(t)\log t)$ and $J_2(t,\lambda) \sim r(t)2M_a m_I/(b(1+\delta))$. Hence $J_2(t,\lambda) = o(J_1(t,\lambda))$ and $J(t, e^{-\lambda t^{-x}}) \sim J_1(t,\lambda)$, $t \to \infty$. On the other hand, for $0 < T < t\varepsilon$

$$I(t,\lambda) = \int_0^{t\varepsilon} Q\big(u; e^{-\lambda t^{-x}}\big)\, du = \int_0^T + \int_T^{t\varepsilon} = I_1(t,\lambda) + I_2(t,\lambda).$$

For large enough T, using Corollary 5.1, one obtains

$$I_2(t,\lambda) \sim \int_T^{t\varepsilon} \frac{\lambda m_I M t^{-x}/M_a}{1 + \lambda M b u t^{-x}/(2M_a^2)}\, du = \frac{2M_a m_I}{b} \log \frac{1 + \lambda M b\varepsilon t^{1-x}/(2M_a^2)}{1 + \lambda M b T t^{-x}/(2M_a^2)}$$

$$\sim \frac{2M_a m_I}{b}(1-x)\log t, \qquad t \to \infty.$$

Note that $Q(u; e^{-\lambda t^{-x}}) \le m_I M_1(u)(1 - e^{-\lambda t^{-x}}) \le Ct^{-x}$ for some constant $0 < C < \infty$. Hence for every fixed T we have $I_1(t,\lambda) \le CTt^{-x} \to 0$, $t \to \infty$.

Thus we finally obtain $J(t, e^{-\lambda t^{-x}}) \sim (2M_a m_I/b)(1-x)r(t)\log t$. Using Theorem 3.1(a), one has (as in the previous theorem)

$$\lim_{t\to\infty} \frac{D(t; e^{-\lambda D(t)})}{D(t)} = 1 - x.$$

Therefore,

$$\lim_{t\to\infty} \mathbf{E}\big[e^{-\lambda Y(t)t^{-x}} \;\big|\; Y(t) > 0\big] = x, \qquad x \in [0,1].$$

Then by the continuity theorem for Laplace transforms one obtains

$$\lim_{t\to\infty} \Pr\big\{Y(t)t^{-x} \le u \;\big|\; Y(t) > 0\big\} = x$$

for $0 \le x \le 1$ and $u > 0$. Hence

$$\lim_{t\to\infty} \Pr\bigg\{\frac{\log Y(t)}{\log t} \le x + \frac{\log u}{\log t} \;\bigg|\; Y(t) > 0\bigg\} = x,$$

which is equivalent to the assertion.

Comment 5.2. Similar limiting distributions were obtained in [19, 20, 22, 41] for different models of non-homogeneous branching processes.

Theorem 5.4. *If* $\lim_{t\to\infty} r(t) = r$, $0 < r < \infty$, *then*

$$\lim_{t\to\infty} \Pr\bigg\{\frac{Y(t)}{rt} \le x\bigg\} = \frac{1}{\theta^\rho \Gamma(\rho)} \int_0^x u^{\rho-1} e^{-u/\theta}\, du, \qquad x \ge 0,$$

where $\theta = Mb/(2M_a^2 r)$ *and* $\rho = 2m_I M_a/b$.

Proof. For every $0 < \varepsilon < 1$ there exists $0 < T \to \infty$ such that $T \le t(1-\varepsilon)$ and $T/t \to 0$ as $t \to \infty$. Consider

$$J\big(t, e^{-\lambda/(rt)}\big) = \left(\int_0^T + \int_T^{t(1-\varepsilon)} + \int_{t(1-\varepsilon)}^t\right) r(t-u) Q\big(u; e^{-\lambda/(rt)}\big)\, du = J_1(t,\lambda) + J_2(t,\lambda) + J_3(t,\lambda).$$

For some constant $0 < C < \infty$ one has

$$J_1(t,\lambda) \le C \int_0^T Q\big(u; e^{-\lambda/(rt)}\big)\, du.$$

Since $Q(u; e^{-\lambda t^{-x}}) \le m_I M_1(u)(1 - e^{-\lambda/(rt)}) \le C_1/t$ for some constant $0 < C_1 < \infty$, it follows that $J_1(t,\lambda) = O(T/t)$. For large enough t one has

$$(r-\varepsilon) \int_T^{t(1-\varepsilon)} Q\big(u; e^{-\lambda/(rt)}\big)\, du \le J_2(t,\lambda) \le (r+\varepsilon) \int_T^t Q\big(u; e^{-\lambda/(rt)}\big)\, du.$$

By Corollary 5.1 for large enough T one obtains

$$\int_T^{t(1-\varepsilon)} Q\big(u; e^{-\lambda/(rt)}\big)\, du \sim \int_T^{t(1-\varepsilon)} \frac{\lambda m_I M/(M_a rt)}{1 + \lambda Mbu/(2M_a^2 rt)}\, du$$

$$= \frac{2m_I M_a}{b} \bigg\{\log\bigg[1 + \lambda \frac{Mb}{2rM_a^2}(1-\varepsilon)\bigg] - \log\bigg[1 + \lambda \frac{Mb}{2rM_a^2} \frac{T}{t}\bigg]\bigg\}.$$

Now for some $0 < C < \infty$ one has

$$J_3(t,\lambda) \le C \int\limits_{t(1-\varepsilon)}^{t} Q\big(u; e^{-\lambda/(rt)}\big)\,du \sim C \int\limits_{t(1-\varepsilon)}^{t} \frac{\lambda m_I M/(M_a rt)}{1+\lambda Mbu/(2M_a^2 rt)}\,du$$

$$= C m_I \frac{M}{M_a}\left\{\log\left[1+\lambda\frac{Mb}{2rM_a^2}\right] - \log\left[1+\lambda\frac{Mb}{2rM_a^2}(1-\varepsilon)\right]\right\}.$$

Since $J_1(t,\lambda) \to 0$ and $J_3(t,\lambda) \to 0$, we prove that

$$\lim_{t\to\infty} J\big(t, e^{-\lambda/R(t)}\big) = \frac{2M_a m_I}{b}\log\left[1+\frac{\lambda Mb}{2M_a^2 r}\right].$$

Therefore,

$$\lim_{t\to\infty} \mathbf{E}\big[e^{-\lambda Y(t)/(rt)}\big] = \lim_{t\to\infty} e^{-J(t,\exp(-\lambda/(rt)))} = \left(1+\frac{\lambda Mb}{2M_a^2 r}\right)^{-2m_I M_a/b}.$$

Now by the continuity theorem for Laplace transforms the limiting distribution follows.

Comment 5.3. The limiting distribution corresponds to that obtained by Sevastyanov [29] in the Markov case and by Yanev [39] for Sevastyanov branching processes with homogeneous Poisson immigration.

Theorem 5.5. *If $r(t) = L(t)t^\delta$, $\delta > 0$, where $L(t)$ is an s.v.f. as $t\to\infty$, then*

$$\lim_{t\to\infty} \frac{Y(t)}{A(t)} = 1 \qquad \textit{in probability.}$$

Proof. Assuming $T/t \to 0$ and $\lambda > 0$, we have

$$J\big(t, e^{-\lambda/A(t)}\big) = \left(\int\limits_0^T + \int\limits_T^{t-T} + \int\limits_{t-T}^{t}\right) r(t-u)Q\big(u; e^{-\lambda/A(t)}\big)\,du = J_1(t,\lambda) + J_2(t,\lambda) + J_3(t,\lambda).$$

By the MVT we get for some $t_1 \in [t-T, t]$

$$J_1(t,\lambda) = r(t_1)\int\limits_0^T Q\big(u; e^{-\lambda/A(t)}\big)\,du.$$

Since $Q(u; e^{-\lambda t^{-x}}) \le m_I M_1(u)(1-e^{-\lambda/A(t)}) \le C/A(t)$ for some $0 < C < \infty$, it follows that $J_1(t,\lambda) = O(T/t)$. If $\overline{r}(T) = \sup_{0\le x\le T} r(x)$, then as in the previous case one obtains

$$J_3(t,\lambda) \le \overline{r}(T)\int\limits_{t-T}^{t} Q\big(u; e^{-\lambda/A(t)}\big)\,du \le \frac{\overline{r}(T)TC}{A(t)} = O\left(\frac{T}{t}\right).$$

Applying Corollary 5.1 with $h(t) = A(t) \sim m_I M r(t)t/(M_a(\delta+1))$, one can prove that

$$J_2(t,\lambda) = \int\limits_T^{t-T} r(t-u)Q\big(u; e^{-\lambda/A(t)}\big)\,du \sim \int\limits_T^{t-T} r(t-u)\frac{\lambda(\delta+1)}{1+\lambda(\delta+1)(b/(2m_I M_a))u/(r(t)t)}\frac{1}{r(t)t}\,du$$

$$= \frac{2m_I M_a}{b}\int\limits_T^{t-T} r(t-u)\,d_u \log\left\{1+\lambda(\delta+1)\frac{b}{2m_I M_a}\frac{u}{r(t)t}\right\} = J_{21}(t,\lambda) + J_{22}(t,\lambda),$$

where

$$J_{21}(t,\lambda) = \frac{2m_I M_a}{b}\left\{r(T)\log\left[1+\lambda(\delta+1)\frac{b}{2m_I M_a}\frac{t-T}{r(t)t}\right]\right.$$
$$\left.- r(t-T)\log\left[1+\lambda(\delta+1)\frac{b}{2m_I M_a}\frac{T}{r(t)t}\right]\right\}$$
$$\sim \frac{2m_I M_a}{b}\left\{r(T)\lambda(\delta+1)\frac{b}{2m_I M_a}\frac{t-T}{r(t)t} - r(t-T)\lambda(\delta+1)\frac{b}{2m_I M_a}\frac{T}{r(t)t}\right\}$$
$$= \lambda(\delta+1)\frac{1}{r(t)}\left\{r(T)\left(1-\frac{T}{t}\right) - r(t-T)\frac{T}{t}\right\} \to 0$$

and

$$J_{22}(t,\lambda) = -\frac{2m_I M_a}{b}\int_T^{t-T}\log\left\{1+\lambda(\delta+1)\frac{b}{2m_I M_a}\frac{u}{r(t)t}\right\}dr(t-u)$$
$$\sim -\lambda(\delta+1)\int_T^{t-T}\frac{u}{r(t)t}\,dr(t-u) = -\lambda(\delta+1)\int_{T/t}^{1-T/t} x\,d\frac{r(t(1-x))}{r(t)}$$
$$\sim -\lambda(\delta+1)\int_{T/t}^{1-T/t} x\,d(1-x)^\delta = \lambda(\delta+1)\delta\int_{T/t}^{1-T/t} x(1-x)^{\delta-1}\,dx$$
$$\to \lambda(\delta+1)\delta\int_0^1 x(1-x)^{\delta-1}\,dx = \lambda(\delta+1)\delta B(2,\delta) = \lambda.$$

Therefore, $\lim_{t\to\infty} J(t, e^{-\lambda/A(t)}) = \lambda$ and

$$\lim_{t\to\infty}\mathbf{E}\left[e^{-\lambda Y(t)/A(t)}\right] = \lim_{t\to\infty} e^{-J(t,\exp(-\lambda/A(t)))} = e^{-\lambda}.$$

By the continuity theorem for Laplace transforms one obtains the convergence to 1 *in distribution* and then *in probability.*

Remark 5.3. The obtained result $Y(t)/\mathbf{E}[Y(t)] \to 1$ *in probability* can be interpreted as a law of large numbers. One can conjecture that the central limit theorem is also available, that is,

$$\frac{Y(t)-\mathbf{E}[Y(t)]}{\sqrt{\mathrm{Var}[Y(t)]}} \to N(0,1), \qquad t\to\infty.$$

Theorem 5.6. *If* $r(t) = r/(t+1)$, $r > 0$, *then for* $x \ge 0$

$$\lim_{t\to\infty}\Pr\left\{\frac{\log Y(t)}{\log t} \le x \;\middle|\; Y(t) > 0\right\} = \frac{x}{2}\mathbf{1}_{\{0\le x\le 1\}} + \frac{1}{2}\mathbf{1}_{\{x\ge 1\}}, \tag{5.1}$$

$$\lim_{t\to\infty}\Pr\left\{\frac{Y(t)D(t)}{\log t} \le x \;\middle|\; Y(t) > 0\right\} = \frac{1}{2} + \frac{1}{2}\left(1-\exp\left(-\frac{M_a}{Mm_I}x\right)\right). \tag{5.2}$$

Proof. For $0 < x < 1$ and $\lambda > 0$ one has

$$J(t, e^{-\lambda t^{-x}}) = \left(\int_0^T + \int_T^t \right) r(t-u) Q\big(u; e^{-\lambda t^{-x}}\big)\, du = J_1(t,\lambda) + J_2(t,\lambda).$$

By Corollary 5.1 one has for large enough T

$$J_2(t,\lambda) \sim \frac{r m_I M \lambda}{M_a t^x} \int_T^t \frac{du}{(t+1-u)(1 + u\lambda t^{-x} Mb/(2M_a^2))}, \qquad t \to \infty.$$

By decomposition into partial fractions one can calculate that

$$S(t,\lambda) = \frac{r m_I M \lambda}{M_a t^x} \int_T^t \frac{1}{t+1-u} \frac{1}{1 + u\lambda t^{-x} Mb/(2M_a^2)}\, du$$

$$= \frac{r m_I M \lambda t^{-x}/M_a}{1 + Mb\lambda t^{-x}(t+1)/(2M_a^2)} \log \frac{(t+1-T)(1 + Mb\lambda t^{1-x}/(2M_a^2))}{1 + Mb\lambda t^{-x} T/(2M_a^2)}.$$

Note that $0 \le Q(u; e^{-\lambda t^{-x}}) \le 1$. Then for any fixed T one can obtain

$$J_1(t,\lambda) = \int_0^T \frac{rQ(u; e^{-\lambda t^{-x}})}{t+1-u}\, du \le \frac{rT}{t+1-T} = O\left(\frac{1}{t}\right), \qquad t \to \infty.$$

Therefore, as $t \to \infty$

$$J(t, e^{-\lambda t^{-x}}) \sim S(t,\lambda) \sim \frac{r 2 M_a m_I}{bt} (2-x) \log t \sim \frac{2-x}{2} D(t).$$

Since $D(t; e^{-\lambda t^{-x}}) \sim J(t, e^{-\lambda t^{-x}})$, it follows that

$$\mathbf{E}\big[e^{-\lambda Y(t) t^{-x}} \,\big|\, Y(t) > 0\big] = 1 - \frac{D(t; e^{-\lambda t^{-x}})}{D(t)} \to \frac{x}{2}, \qquad x \in [0,1],$$

and by the continuity theorem for Laplace transforms we prove (5.1). The proof of (5.2) is similar, and we omit it.

Comment 5.4. Under the condition $r(t) = r/(t+1)$, using different normalizations, we obtained two limiting distributions singular to each other. One can conclude that the non-degenerate sample paths are of two types:

- half of them grows up linearly with a slope being an exponentially distributed random variable;
- the logarithm of the other half of them grows up as $\log t$ with a coefficient being a random variable uniformly distributed on the unit interval.

Similar effects were also discovered in [19, 20, 41].

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