Nikolay Moshchevitin, Badly approximable vectors in affine subspaces: Jarník-type result, *Чебышевский сб.*, 2011, том 12, выпуск 2, 77–84

ЧЕБЫШЕВСКИЙ СБОРНИК
Том 12 Выпуск 2 (2011)

Труды VIII Международной конференции
Алгебра и теория чисел: современные проблемы и приложения,
посвященной 190-летию Пафнутия Львовича Чебышева и
120-летию Ивана Матвеевича Виноградова

BADLY APPROXIMABLE VECTORS IN AFFINE SUBSPACES: JARNÍK-TYPE RESULT

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Abstract. Consider irrational affine subspace $A \subset \mathbb{R}^d$ of dimension a. We prove that the set

$$\{\xi = (\xi_1, ..., \xi_d) \in A : \quad q^{1/a} \cdot \max_{1 \leqslant i \leqslant d} ||q\xi_i|| \to \infty, \quad q \to \infty\}$$

is an α-winning set for every $\alpha \in (0, 1/2]$

This simple short communication may be considered as a supplement to our paper [9].

1. Jarník's result in simultaneous Diophantine approximations. All numbers in this paper are real. Notation $|| \cdot ||$ stands for the distance to the nearest integer. In 1938 V. Jarník (see [1], Satz 9 and [2], Statement **E**) proved the following result.

Theorem 1. (V. Jarník) *Suppose that among numbers $\xi_1, ..., \xi_d$ there are at least two numbers which are linearly independent over $\mathbb{Z}$, together with 1. Then*

$$\limsup_{t \to +\infty} \left(t \cdot \min_{q \in \mathbb{Z},\, 1 \leqslant q \leqslant t} \ \max_{1 \leqslant i \leqslant d} ||q\xi_i|| \right) = +\infty.$$

Nothing more can be said in a general situation. In 1926 A. Khintchine [3] proved the following result.

Theorem 2. (A. Khintchine) *Let $\psi(t)$ increases to infinity as $t \to +\infty$. Then there exist two algebraically independent real numbers ξ_1, ξ_2 such that for all t large enough one has*

$$t \cdot \min_{q \in \mathbb{Z},\, 1 \leqslant q \leqslant t} \ \max_{i=1,2} ||q\xi_i|| \leqslant \psi(t).$$

A general form of such a result one can find in Jarník's paper [2]. A corresponding $\limsup$ result is due to J. Lesca [6]:

*supported by grants NSh-8784.2010.1 and RFBR: 09-01-00371a

Theorem 3. (J. Lesca) *Let $d \geqslant 2$. Let $\psi(t)$ be a positive continuous function in t such that the function $t \mapsto \psi(t)/t$ is a decreasing function. Suppose that*

$$\limsup_{t\to\infty} \psi(t) = +\infty.$$

Then the set of all vectors $\xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d$, containing of algebraically independent elements, such that

$$t \cdot \min_{q\in\mathbb{Z},\, 1\leqslant q\leqslant t} \max_{1\leqslant i\leqslant d} ||q\xi_i|| \leqslant \psi(t)$$

for all t large enough, being intersected with a given open set $\mathcal{G} \subset \mathbb{R}^d$ is of cardinality continuum.

We would like to note that Jarník's Theorem 1 as well as some other theorems by Khintchine and V. Jarník were discussed and generalized in author's survey [7]. In particular in [7], Secton 4.1 (see also [8]) one can find an improvement of Theorem 1 in terms of the best approximation vectors.

2. Affine subspaces. Let $\mathbb{R}^d$ be a Euclidean space with the coordinates $(x_1, ..., x_d)$, let $\mathbb{R}^{d+1}$ be a Euclidean space with the coordinates $(x_0, x_1, ..., x_d)$. Consider an affine subspace $A \subset \mathbb{R}^d$. Let $a = \dim A \geqslant 1$. Define the affine subspace $\mathcal{A} \subset \mathbb{R}^{d+1}$ in the following way:

$$\mathcal{A} = \{\mathbf{x} = (1, x_1, ..., x_d) : \ (x_1, ..., x_d) \in A\}.$$

We define *linear* subspace $\mathfrak{A} = \operatorname{span} \mathcal{A}$, as the smallest linear subspace in $\mathbb{R}^{d+1}$ containing $\mathcal{A}$. So $\dim \mathfrak{A} = a + 1$.

Consider a sublattice $\Gamma(A) = \mathfrak{A} \cap \mathbb{Z}^{d+1}$ of the integer lattice $\mathbb{Z}^{d+1}$. We see that

$$0 \leqslant \dim \Gamma(A) \leqslant a + 1.$$

Of course here for a lattice $\Gamma \subset \mathbb{Z}^{d+1}$ by $\dim \Gamma$ we mean the dimension of the linear subspace $\operatorname{span} \Gamma$.

In the case $\dim \Gamma(A) = a + 1 = \dim \mathfrak{A}$ we define A to be a *completely rational* affine subspace in $\mathbb{R}^d$. For a completely rational affine subspace A by $d(A)$ we denote the fundamental $(a + 1)$-dimensional volume of the lattice $\Gamma(A)$.

We see from Dirichlet principle that for any completely rational affine subspace A of dimension a there exisis a positive constant $\gamma = \gamma(A)$ such that for any $\xi = (\xi_1, ..., \xi_d) \in A$ the inequality

$$\max_{1\leqslant i\leqslant d} ||q\xi_i|| \leqslant \frac{\gamma}{q^{1/a}}$$

has infinitely many solutions in positive integers q.

One can easily see that for any affine subspace A of dimension a the set

$$\Omega = \{\xi = (\xi_1, ..., \xi_d) \in A : \ \inf_{q\in\mathbb{Z}_+} q^{1/a} \cdot \max_{1\leqslant i\leqslant d} ||q\xi_i|| \ > 0\}$$

is an 1/2-winning set in A. Here we do not want to discuss the definitions (α,β)-games and (α,β)-winnning set or α-winning set. This definitions were given in W.M. Schmidt's paper [10]. All the definitions and basic properties of wining sets one can find in the book [11], Chapter 3. In particular, every α-winning set in A has full Hausdorff dimension. A countable intersection of α-winning sets inn A is also an α-winning set.

In the case when A is not a completely rational subspace the result about winning property of the set Ω admits a small improvement. This improvement is related to Jarník's result cited behind.

Theorem 4. *Let*

$$0<\alpha<1,\ \ 0<\beta<1,\ \ \gamma=1+\alpha\beta-2\alpha>0.$$

Suppose that $\dim\Gamma(A)<a$. *Then the set*

$$\Omega^*=\{\xi=(\xi_1,...,\xi_d)\in A:\quad q^{1/a}\cdot\max_{1\leqslant i\leqslant d}||q\xi_i||\to\infty,\quad q\to\infty\} \tag{1}$$

is (α,β)*-winning set in* A*. In particular, it is an* α*-winning set for every* $\alpha\in(0,1/2]$.

Here we should note that certain results concerning badly approximable vectors in affine subspaces one can find in [4, 5, 9, 12].

3. Lemmata. Consider the set of all $(a+1)$-dimensional complete sublattices of the integer lattice $\mathbb{Z}^{d+1}$. It is a countable set. One can easily see that for any positive H there exist not more than a finite number of such sublattices Γ with the fundamental volume $\det\Gamma\leqslant H$. Hence we can order the set $\{V_\nu\}_{\nu=1}^\infty$ of all a-dimensional affine subspaces in $\mathbb{R}^d$ in such a way that values $d_\nu=d(V_\nu)=\det\Gamma(V_\nu)$ form an increasing sequence:

$$1=d_1\leqslant d_2\leqslant\dots\leqslant d_\nu\leqslant d_{\nu+1}\leqslant\cdots.$$

We see that

$$d_\nu\to\infty,\ \ \nu\to\infty. \tag{2}$$

Some of consequtive values of d_ν may be equal. We define a sequence d_{ν_k} of all different elements from the sequence $\{d_\nu\}$:

$$1=d_{\nu_1}=...=d_{\nu_2-1}<d_{\nu_2}=....=d_{\nu_2-1}<d_{\nu_3}=...<d_{\nu_k}=d_{\nu_k+1}=...$$
$$...=d_{\nu_{k+1}-1}<d_{\nu_k}=...$$

(of course $\nu_1=1$). For V_j we define the affine subspace $\mathcal{V}_j\subset\mathbb{R}^{d+1}$ as

$$\mathcal{V}_j=\{\mathbf{x}=(1,x_1,...,x_d):\ (x_1,...,x_d)\in V_j\}$$

and consider linear subspace $\mathfrak{V}_j=\operatorname{span}\mathcal{V}_j$.

In the sequel for $\xi=(x_1,...,x_d)\in A$ we consider a-dimensional ball

$$B(\xi,\rho)=\{\xi'=(\xi_1',...,\xi_d')\in A:\ \max_{1\leqslant i\leqslant d}|\xi_i-\xi_i'|\leqslant\rho\}$$

and d-dimensional ball

$$\hat{B}(\xi,\rho)=\{\xi'=(\xi'_1,...,\xi'_d)\in\mathbb{R}^d:\ \max_{1\leqslant i\leqslant d}|\xi_i-\xi'_i|\leqslant\rho\}.$$

Obviously

$$B(\xi,\rho)=\hat{B}(\xi,\rho)\cap A.$$

Lemma 1. *Suppose that* $U,V\subset\mathbb{R}^d$ *are two affine subspaces. Put* $L=U\cap V$ *and suppose that* $\dim U>\dim L$. *Suppose that affine subspace* $L'\subset U$ *has dimension* $\dim L'=\dim U-1$, *and* $L'\cap L=\varnothing$. *Define* $\overline{U}\subset U$ *to be a half-subspace with the boundary* L' *and such that* $\overline{U}\cap L=\varnothing$. *Then* $\mathrm{dist}(\overline{U},V)>0$.

Proof. In affine subspace aff $(U\cup V)$ of dimension $w=\dim U+\dim V-\dim L$ there exists an affine subspace $L''\supset L'$ with dimension $\dim L''=w-1$ such that $L''\cap V=\varnothing$. So dist $(L'',U)>0$. The subspace L'' divides aff $(U\cup V)$ into two parts, and lemma follows.□

Corollary. *Consider two affine subspaces* $A,V\subset\mathbb{R}^d$. *Suppose that for* $\xi\in A$ *the ball* $B(\xi,\rho)\subset A$ *satisfies the property*

$$\mathrm{dist}(B(\xi,\rho),A\cap V)\geqslant\varepsilon>0.$$

Then there exists positive $\delta=\delta(A,V,\xi,\varepsilon)$ *such that for any* $\xi'\in B(\xi,\rho)$ *one has*

$$\hat{B}(\xi',\delta)\cap V=\varnothing.$$

Proof. From the conditions of our Corolary we see that $\dim(A\cap V)<\dim A$. So we can take a subspace L' of dimension $\dim L'=\dim A-1$ which separates the ball $B(\xi,\rho)$ from the subspace $A\cap V$ in A. Now we use Lemma 1.□

Lemma 2. *Let* $\rho>0$ *and* $\xi\in A$. *Consider a ball* $\hat{B}(\xi,\rho)\subset\mathbb{R}^d$ *such that*

$$\hat{B}(\xi,\rho)\cap\mathfrak{V}_j=\varnothing,\quad 1\leqslant j\leqslant n. \tag{3}$$

Define $k=k(n)$ *from the condition*

$$\nu_k\leqslant n<\nu_{k+1}. \tag{4}$$

Put

$$\kappa=\kappa_{d,\xi}=(2\sqrt{d})^a\times\sqrt{1+(|\xi_1|+1)^2+...+(|\xi_d|+1)^2},\ \ \sigma=\sigma_{a,d,\xi}=\frac{1}{\kappa_{d,\xi}(a+1)!} \tag{5}$$

and

$$T=(\sigma d_{\nu_n}\rho^{-a})^{\frac{1}{a+1}}.$$

Then the set of all rational points $\left(\frac{b_1}{q},...,\frac{b_d}{q}\right)\in\hat{B}(\xi,\rho)$ *with* $q\leqslant T$ *lie in a certain* $(a-1)$*-dimensional affine subspace.*

Proof. We may suppose that the set of rational points from $\hat{B}(\xi,\rho)$ with $q \leqslant T$ consists of more than a points (otherwise there is nothing to prove). We take arbitrary $a+1$ points

$$\left(\frac{b_{1,j}}{q_j},...,\frac{b_{d,j}}{q_j}\right) \in \hat{B}(\xi,\rho), \quad 1 \leqslant q_j \leqslant T, \quad \text{g.c.d.}(q_j, b_{1,j}, ..., b_{d,j}) = 1, \quad 1 \leqslant j \leqslant a+1$$

and prove that primitive integer vectors

$$\mathbf{b}_j = (q_j, b_{1,j}, ..., b_{d,j}), \quad 1 \leqslant j \leqslant a+1 \tag{6}$$

are linearly dependent. Then the lemma will be proved.

All integer vectors (6) belong to the cylinder

$$C = C_\xi(T,\rho) = \{\mathbf{x} = (x_0, x_1, ..., x_d) \in \mathbb{R}^{d+1}: \ \ 0 \leqslant x_0 \leqslant T, \ \ \max_{1\leqslant j\leqslant d} |x_0\xi_j - x_j| \leqslant \rho T.\}$$

Suppose that they are linearly independent. Then $\mathfrak{L} = \text{span}(\mathbf{b}_1, ..., \mathbf{b}_{a+1})$ is an $(a+1)$-dimensional completely rational linear subspace. By D we denote the fundamental $(a+1)$-dimensional volume of the lattice $\mathfrak{L} \cap \mathbb{Z}^{d+1}$. From (3) we see that

$$\mathfrak{L} \neq \mathfrak{V}_j, \quad 1 \leqslant j \leqslant n.$$

From (4) we see that

$$D \geqslant d_{\nu_n}. \tag{7}$$

Now we consider the section $\mathfrak{L}\cap C$ which is an $(a+1)$-dimensional convex polytope. As it is inside C, its $(a+1)$-dimensional measure is less than

$$(2\sqrt{d}\rho T)^a \times T\sqrt{1 + (|\xi_1|+1)^2 + ... + (|\xi_d|+1)^2} = \kappa \rho^a T^{a+1} = \kappa\sigma d_{\nu_n}.$$

But the section $\mathfrak{L}\cap C$ consist of $a+1$ independent points from the lattice $\mathfrak{L}\cap \mathbb{Z}^{d+1}$. For the fundamental volume of this lattice we have lower bound (7). That is why

$$\frac{d_{\nu_n}}{(a+1)!} \leqslant \frac{D}{(a+1)!} < \kappa\sigma d_{\nu_n} = \frac{d_{\nu_n}}{(a+1)!}.$$

This is a contradiction. Lemma is proved.□

Lemma 3.(W.M. Schmidt's escaping lemma, Lemma 1B, [11], Chapter 3) *Let t be such that*

$$(\alpha\beta)^t < \frac{\gamma}{2}.$$

Suppose a ball $B_j \subset A$ with the radius ρ_j occurs in the game (as a Black ball). Suppose V is an $(d-1)$-dimensional affine subspace passing through the center of the ball B_j. Then White can play in such a way that the ball B_{j+t} is contained in the halfspace Π such that the boundary of Π is parallel to the subspace V and the distance between Π and V is equal to $\frac{\rho_j\gamma}{2}$.

Corollary. *Suppose a ball $B_j \subset A$ with the radius ρ_j occurs in the game (as a Black ball). Suppose that $V, V' \subset A$ are two proper affine subspaces of A. Then White can play in such a way that the distance from the ball B_{j+2t} to each of subspaces V, V' is greater than $\frac{\rho_{j+t}\gamma}{2}$ (here ρ_{j+t} is the radius of the ball B_{j+t}).*

4. Proof of Theorem 4. Suppose that $t = t(\alpha, \beta)$ satisfies the condition of Lemma 3. Put $j_k = 2tk$ and $R_0 = 1$. Suppose that the first Black ball $B_0 \subset A$ with the radius ρ_0 lies inside the box $\{\xi \in \mathbb{R}^d : \max_{1\leqslant i\leqslant d} |\xi_i| \leqslant W\}$. We shall prove that White can play in such a way that for any $\xi \in B_{j_r}$ one has

$$\max_{1\leqslant i\leqslant d} ||q\xi_i|| \geqslant \frac{(\alpha\beta)^t\gamma\rho_0}{2} R^{-\frac{(a+1)r}{a}} \cdot q, \quad \forall q < R_r \tag{8}$$

with a certain R_r which we define later in the inductive step.

We shall prove it by induction in r.

The base of induction is obvious.

Suppose that the ball $B_{j_{r-1}} = B(\xi_{j_{r-1}}, \rho_{j_{r-1}}) \in A, \xi_{j_{r-1}} = (\xi_{j_{r-1},1}, ..., \xi_{j_{r-1},d})$ which occurs as a Black ball satisfies the condition specified. Note that $\rho_{j_{r-1}} = \rho_0(\alpha\beta)^{j_{r-1}}$. Consider the ball $\hat{B}_{j_{r-1}} = \hat{B}(\xi_{j_{r-1}}, 2\rho_{j_{r-1}}) \in \mathbb{R}^d$. Define k_r as the maximal k such that $\hat{B}_{j_{r-1}} \cap \mathfrak{V}_j = \varnothing, 1 \leqslant j \leqslant \nu_k$. Then we apply Lemma 2 to see that all rational points $\left(\frac{b_1}{q}, ..., \frac{b_d}{q}\right) \in \hat{B}_{j_{r-1}}$ with

$$q \leqslant \left(\sigma_{a,d,\xi_{j_{r-1}}}(2\rho_0)^{-a}\right)^{\frac{1}{a+1}} \left(\frac{1}{\alpha\beta}\right)^{\frac{2at(r-1)}{a+1}} d_{\nu_{k_{r-1}}}^{\frac{1}{a+1}}$$

lie in a certain $(a-1)$-dimensional affine subspace. We denote this subspace by V'_r. As $\max_{1\leqslant i\leqslant d} |\xi_{j_{r-1},i}| \leqslant W$ we see that

$$\sigma_{a,d,\xi_{j_{r-1}}} \geqslant \Sigma_{a,d,W} = \frac{1}{(2\sqrt{d})^a \sqrt{1+(W+1)^2 d}\,(a+1)!}.$$

We put

$$R_r = \left(\Sigma_{a,d,W}(2\rho_0)^{-a}\right)^{\frac{1}{a+1}} \left(\frac{1}{\alpha\beta}\right)^{\frac{2at(r-1)}{a+1}} d_{\nu_{k_{r-1}}}^{\frac{1}{a+1}}. \tag{9}$$

By Corollary to Lemma 3 White can play in such a way that

$$\operatorname{dist}(B_{j_r}, V_r) \geqslant \frac{\gamma\rho_{j_{r-1}+t}}{2} \tag{10}$$

and

$$\operatorname{dist}(B_{j_r}, V'_r) \geqslant \frac{\gamma\rho_{j_{r-1}+t}}{2} \tag{11}$$

So the inductive step is discribed and we must show that (8) is valid. But it is clear from (11) that for any $\xi \in B_{j_r}$ one has

$$\max_{1\leqslant i\leqslant d} ||q\xi_i|| \geqslant \frac{1}{2}\gamma\rho_{j_{r-1}+t} q = \frac{\gamma\rho_0}{2} (\alpha\beta)^{(2t-1)r} q, \quad \forall q < R_r. \tag{12}$$

Moreover by Corollary to Lemma 1 from (10) we see that

$$k_r \to +\infty, \quad r \to +\infty.$$

Hence

$$d_{\nu_{k_r}} \to +\infty, \quad r \to +\infty. \tag{13}$$

Consider a point $\xi \in \bigcap_j B_j$. For positive integer q define r from the condition

$$R_{r-1} \leqslant q < R_r.$$

Then we make use of $\xi \in B_{j_r}$. From the inequality $q \geqslant R_{r-1}$ and (9) we see that

$$\alpha\beta \geqslant \omega_1 q^{-\frac{a+1}{2atr}} d_{\nu_{k_{r-2}}}^{\frac{1}{2atr}},$$

where $\omega_1 = \omega_1(a, d, W, \alpha, \beta, t) > 0$. We substitute this estimate into (12) to see that

$$\max_{1\leqslant i\leqslant d} ||q\xi_i|| \geqslant \omega_2 q^{-1/a} d_{\nu_{k_{r-2}}}^{1/a}, \quad R_{r-1} \leqslant q < R_r,$$

with positive $\omega_2 = \omega_2(a, d, W, \alpha, \beta, t)$. From (13) for $\xi \in \bigcap_j B_j$ we deduce that

$$q^{1/a} \cdot \max_{1\leqslant i\leqslant d} ||q\xi_i|| \to +\infty, \quad q \to \infty.$$

So White can enforce Black to reach a point ξ with the desired properties. Theorem 4 is proved.□

REFERENCES

[1] V. Jarník, Zum Khintchineschen " Übertragungssatz", Travaux de L'Institut Mathematique de Tbilissi, T 3 (1938), 193 - 212.

[2] V. Jarník, Eine Bemerkung uber diophantische approximationen, Math. Zeitschr. 72, 187 - 191 (1959).

[3] A.Y. Khinchine, Über eine klasse linear Diophantine Approximationen, Rendiconti Circ. Math. Palermo, 1926, 50, p.170 - 195.

[4] D. Kleinbock, Extremal supspaces and their submanifolds, GAFA 13:2 (2003), 437 - 466.

[5] D. Kleinbock, An extension of quantitative nondivergence and applications to Diophantine exponents, Trans. Amer. Math. Soc. 360 (2008), 6497-6523.

[6] J. Lesca, Sur un résultat de Jarník, Acta Arith. 11 (1966), 359–364.

[7] N.G. Moshchevitin, Khintchine's singular Diophantine systems and their applications, Russian Mathematical Surveys. 65:3 43 - 126 (2010).

[8] N.G. Moshchevitin, Best simultaneous approximations: Norms, signatures, and asymptotic directions, Math. Notes 67:5 (2000), 618-624.

[9] N. Moshchevitin, On Kleinbock's Diophantine result, Proc. Math. Debrecen (to appear), preprint available at arXiv:0906.1541 (2009, 2011).

[10] W.M. Schmidt, On badly approximable numbers and certain games, Trans. Amer. Math. Soc., 623 (1966), p. 178 – 199.

[11] W.M. Schmidt, Diophantine Approximations, Lect. Not. Math., 785 (1980).

[12] Y. Zhang, Diophantine exponents of affine subspaces: The simultaneous approximation case, J. Number Theory, 129:8 (2009), 1976-1989.

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Поступило 28.10.2011