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RELAXATION OF CONVEX VARIATIONAL PROBLEMS WITH LINEAR GROWTH DEFINED ON CLASSES OF VECTOR-VALUED FUNCTIONS

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Abstract. For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and a function $u_0 \in W_1^1(\Omega; \mathbb{R}^N)$, the following minimization problem is considered:

$$(\mathcal{P}): \int_\Omega f(\nabla u)\ dx \to \min \text{ in } u_0 + \overset{\circ}{W}{}_1^1(\Omega; \mathbb{R}^N),$$

where f: $\mathbb{R}^{nN} \to [0, \infty)$ is a strictly convex integrand. Let $\mathcal{M}$ denote the set of all L^1-cluster points of minimizing sequences of problem $(\mathcal{P})$. It is shown that the geometric relaxation of problem $(\mathcal{P})$ coincides with the relaxation based on the notion of the extended Lagrangian; moreover, it is proved that the elements u of $\mathcal{M}$ are in one-to-one correspondence with the solutions of the relaxed problems.

§1. Introduction

In this paper, we are concerned with variational problems of linear growth defined on spaces of vector-valued functions; such problems are usually handled by introducing a suitable relaxed version of the problem or by passing to some dual variational formulation. There exist two — at least formally — different approaches to a reasonable concept of relaxation, the first one being preferred in connection with problems of minimal surface type, and the second one occurring in the theory of perfect plasticity. For experts in the theory of relaxation it might be obvious that both points of view lead to the same result, but we did not find a rigorous proof of this equivalence in the literature and, therefore, we sketch the arguments in the present paper.

To be more precise, first we fix our notation. In what follows Ω and $\hat{\Omega}$ denote bounded Lipschitz domains in $\mathbb{R}^n$, $n \geqslant 2$, such that $\Omega \Subset \hat{\Omega}$. Given boundary values

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u_0 of Sobolev class $W_1^1(\Omega;\mathbb{R}^N)$, we may extend u_0 to the domain $\widehat{\Omega}$ in such a way that $u_0 \in \overset{\circ}{W}{}_1^1(\widehat{\Omega};\mathbb{R}^N)$; we denote

$$BV_{u_0}(\Omega;\mathbb{R}^N) := \{u \in BV(\widehat{\Omega};\mathbb{R}^N) : u = u_0 \text{ on } \widehat{\Omega} \setminus \Omega\}, \tag{1.1}$$

where $BV(\widehat{\Omega};\mathbb{R}^N)$ is the space of functions of bounded variation (see, e.g., [Giu] or [AFP]). Suppose we are given a strictly convex (in the sense of the standard definition) function f: $\mathbb{R}^{nN} \to [0,\infty)$ of linear growth. By the latter we mean that f satisfies the condition

$$a|Z| - b \leqslant f(Z) \leqslant A|Z| + B \quad \text{for all } Z \in \mathbb{R}^{nN} \tag{1.2}$$

with some positive constants a, A, b, B, and we further assume that $f(0) = 0$. As a matter of fact, in general, the variational problem

$$J[u] := \int_\Omega f(\nabla u)\, dx \to \min \quad \text{in } u_0 + \overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N) \tag{$\mathcal{P}$}$$

fails to have a solution, but from (1.2) it follows that the set

$$\mathcal{M} := \{u \in BV(\Omega;\mathbb{R}^N) : \\ u \text{ is an } L^1\text{-cluster point of some minimizing sequence to problem } (\mathcal{P})\}$$

of generalized minimizers is nonempty. Next we recall the concept of the "geometric" relaxation of problem $(\mathcal{P})$ (introduced by DeGiorgi in [Gio]). This concept involves the Dirichlet boundary data u_0 via the space $BV_{u_0}(\Omega;\mathbb{R}^N)$ (see (1.1)), which was discussed in the paper [GMS] of Giaquinta, Modica, and Souček. For $w \in BV(\widehat{\Omega};\mathbb{R}^N)$, let

$$\widehat{J}\,[w,\widehat{\Omega}] = \int_{\widehat{\Omega}} f(\nabla^a w)\, dx \; + \int_{\widehat{\Omega}} f_\infty\left(\frac{\nabla^s w}{|\nabla^s w|}\right) d|\nabla^s w|,$$

where $\nabla^a w$ denotes the absolutely continuous part of ∇w with respect to Lebesgue measure, $\nabla^s w$ is the singular part of ∇w, and f_∞ denotes the recession function of f. With these definitions, the lower semicontinuity theorem of Reschetnyak (see [Re]) immediately implies the following existence result [GMS, Theorem 1.3].

Theorem 1.1. *There exists a minimum point for the problem*

$$\widehat{J}\,[\cdot,\widehat{\Omega}] \to \min \quad \textit{in } BV_{u_0}(\Omega;\mathbb{R}^N).$$

We fix a function $w \in BV_{u_0}(\Omega;\mathbb{R}^N)$. For vectors $\eta \in \mathbb{R}^N$, $\zeta \in \mathbb{R}^n$, we denote by $\eta \otimes \zeta$ the matrix with the components $\eta^i\zeta_\alpha$, $i = 1,\dots,N$, $\alpha = 1,\dots,n$. If μ is a vector-valued measure and A is a measurable subset of $\mathbb{R}^n$, then we use the symbol $\mu \llcorner A$ for the measure $B \mapsto \mu(A \cap B)$. Observing (see [AFP, Theorem 3.77, p. 171]) that

$$\nabla^s w \llcorner \partial\Omega = (u_0 - w) \otimes \nu \, \mathcal{H}^{n-1} \llcorner \partial\Omega,$$

where ν denotes the outward unit normal to $\partial\Omega$, we may write

$$\begin{aligned}\hat{J}\,[w,\hat{\Omega}] &= \int_\Omega f(\nabla^a w)\,dx + \int_\Omega f_\infty\left(\frac{\nabla^s w}{|\nabla^s w|}\right) d|\nabla^s w| \\ &\quad + \int_{\partial\Omega} f_\infty((u_0 - w)\otimes\nu)\,d\,\mathcal{H}^{n-1} + \int_{\hat{\Omega}\setminus\Omega} f(\nabla u_0)\,dx,\end{aligned} \tag{1.3}$$

and since the last integral on the right-hand side of (1.3) is constant, we drop this term and introduce the energy $K\colon BV(\Omega;\mathbb{R}^N) \to \mathbb{R}$ by

$$K[u] := \int_\Omega f(\nabla^a u)\,dx + \int_\Omega f_\infty\left(\frac{\nabla^s u}{|\nabla^s u|}\right) d|\nabla^s u| + \int_{\partial\Omega} f_\infty((u_0 - u)\otimes\nu)\,d\mathcal{H}^{n-1}, \tag{1.4}$$

whence

$$\hat{J}\,[w,\hat{\Omega}] = K[w_{|\Omega}] + \text{const}$$

for any $w \in BV_{u_0}(\Omega;\mathbb{R}^N)$. Conversely, given $u \in BV(\Omega;\mathbb{R}^N)$, we let $\hat{u}$ denote the extension via u_0 to the domain $\hat{\Omega}$, i.e., we put

$$\hat{u} = \begin{cases} u & \text{on } \Omega, \\ u_0 & \text{on } \hat{\Omega}\setminus\Omega, \end{cases}$$

obtaining

$$\hat{J}\,[\hat{u},\hat{\Omega}] = K[u] + \text{const}.$$

The properties of the functional K defined in (1.4) are summarized below.

Theorem 1.2.

i) *The minimization problem*

$$K[w] \to \min \ \text{in } BV(\Omega;\mathbb{R}^N) \tag{$\hat{\mathcal{P}}$}$$

admits at least one solution.

ii) $\inf_{w \in u_0 + \mathring{W}_1^1(\Omega;\mathbb{R}^N)} J[w] = \inf_{w \in BV(\Omega;\mathbb{R}^N)} K[w].$

iii) *A function u is a minimizer of the energy K if and only if u is a generalized minimizer, i.e., $u \in \mathcal{M}$.*

We note that the proof of Theorem 1.2 is based on Theorem 1.1 together with some minor adjustments of the arguments given in [Giu] for the minimal surface case. A short outline will be presented in the next section.

Theorem 1.2 has a nice interpretation. Consider the variational problem

$$\int_\Omega f(\nabla w)\,dx + \int_{\partial\Omega} f_\infty\big((u_0 - w)\otimes\nu\big)\,d\mathcal{H}^{n-1} \to \min \quad \text{in } W_1^1(\Omega;\mathbb{R}^N), \qquad (\mathcal{P}')$$

and let $\mathcal{M}' = \mathcal{M}\cap W_1^1(\Omega;\mathbb{R}^N)$. As was shown in [B1, 2], at least one generalized minimizer of Sobolev class W_1^1 will exist if in addition to our hypothesis on f stated after formula (1.1) we require that $f \in C^2(\mathbb{R}^{nN})$ and that the ellipticity condition

$$\lambda\left(1+|Z|^2\right)^{-\frac{3}{2}}|Y|^2 \leqslant D^2 f(Z)(Y,Y) \leqslant \Lambda\left(1+|Z|^2\right)^{-\frac{1}{2}}|Y|^2$$

be valid for all Y, $Z \in \mathbb{R}^{nN}$ (λ, Λ are positive constants). Now we have

$$\mathcal{M}' = \left\{u \in W_1^1(\Omega;\mathbb{R}^N) : u \text{ is a solution of } (\mathcal{P}')\right\},$$

and since the strict convexity of f ensures that solutions of $(\mathcal{P}')$ may only differ by a constant, we see that $\mathcal{M}'$ is uniquely determined up to constants. To prove the above description of $\mathcal{M}'$, choose $u^* \in \mathcal{M}'$. By Theorem 1.2, iii), u^* is K-minimizing, and from (1.4) together with $\nabla^s u^* = 0$ we deduce that u^* is a solution of $(\mathcal{P}')$. Conversely, consider a solution v^* of $(\mathcal{P}')$ and a J-minimizing sequence $\{u_m\}$ in $u_0 + \mathring{W}_1^1(\Omega;\mathbb{R}^N)$. The minimality of v^* implies that

$$K[v^*] = \int_\Omega f(\nabla v^*)\,dx + \int_{\partial\Omega} f_\infty\big((u_0 - v^*)\otimes\nu\big)\,d\mathcal{H}^{n-1} \leqslant \int_\Omega f(\nabla u_m)\,dx,$$

and Theorem 1.2, ii), shows that v^* is K-minimizing; consequently, $v^* \in \mathcal{M}$ by Theorem 1.2, iii).

In order to introduce the notion of relaxation via some suitable Lagrangian function, we follow the lines of Seregin [Se] (see [FS] for a complete list of references) and of Strang and Temam [ST]. We start with observing that

$$J[u] = \sup_{\tau\in L^\infty(\Omega;\mathbb{R}^{nN})} l(u,\tau) \quad \text{for all} \quad u \in W_1^1(\Omega;\mathbb{R}^N),$$

where

$$l(u,\tau) = \int_\Omega \tau : \nabla u\,dx - \int_\Omega f^*(\tau)\,dx, \quad u \in W_1^1(\Omega;\mathbb{R}^N), \quad \tau \in L^\infty(\Omega;\mathbb{R}^{nN}),$$

is the Lagrangian, and f^* denotes the conjugate function of f (see [ET]). We remark that the dual problem to $(\mathcal{P})$, i.e., the problem

$$R \to \max \text{ on } L^\infty(\Omega;\mathbb{R}^{nN}), \quad R[\tau] := \inf_{u \in u_0 + \mathring{W}_1^1(\Omega;\mathbb{R}^N)} l(u,\tau), \tag{$\mathcal{P}^*$}$$

admits a solution; moreover, we have

$$\inf_{w \in u_0 + \mathring{W}_1^1(\Omega;\mathbb{R}^N)} J[w] = \max_{\tau \in L^\infty(\Omega;\mathbb{R}^{nN})} R[\tau].$$

For $u \in u_0 + \mathring{W}_1^1(\Omega;\mathbb{R}^N)$ and for tensors τ belonging to the space

$$\mathcal{U} := \left\{\sigma \in L^\infty(\Omega;\mathbb{R}^{nN}) : \operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^N)\right\},$$

it is easy to check that

$$l(u,\tau) = \int_\Omega \operatorname{div}\tau \cdot (u_0 - u)\,dx - \int_\Omega f^*(\tau)\,dx + \int_\Omega \tau : \nabla u_0\,dx =: \widetilde{l}(u,\tau),$$

and the extended Lagrangian $\widetilde{l}(u,\tau)$ makes sense if $u \in BV(\Omega;\mathbb{R}^N)$. Finally, we use the extended Lagrangian to define

$$\widetilde{J}\,[w] := \sup_{\tau \in U} \widetilde{l}(w,\tau), \quad w \in BV(\Omega;\mathbb{R}^N). \tag{1.5}$$

In essence, the next result is due to Seregin (see, e.g., [Se]; we have only added the fact that the $\widetilde{J}$-minimizers lie in the set $\mathcal{M}$), but it can be reduced to Theorem 1.2 with the help of Theorem 1.4 below.

Theorem 1.3.

i) *The minimization problem*

$$\widetilde{J}[w] \to \min \textit{ in } BV(\Omega;\mathbb{R}^N) \tag{$\widetilde{\mathcal{P}}$}$$

admits a solution.

ii) $\displaystyle\inf_{w \in u_0 + \mathring{W}_1^1(\Omega;\mathbb{R}^N)} J[w] = \inf_{w \in BV(\Omega;\mathbb{R}^N)} \widetilde{J}[w].$

iii) *A function u is a minimizer of the energy $\widetilde{J}$ if and only if u is a generalized minimizer, i.e., $u \in \mathcal{M}$.*

So, in accordance with Theorems 1.2 and 1.3, problem $(\widehat{\mathcal{P}})$ (as well as problem $(\widetilde{\mathcal{P}})$) is a suitable relaxed version of the original problem $(\mathcal{P})$ in the sense that we have the properties stated in ii) and iii). Now, in order to get a complete picture of the situation, we formulate our main result.

Theorem 1.4. *On the space $BV(\Omega;\mathbb{R}^N)$ the functionals K and $\widetilde{J}$ defined in* (1.4) *and* (1.5) *coincide.*

The rest of the paper is organized as follows. In §2 we give a short proof of Theorem 1.2, and in §3 we prove the identity $\widetilde{J} = K$. The main tool used there is the level-set approximation Lemma 3.2, which we combine with the representation formula for convex functions of a measure due to Demengel and Temam [DT].

§2. Proof of Theorem 1.2

Part i) of Theorem 1.2 is an immediate consequence of Theorem 1.1 and the definition of the functional K. For the case of the minimal surface problem, i.e., for $N = 1$ and $J[u] = \int_\Omega \sqrt{1+|\nabla u|^2}\,dx$, part ii) was established in [Giu, Proposition 14.3, p. 161]. In the general case, first we observe that $K[w] = J[w]$ for $w \in u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)$, whence

$$\inf_{u_0+\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)} J \geqslant \inf_{BV(\Omega;\mathbb{R}^N)} K.$$

For the reverse inequality, we use a lemma that occurs (in slightly different forms) in many textbooks; see, e.g., [Giu] or [AFP]. Regrettably, we could not find an explicit reference for the statement given below.

Lemma 2.1. *For $w \in BV(\Omega;\mathbb{R}^N)$, consider the extension*

$$\widehat{w} = \begin{cases} w & \text{on } \Omega, \\ u_0 & \text{on } \widehat{\Omega} \setminus \Omega. \end{cases}$$

There exists a sequence $\{w_m\}$ in $u_0 + C_0^\infty(\Omega;\mathbb{R}^N)$ such that, as $m \to \infty$, we have

a) $w_m \to \widehat{w}$ *in* $L^1(\widehat{\Omega};\mathbb{R}^N)$,

b) $\displaystyle\int_{\widehat{\Omega}} \sqrt{1+|\nabla w_m|^2}\,dx \to \int_{\widehat{\Omega}} \sqrt{1+|\nabla \widehat{w}|^2}$

(we extend w_m by u_0 to $\widehat{\Omega}$).

Assuming that Lemma 2.1 is true, we fix $w \in BV(\Omega,\mathbb{R}^N)$ and define $\{w_m\}$ as above. Then from Reschetnyak's continuity theorem (see [Re] and compare also with [AG, Theorem 2.1 and Proposition 2.2]) we deduce that

$$\widehat{J}[\widehat{w},\widehat{\Omega}] = \lim_{m\to\infty} \widehat{J}[w_m,\widehat{\Omega}],$$

whence $K[w] = \lim_{m\to\infty} K[w_{m|\Omega}]$. Since, clearly,

$$K[w_{m|\Omega}] \;=\; J[w_{m|\Omega}] \;\geqslant \inf_{u_0+\mathring{W}_1^1(\Omega;\mathbb{R}^N)} J,$$

this proves part ii) of the theorem. Suppose that $u \in BV(\Omega;\mathbb{R}^N)$ is K-minimizing. Applying Lemma 2.1 with w replaced by u, for the corresponding approximating sequence $\{w_m\} \in BV(\widehat{\Omega};\mathbb{R}^N)$ we get

$$J[w_{m|\Omega}] \;=\; K[w_{m|\Omega}] \xrightarrow{m\to\infty} K[u],$$

which yields

$$J[w_{m|\Omega}] \to \inf_{u_0+\mathring{W}_1^1(\Omega;\mathbb{R}^N)} J$$

by ii). Hence, $\{w_{m|\Omega}\}$ is a J-minimizing sequence of class $u_0 + \mathring{W}_1^1(\Omega;\mathbb{R}^N)$ and such that $w_{m|\Omega} \to u$ in $L^1(\Omega;\mathbb{R}^N)$. This implies $u \in \mathcal{M}$. Conversely, suppose that $u \in \mathcal{M}$ is the L^1-limit of some J-minimizing sequence $\{u_m\} \in u_0 + \mathring{W}_1^1(\Omega;\mathbb{R}^N)$. Since K is lower semicontinuous with respect to this convergence, we find

$$K[u] \;\leqslant\; \liminf_{m\to\infty} K[u_m] \;=\; \liminf_{m\to\infty}\; J[u_m] \;=\; \inf_{u_0+\mathring{W}_1^1(\Omega;\mathbb{R}^N)} J,$$

and ii) shows that u is K-minimizing.

Proof of Lemma 2.1. First, we recall the definition of the measure

$$B \longmapsto \int_B \sqrt{1+|\nabla u|^2},$$

where B is a Borel subset of $\widehat{\Omega}$ and u is a function in $BV(\widehat{\Omega};\mathbb{R}^N)$. In agreement with the general concept of applying a convex function to a measure (see [DT]), we put

$$\int_B \sqrt{1+|\nabla u|^2} \;:=\; \int_B \sqrt{1+|\nabla^a u|^2}\,dx + |\nabla^s u|(B). \tag{2.1}$$

It is easily seen that statement $b)$ of Lemma 2.1 implies the weaker condition

$$\int_{\widehat{\Omega}} |\nabla w_m|\,dx \longrightarrow \int_{\widehat{\Omega}} |\nabla \widehat{w}|;$$

for instance, we may quote [AG, Proposition 2.2], with the choice $F(P) := |P|$. It should also be noted that a version of our approximation lemma involving condition

b) occurs in [AG, Proposition 2.3]. In the course of the proof of Lemma 2.1 we replace (2.1) by the following equivalent representation, which can be deduced, e.g., by applying [DT, Proposition 1.2] to the function $f_0(P) = \sqrt{1+|P|^2} - 1$. For u and B as in (2.1) we have

$$\int_B \sqrt{1+|\nabla u|^2} = \mathcal{L}^n(B) - \mathcal{L}^n(\widehat{\Omega}) + \sup_{\tau \in C_0^\infty(\widehat{\Omega};\mathbb{R}^{nN}),|\tau|\leqslant 1} \left\{ \int_B \tau : \nabla u + \int_{\widehat{\Omega}} \sqrt{1-|\tau|^2}\, dx \right\}. \tag{2.2}$$

For notational simplicity, we restrict ourselves to the case where $\Omega = B_1 = B_1(0)$ and $\widehat{\Omega} = B_2 = B_2(0)$ (the general situation can be reduced to this special setting via a covering argument; for the details we refer to [B2]). We fix $w \in BV(B_1;\mathbb{R}^N)$ and put

$$\widehat{w}(z) = \begin{cases} w(z) & \text{if } |z| < 1, \\ u_0(z) & \text{if } |z| \geqslant 1, \end{cases}$$

where we agree that $\widehat{w}(z) = 0$ for $|z| \geqslant 2$. For $\delta \in (0,1)$ close to 1, let

$$v_\delta(z) := (\widehat{w} - u_0)\left(\frac{z}{\delta}\right) + u_0(z). \tag{2.3}$$

Note that $v_\delta(z) = u_0(z)$ for $|z| \geqslant \delta$. Let $\tau \in C_0^\infty(B_2;\mathbb{R}^{nN})$, $|\tau| \leqslant 1$. Putting $\widetilde{\tau}(z) := \tau(\delta z)$, we deduce that

$$\begin{aligned} &\int_{B_2} \nabla v_\delta : \tau + \int_{B_2} \sqrt{1-|\tau|^2}\, dx \\ &= \delta^{n-1}\Bigg[\int_{B_{2/\delta}} \widetilde{\tau} : \nabla \widehat{w} + \int_{B_{2/\delta}} \sqrt{1-|\widetilde{\tau}|^2}\, dx \Bigg] \\ &\quad + \delta^{n-1}\Bigg[\delta^{1-n} \int_{B_2} \sqrt{1-|\tau|^2}\, dx - \int_{B_{2/\delta}} \sqrt{1-|\widetilde{\tau}|^2}\, dx\Bigg] \\ &\quad - \int_{B_2} \left[\delta^{-1}\nabla u_0\left(\frac{y}{\delta}\right) - \nabla u_0(y)\right] : \tau(y)\, dy =: \text{(I)} + \text{(II)} - \text{(III)}, \end{aligned}$$

where

$$\text{(I)} \leqslant \delta^{n-1} \int_{B_{2/\delta}} \sqrt{1+|\nabla \widehat{w}|^2},$$

$$\big|\text{(III)}\big| \leqslant \varepsilon \text{ if } \delta \text{ is sufficiently close to 1},$$

$$\big|\text{(II)}\big| = \left| \int_{B_2} \sqrt{1-|\tau|^2}\, dx - \frac{1}{\delta} \int_{B_2} \sqrt{1-|\tau|^2}\, dx \right| \leqslant (1/\delta - 1)\, \mathcal{L}^n(B_2).$$

The explicit formula (2.1) for the measure $\int \sqrt{1+|\nabla \hat{w}|^2}$ (in accordance with our convention, we regard this as a measure on $\mathbb{R}^n$) and the properties of $\hat{w}$ (there is no mass of $\nabla \hat{w}$ on ∂B_2) show that

$$\lim_{\delta \uparrow 1} \delta^{n-1} \int_{B_{2/\delta}} \sqrt{1+|\nabla \hat{w}|^2} = \int_{B_2} \sqrt{1+|\nabla \hat{w}|^2}.$$

Thus,

$$\int_{B_2} \sqrt{1+|\nabla v_\delta|^2} = \sup\left\{ \int_{B_2} \nabla v_\delta : \tau + \int_{B_2} \sqrt{1-|\tau|^2}dx : \tau \in C_0^\infty(B_2;\mathbb{R}^{nN}), |\tau| \leqslant 1 \right\}$$
$$\leqslant \varepsilon + (1/\delta - 1)\mathcal{L}^n(B_2) + \delta^{n-1} \int_{B_{2/\delta}} \sqrt{1+|\nabla \hat{w}|^2}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\limsup_{\delta \uparrow 1} \int_{B_2} \sqrt{1+|\nabla v_\delta|^2} \leqslant \int_{B_2} \sqrt{1+|\nabla \hat{w}|^2}.$$

The reverse inequality follows from the convergence $v_\delta \longrightarrow \hat{w}$ in $L^1(B_2)$ together with the lower semicontinuity of $\int_{B_2} \sqrt{1+|\cdot|^2}$.

So, given $\varepsilon > 0$, we can find δ close to 1 such that the function $v = v_\delta$ defined in (2.3) satisfies

$$\begin{aligned} v(z) = u_0(z) \quad \text{for } |z| \geqslant \delta, \\ \int_{B_2} |v - \hat{w}|\, dz \leqslant \varepsilon, \\ \left| \int_{B_2} \sqrt{1+|\nabla v|^2} - \int_{B_2} \sqrt{1+|\nabla \hat{w}|^2} \right| \leqslant \varepsilon. \end{aligned} \tag{2.4}$$

Let $(\dots)^\rho$ denote the mollification of a function with respect to a smoothing kernel ω_ρ. With v as in (2.4), we put

$$w_\rho := u_0 + (v - u_0)^\rho,$$

i.e., w_ρ equals "u_0 + the mollification of the scaled function $z \mapsto (\hat{w} - u_0)(z/\delta)$".

We have $(v - u_0)^\rho(z) = 0$ if $|z| \geqslant \delta + \rho$, and if we assume that $\rho + \delta < 1$, then the function $(v - u_0)^\rho$ is of class $C_0^\infty(B_1;\mathbb{R}^N)$. Moreover, from (2.4) we see that

$$\int_{B_2} |w_\rho - \hat{w}|\, dx \leqslant 2\varepsilon \quad \text{if } \rho \leqslant \rho(\varepsilon, \delta). \tag{2.5}$$

The inequality

$$\int_{B_2} \sqrt{1+|\nabla w_\rho|^2}\,dx \leqslant \int_{B_2} \sqrt{1+|\nabla v^\rho|^2}\,dx + \int_{B_2} |\nabla u_0 - \nabla u_0^\rho|\,dx$$

shows that it suffices to discuss

$$\int_{B_2} \sqrt{1+|\nabla v^\rho|^2}\,dx$$
$$= \sup\left\{ \int_{B_2} \tau : \nabla v^\rho\,dx + \int_{B_2} \sqrt{1-|\tau|^2}\,dx : \tau \in C_0^\infty(B_2;\mathbb{R}^{nN}), |\tau| \leqslant 1 \right\}.$$

We fix a smooth tensor τ with compact support in B_2. Recalling that $v = u_0$ for $|z| \geqslant \delta$ and $|\nabla v|(\mathbb{R}^n - B_2) = 0$, we get

$$\int_{B_2} \tau : \nabla v^\rho\,dx = \int_{B_2} \tau^\rho : \nabla v = \int_{B_{2+\rho}} \tau^\rho : \nabla v,$$

whence

$$\int_{B_2} \tau : \nabla v^\rho + \int_{B_2} \sqrt{1-|\tau|^2}dx$$
$$= \int_{B_{2+\rho}} \tau^\rho : \nabla v$$
$$+ \int_{B_{2+\rho}} \sqrt{1-|\tau^\rho|^2}dx + \int_{B_2} \sqrt{1-|\tau|^2}dx - \int_{B_{2+\rho}} \sqrt{1-|\tau^\rho|^2}dx$$
$$\leqslant \int_{B_{2+\rho}} \sqrt{1+|\nabla v|^2} + \int_{B_2} \sqrt{1-|\tau|^2}\,dx - \int_{B_{2+\rho}} \sqrt{1-|\tau^\rho|^2}dx$$

(observe that $\tau^\rho \in C_0^\infty(B_{2+\rho};\mathbb{R}^{nN})$, $|\tau^\rho| \leqslant 1$). As was mentioned above, it is immediate (see (2.1)) that

$$\lim_{\rho\downarrow 0} \int_{B_{2+\rho}} \sqrt{1+|\nabla v|^2} = \int_{B_2} \sqrt{1+|\nabla v|^2}.$$

Since the function $P \mapsto \sqrt{1-|P|^2}$, $|P| \leqslant 1$, is concave, the Jensen inequality yields

$$\int_{B_2} \sqrt{1-|\tau|^2}dx - \int_{B_2+\rho} \sqrt{1-|\tau^\rho|^2}dx \leqslant \int_{B_2} \sqrt{1-|\tau|^2}dx - \int_{B_2+\rho} \left(\sqrt{1-|\tau|^2}\right)^\rho dx.$$

Finally,

$$\int_{B_2} \sqrt{1-|\tau|^2}\,dx \leqslant \int_{B_{2+\rho}} \left(\sqrt{1-|\tau|^2}\right)^\rho (y)\,dy,$$

and we get

$$\int_{B_2} \sqrt{1+|\nabla v^\rho|^2}\,dx \leqslant \int_{B_2+\rho} \sqrt{1+|\nabla v|^2}.$$

Consequently, for $\rho \leqslant \rho(\varepsilon, \delta)$ we have

$$\int_{B_2} \sqrt{1+|\nabla w_\rho|^2}\,dx \leqslant 2\varepsilon + \int_{B_2} \sqrt{1+|\nabla v|^2}. \tag{2.6}$$

Now we take $\varepsilon = 1/m$, $m \in \mathbb{N}$ and calculate $\delta_m \uparrow 1$ by (2.4). A sequence $\rho_m = \rho_m(\delta_m)$ is defined in such a way that (2.5) and (2.6) be true. Let $w_m := w_{\rho_m}$. Our construction implies that $w_m - u_0 \in C_0^\infty(B_1; \mathbb{R}^N)$, (2.5) gives $w_m \to w$ in $L^1(B_2; \mathbb{R}^N)$, and from (2.4) and (2.6) we deduce the inequality

$$\limsup_{m\to\infty} \int_{B_2} \sqrt{1+|\nabla w_m|^2}\,dx \leqslant \int_{B_2} \sqrt{1+|\nabla \widehat{w}|^2}.$$

Since the reverse inequality is a consequence of lower semicontinuity, this completes the proof of Lemma 2.1. •

§3. Proof of Theorem 1.4

Under our assumptions concerning the integrand f, we have the following statement.

Lemma 3.1. *The conjugate function f^* is essentially smooth, i.e., f^* is a proper convex function, and for $D := \operatorname{int}(\operatorname{dom} f^*)$ we have*

a) *D is nonempty.*

Moreover,

b) f^* *is differentiable everywhere in* D, *and*
c) $\lim_{i\to\infty} |\nabla f^*(Q_i)| = +\infty$ *whenever* $\{Q_i\}$ *is a sequence in* D *converging to a boundary point* Q *of* D.

Proof of Lemma 3.1. Since f is a convex and finite function on $\mathbb{R}^{nN}$, f is continuous (see [Ro, Corollary 10.1.1, p. 83]). For a proper convex function, closedness is the same as lower semicontinuity (see [Ro, p. 52]); in particular, f is closed. Now, f is a strictly convex function, whence, clearly, f is essentially strictly convex, which allows us to apply [Ro, Theorem 26.3, p. 253] to show that f^* is essentially smooth. •

Remark 3.1. From the linear growth condition (1.2) it follows that $\operatorname{dom} f^*$ is a bounded set (cf., e.g., [DT, 1.2]); moreover, we have $f^* \geqslant 0$ and $f^*(0) = 0$.

Let $L(c) := \{Q \in \mathbb{R}^{nN} : f^*(Q) \leqslant c\}$, $c \in \mathbb{R}$. We claim that the definition (1.5) of $\widetilde{J}$ can be rewritten as

$$\widetilde{J}[w] = \sup\left\{\widetilde{l}(w,\tau) : \tau \in C^\infty(\overline{\Omega};\mathbb{R}^{nN}),\ \tau(x) \in L(c) \text{ on } \overline{\Omega} \text{ for some } c = c(\tau) \in \mathbb{R}\right\},\quad w \in BV(\Omega;\mathbb{R}^N). \tag{3.1}$$

Let us fix some point $Q_0 \in D$ and a function $w \in BV(\Omega;\mathbb{R}^N)$. For $\varepsilon > 0$, we choose $\tau = \tau_\varepsilon \in \mathcal{U}$ such that

$$\widetilde{J}[w] - \widetilde{l}(w,\tau) < \varepsilon;$$

in particular, we may assume that $\tau(x) \in \operatorname{dom} f^*$ a.e. For a sequence $\lambda_k \in (0,1)$ satisfying $\lambda_k \to 1$ as $k \to \infty$, we put

$$\tau_k := (1-\lambda_k)Q_0 + \lambda_k \tau.$$

Since Q_0 belongs to the interior of $\operatorname{dom} f^*$, we can find an open ball B around Q_0 compactly included in D. Let C denote the union of all segments $\overline{P\tau(x)}$ with $P \in B$. Clearly, C is included in the convex set $\operatorname{dom} f^*$, and any point $Q \in \overline{Q_0\tau(x)}$ (different from $\tau(x)$ if $\tau(x) \in \partial \operatorname{dom} f^*$) belongs to the interior of $\operatorname{dom} f^*$. Therefore, $\tau_k(x) \in D$ for almost all x and any k. Next, we prove the existence of numbers $\gamma_k > 0$ such that

$$\operatorname{dist}(\tau_k, \partial \operatorname{dom} f^*) \geqslant \gamma_k \quad \text{for all } k \in \mathbb{N} \tag{3.2}$$

almost everywhere. Let

$$\varepsilon_k := \frac{1}{2}\min\{\operatorname{dist}(B, \partial \operatorname{dom} f^*), (1-\lambda_k)\operatorname{rad}(B)\}.$$

Case 1. $\operatorname{dist}(\tau(x), \partial \operatorname{dom} f^*) \leqslant \varepsilon_k$.

Then

$$\begin{aligned}\operatorname{dist}(\tau_k(x), \partial \operatorname{dom} f^*) &\geqslant |\tau_k(x) - \tau(x)| - \operatorname{dist}(\tau(x), \partial \operatorname{dom} f^*)\\ &\geqslant (1-\lambda_k)|\tau(x) - Q_0| - \varepsilon_k,\end{aligned}$$

and $|\tau(x) - Q_0| \geqslant \operatorname{rad}(B)$, because $\tau(x) \in B$ would imply

$$\operatorname{dist}(\tau(x), \partial \operatorname{dom} f^*) \geqslant \operatorname{dist}(B, \partial \operatorname{dom} f^*) > \varepsilon_k,$$

which contradicts our assumption in Case 1. Thus,

$$\operatorname{dist}\left(\tau_k(x), \partial \operatorname{dom} f^*\right) \geqslant (1-\lambda_k)\operatorname{rad}(B) - \varepsilon_k \geqslant \frac{1}{2}(1-\lambda_k)\operatorname{rad}(B).$$

Case 2. $\operatorname{dist}\left(\tau(x), \partial \operatorname{dom} f^*\right) \geqslant \varepsilon_k$.

By the choice of ε_k, the open ball B' of radius ε_k around Q_0 is included in D, and the same is true for the ball B'' of radius ε_k centered at $\tau(x)$. Let Z denote the union of all segments $\overline{PQ}$ with $P \in B'$, $Q \in B''$. Clearly, $Z \subset \operatorname{int}(\operatorname{dom} f^*)$, and it is immediate that the distance of the segment $\overline{Q_0\tau(x)}$ to $\partial \operatorname{dom} f^*$ is bounded from below by ε_k, whence $\operatorname{dist}\left(\tau_k(x), \partial \operatorname{dom} f^*\right) \geqslant \varepsilon_k$. This implies (3.2) with $\gamma_k = \varepsilon_k$.

We note that $K_k := \{Q \in \operatorname{dom} f^* : \operatorname{dist}(Q, \partial \operatorname{dom} f^*) \geqslant \gamma_k\}$ is a compact set included in $\operatorname{int}(\operatorname{dom} f^*)$. Therefore, the continuity of f^* on $\operatorname{int}(\operatorname{dom} f^*)$ implies the existence of a real number c_k such that $f^* \leqslant c_k$ on K_k, and from (3.2) we deduce that

$$\tau_k \in L(c_k) \quad \text{almost everywhere.} \tag{3.3}$$

From

$$f^*(\tau_k) \leqslant (1-\lambda_k) f^*(Q_0) + \lambda_k f^*(\tau)$$

and the choice of τ we get

$$\begin{aligned}\widetilde{l}(w, \tau_k) &\geqslant \lambda_k\, \widetilde{l}(w, \tau) + (1-\lambda_k)\left(-|\Omega| f^*(Q_0) + \int_\Omega Q_0 : \nabla u_0\, dx\right)\\ &\geqslant \lambda_k(\widetilde{J}[w] - \varepsilon) + (1-\lambda_k)\left(-|\Omega| f^*(Q_0) + \int_\Omega Q_0 : \nabla u_0\, dx\right).\end{aligned}$$

Thus,

$$\widetilde{J}\,[w] \leqslant 2\varepsilon + \widetilde{l}(w, \tau_k) \tag{3.4}$$

for all $k \gg 1$. We fix such an integer k. In order to verify our claim (3.1), we apply a modification of the approximation lemma of [FS] (Lemma A.1.1 therein) to the tensor $\sigma := \tau_k$. We need this because it is not clear that the construction provided in [FS] preserves condition (3.3).

Lemma 3.2. *Suppose that $\sigma \in \mathcal{U}$ satisfies $\sigma(x) \in L(c)$ for some $c \in \mathbb{R}$. Then there is a sequence $\sigma_m \in C^\infty(\overline{\Omega}; \mathbb{R}^N)$ such that*

i) $\sigma_m \to \sigma$ *a.e. and in $L^t(\Omega; \mathbb{R}^{nN})$ for all $t < \infty$;*

ii) $\operatorname{div} \sigma_m \to \operatorname{div} \sigma$ *in $L^n(\Omega; \mathbb{R}^N)$;*

iii) $\sigma_m \overset{*}{\rightharpoonup} \sigma$ *in $L^\infty(\Omega; \mathbb{R}^{nN})$;*

iv) $\sigma_m(x) \in L(c)$ *for all $x \in \overline{\Omega}$, $m \in \mathbb{N}$.*

Proof of Lemma 3.2. We use a construction borrowed from [A, p. 170]. Since $\partial\Omega$ is Lipschitz, we can cover $\partial\Omega$ by open sets $V_1, \dots, V_r$ such that, after rotation, V_j takes the form

$$V_j = \{x \in \mathbb{R}^n : |(x_1, \dots, x_{n-1})| < r_j, |x_n - g_j(x_1, \dots, x_{n-1})| < h_j\},$$

where g_j is a Lipschitz function. Moreover, we have

$$\begin{aligned} x_n = g_j(x_1, \dots, x_{n-1}) \quad & \Longrightarrow x \in \partial\Omega, \\ 0 < x_n - g_j(x_1, \dots, x_{n-1}) < h_j \quad & \Longrightarrow x \in \Omega, \\ 0 > x_n - g_j(x_1, \dots, x_{n-1}) > -h_j \quad & \Longrightarrow x \notin \Omega. \end{aligned}$$

Let V_0 denote an open set such that $\overline{V_0} \subset \Omega$ and

$$\overline{\Omega} \subset \bigcup_{j=0}^{r} V_j.$$

Finally, we consider a partition $\{\varphi_j\}$ of unity, i.e., $\varphi_j \in C_0^\infty(V_j)$, $0 \leqslant \varphi_j \leqslant 1$, and $\sum_{j=0}^r \varphi_j \equiv 1$ on $\overline{\Omega}$. Fixing an index $j \geqslant 1$, for $\delta \ll 1$ we put

$$\sigma_j^\delta(x) := \begin{cases} \sigma(x + \delta e_n)\varphi^j(x) & \text{if } x \in \Omega \cap V_j, \\ 0 & \text{if } x \in \Omega - V_j. \end{cases}$$

Observe that $\sigma_j^\delta \equiv 0$ near the upper boundary part of V_j, and the same is true near the "vertical boundary parts", which follows from the support properties of φ_j for a suitable choice of δ. If ω_ρ denotes a smoothing kernel, we let

$$\sigma^{\delta,\rho}(x) \;:=\; \omega_\rho * \Big[\varphi_0\sigma + \sum_{j=1}^{r} \sigma_j^\delta\Big](x), \; x \in \overline{\Omega}.$$

Assuming again the standard representation of the neighborhood V_j, we get

$$\omega_\rho * \sigma_j^\delta(x) = \int_{\mathbb{R}^n} \omega_\rho(y-x)\ \sigma(y+\delta e_n)\ \varphi_j(y)\,dy.$$

For sufficiently small ρ depending on δ, we see that for $y \in B_\rho(x), x \in \overline{\Omega}$, the point $y + \delta e_n$ belongs to Ω, and $\omega_\rho * \sigma_j^\delta(x)$ is well defined. Clearly, we have $\omega_\rho * \sigma_j^\delta \in C^\infty(\overline{\Omega}; \mathbb{R}^{nN})$,

$$\omega_\rho * \sigma_j^\delta \xrightarrow{\rho\downarrow 0} \sigma_j^\delta \ \text{in}\ L^p(\Omega;\mathbb{R}^{nN}) \quad \text{for all } p<\infty,$$

and, moreover (see [A, 1.16 Lemma, p. 18]),

$$\sigma_j^\delta \xrightarrow{\delta\downarrow 0} \sigma\varphi^j \ \text{ in } L^p(\Omega;\mathbb{R}^{nN}) \quad \text{for all } p<\infty.$$

Next, for $x \in \Omega$ we have

$$\operatorname{div}(\omega_\rho * \sigma_j^\delta)(x) = \int_{\mathbb{R}^n} \omega_\rho(y-x)\left[\operatorname{div}\sigma(y+\delta e_n)\,\varphi_j(y) + \sigma(y+\delta e_n)\nabla\varphi_j(y)\right] dy,$$

and, as above,

$$\operatorname{div}(\omega_\rho * \sigma_j^\delta) \xrightarrow{\rho\downarrow 0} \operatorname{div}\sigma(\cdot+\delta e_n)\varphi_j + \sigma(\cdot+\delta e_n)\nabla\varphi_j$$

in $L^n(\Omega;\mathbb{R}^N)$, As $\delta\downarrow 0$, the right-hand side converges to

$$\operatorname{div}\sigma\ \varphi_j + \sigma\ \nabla\varphi_j$$

in $L^n(\Omega;\mathbb{R}^N)$. So, if we first fix a sequence $\delta_m \downarrow 0$, we can find a sequence ρ_m depending on δ_m and possessing the convergence properties i) and ii) for $\sigma^m := \sigma^{\delta_m,\rho_m}$. The boundedness of $\|\sigma^m\|_{L^\infty(\Omega;\mathbb{R}^{nN})}$ implies that $\sigma^m \xrightarrow{*} \widetilde{\sigma}$ in $L^\infty(\Omega;\mathbb{R}^{nN})$ for a subsequence and some tensor $\widetilde{\sigma} \in L^\infty(\Omega;\mathbb{R}^{nN})$, but i) shows that $\widetilde{\sigma} = \sigma$. It remains to prove iv). Applying the Jensen inequality to the measure $\omega_\rho(x-\cdot)\mathcal{L}^n$, we get

$$f^*\left(\sigma^{\delta,\rho}(x)\right) \leqslant \int \omega_\rho(x-y) f^*\Big(\varphi_0\sigma + \sum_{j=1}^r \sigma_j^\delta\Big)(y)\,dy.$$

If we recall the definition of σ_j^δ, we see that f^* is evaluated on the convex combination

$$\varphi_0(y)\sigma(y) + \sum_{j=1}^r \varphi_j(y)\ \sigma(\ldots),$$

where $\sigma(\dots)$ has an obvious meaning for $j = 1, \dots, r$. Then our assumption $\sigma \in L(c)$ a.e. implies that

$$f^*\Big(\varphi_0\sigma + \sum_{j=1}^{r} \sigma_j^\delta\Big)(y) \leqslant c,$$

i.e., $\sigma^{\delta,\rho} \in L(c)$. •

Now we return to inequality (3.4). Lemma 3.2 gives the existence of a sequence $\{\tau_{m,k}\}_{m\in\mathbb{N}}$ in $C^\infty(\overline{\Omega};\mathbb{R}^{nN})$ with values in $L(c_k)$ and such that

$$(*)\quad \widetilde{l}(w,\tau_{m,k}) \xrightarrow{m\to\infty} \widetilde{l}(w,\tau_k).$$

In fact, in order to get $(*)$, we need to know that

$$\int_\Omega f^*(\tau_{m,k})\,dx \xrightarrow{m\to\infty} \int_\Omega f^*(\tau_k)\,dx,$$

but this follows from $\tau_{m,k} \to \tau_k$ a.e. together with the level-set property by quoting Lebesgue's dominated convergence theorem. Consequently, from (3.4) we deduce the inequality

$$\widetilde{J}[w] \leqslant 3\varepsilon + \widetilde{l}(w,\tau_{m,k})$$

at least for $m \gg 1$, and (3.1) is established.

Next, we extend the function $w \in BV(\Omega;\mathbb{R}^N)$ via u_0 to a function $\widehat{w}$ defined on $\widehat{\Omega}$, and consider a tensor $\sigma \in C_0^\infty(\widehat{\Omega};\mathbb{R}^{nN})$, $\sigma(x) \in L(c)$ for some $c \in \mathbb{R}$. Then

$$\int_{\widehat{\Omega}} \operatorname{div}\sigma\cdot(u_0-\widehat{w})\,dx - \int_\Omega f^*(\sigma)\,dx + \int_\Omega \sigma : \nabla u_0\,dx = \widetilde{l}(w,\sigma_{|\Omega}),$$

whence

$$\widetilde{J}[w] \geqslant \sup\Big\{\int_{\widehat{\Omega}} \operatorname{div}\sigma\cdot(u_0-\widehat{w})\,dx - \int_\Omega f^*(\sigma)\,dx + \int_\Omega \sigma : \nabla u_0\,dx : \sigma \in C_0^\infty(\widehat{\Omega};\mathbb{R}^{nN}),\ \sigma \in L(c) \text{ for some } c \in \mathbb{R}\Big\}.$$

Conversely, let $\tau \in C^\infty(\overline{\Omega};\mathbb{R}^{nN})$, $\tau \in L(c)$ for some c, be such that

$$\widetilde{J}[w] \leqslant \widetilde{l}(w,\tau) + \varepsilon.$$

A modification of Lemma 3.2 yields a sequence $\{\tau_m\} \in C_0^\infty(\widehat{\Omega};\mathbb{R}^{nN})$ for which $\tau_m \in L(c)$ and

$$\begin{cases} \tau_m \to \tau & \text{in } L^t(\Omega;\mathbb{R}^{nN}) \text{ and a.e. for all } t<\infty, \\ \operatorname{div}\tau_m \to \operatorname{div}\tau & \text{in any space } L^s(\Omega;\mathbb{R}^N),\ s<\infty, \\ \tau \overset{*}{\rightharpoondown} \tau & \text{in } L^\infty(\Omega;\mathbb{R}^{nN}). \end{cases}$$

To be precise, we use the notation from the proof of Lemma 3.2; in particular, we recall the definition of τ_j^δ, $j = 1,\dots,r$, which are now tensors of class C^∞. Clearly, the definition of $\tau_j^\delta(x)$ makes sense for the points x such that $x+\delta e_n \in \overline{\Omega}$, i.e.,

$$-\delta \leqslant x_n - g_j(x_1,\dots,x_{n-1}),$$

so that

$$\tau_j^\delta \in C^\infty\big(V_j \cap \big[-\delta \leqslant x_n - g_j(x_1,\dots,x_{n-1})\big]\big).$$

Let

$$\Psi_\delta^j(x) := \begin{cases} 1 & \text{if } x\in V_j,\ x_n - g_j(x_1,\dots,x_{n-1}) \geqslant 0, \\ 0 & \text{if } x\in V_j,\ x_n - g_j(x_1,\dots,x_{n-1}) \leqslant -\delta/4, \end{cases}$$

with suitable functions $\Psi_\delta^j \in C_0^\infty(\mathbb{R}^n)$, $0\leqslant \Psi_\delta^j \leqslant 1$. The function $\Psi_\delta^j\ \tau_j^\delta$ is of class $C_0^\infty(\widehat{\Omega};\mathbb{R}^{nN})$, and $\Psi_\delta^j\ \tau_j^\delta = \tau_j^\delta$ on Ω. Finally, we put

$$\tau^\delta := \varphi_0\tau + \sum_{j=1}^r \Psi_\delta^j\ \tau_j^\delta \in C_0^\infty(\widehat{\Omega};\mathbb{R}^{nN}).$$

Then, as $\delta\downarrow 0$, the desired convergence properties of τ^δ on Ω are immediate; moreover, we have $f^*(\tau^\delta) \leqslant \varphi_0 f^*(\tau) + \sum_{j=1}^r \Psi_\delta^j\varphi_j f^*(\tau(\dots)) \leqslant c$ on $\widehat{\Omega}$ (observe that $\varphi_0 + \sum_{j=1}^r \Psi_\delta^j\ \varphi_j \leqslant 1$ on $\widehat{\Omega}$, $f^*(0)=0$). As a consequence, we arrive at the formula

$$\widetilde{J}[w] = \sup\Big\{\int_{\widehat{\Omega}} \operatorname{div}\sigma\cdot(u_0-\widehat{w})\,dx - \int_\Omega f^*(\sigma)\,dx + \int_\Omega \sigma : \nabla u_0\,dx : \sigma\in C_0^\infty(\widehat{\Omega};\mathbb{R}^{nN}),$$
$$\sigma\in L(c) \text{ for some } c\in\mathbb{R}\Big\},$$

and integration by parts shows that, for any $w \in BV(\Omega;\mathbb{R}^N)$,

$$\widetilde{J}[w] = \sup\left\{\int_{\overline{\Omega}} \sigma : \nabla\widehat{w}\,dx - \int_{\Omega} f^*(\sigma)\,dx : \sigma \in C_0^\infty(\widehat{\Omega};\mathbb{R}^{nN}),\ \sigma \in L(c) \text{ for some } c \in \mathbb{R}\right\}. \tag{3.5}$$

Clearly, by a smoothing argument, in (3.5) the space $C_0^\infty(\widehat{\Omega};\mathbb{R}^{nN})$ can be replaced by $C_0^0(\widehat{\Omega};\mathbb{R}^{nN})$. We fix $w \in BV(\Omega;\mathbb{R}^N)$ and a tensor $\sigma \in C_0^0(\widehat{\Omega};\mathbb{R}^{nN})$ such that $\sigma \in L(c)$. Then $f^* \circ \sigma$ is of class $L^1(\widehat{\Omega})$. Conversely, assume that $f^* \circ \sigma$ is integrable on $\widehat{\Omega}$ (which implies $\sigma(x) \in \operatorname{dom} f^*$). Then, using the arguments presented after the proof of Lemma 3.1, we can construct tensors $\sigma_k := (1-\lambda_k)Q_0 + \lambda_k\sigma$ as before, which we multiply by some function $\eta \in C_0^0(\widehat{\Omega})$ such that $0 \leqslant \eta \leqslant 1$ and $\eta \equiv 1$ on a neighborhood of $\overline{\Omega}$. Then the tensors $\tau_k := \eta\sigma_k$ are of class $C_0^0(\widehat{\Omega};\mathbb{R}^N)$, and

$$f^*(\tau_k) \;=\; f^*\big((1-\eta)\,0 + \eta\,\sigma_k\big) \;\leqslant\; \eta\, f^*(\sigma_k) \;\leqslant\; c_k$$

provided $\sigma_k \in L(c_k)$. We have

$$\int_{\overline{\Omega}} \tau_k : \nabla\widehat{w} \;=\; \int_{\overline{\Omega}} \sigma_k : \nabla\widehat{w} \;\xrightarrow{k\to\infty}\; \int_{\overline{\Omega}} \sigma : \nabla\widehat{w}.$$

On Ω we estimate

$$0 \leqslant f^*(\tau_k) \;=\; f^*(\sigma_k) \;\leqslant\; (1-\lambda_k)f^*(Q_0) + \lambda_k f^*(\sigma) \;\leqslant\; \max\{f^*(Q_0), f^*(\sigma)\} \in L^1(\Omega).$$

Hence, we have

$$\int_{\Omega} f^*(\tau_k)\,dx \;\longrightarrow\; \int_{\Omega} f^*(\sigma)\,dx$$

by dominated convergence because $f^*(\tau_k) \to f^*(\sigma)$ a.e. on Ω. Altogether, we find

$$\int_{\overline{\Omega}} \sigma : \nabla\widehat{w} - \int_{\Omega} f^*(\sigma)\,dx = \lim_{k\to\infty}\left\{\int_{\overline{\Omega}} \tau_k : \nabla\widehat{w} - \int_{\Omega} f^*(\tau_k)\,dx\right\},$$

and (3.5) implies that

$$\begin{aligned}\widetilde{J}[w] &= \sup\left\{\int_{\widehat{\Omega}} \mathbf{1}_{\overline{\Omega}}\,\sigma : \nabla\widehat{w} - \int_{\Omega} f^*(\sigma)\,dx :\ \sigma \in C_0^0(\widehat{\Omega};\mathbb{R}^{nN}),\ f^* \circ \sigma \in L^1(\widehat{\Omega})\right\} \\ &\geqslant \sup\left\{\int_{\widehat{\Omega}} \mathbf{1}_{\overline{\Omega}}\,\sigma : \nabla\widehat{w} - \int_{\widehat{\Omega}} f^*(\sigma)\,dx :\ \sigma \in C_0^0(\widehat{\Omega};\mathbb{R}^{nN}),\ f^* \circ \sigma \in L^1(\widehat{\Omega})\right\}\end{aligned} \tag{3.6}$$

(the inequality is a consequence of $f^* \geqslant 0$). Consider a tensor σ that realizes the first supremum up to given $\varepsilon > 0$. Then we take $\eta_k \in C_0^0(\widehat{\Omega})$ with $0 \leqslant \eta_k \leqslant 1$ and such that $\eta_k \equiv 1$ on $\overline{\Omega}$ and $\eta_k \longrightarrow \mathbf{1}_{\overline{\Omega}}$. Let $\sigma_k := \eta_k \sigma$. Since

$$\int_{\widehat{\Omega}} \mathbf{1}_{\overline{\Omega}}\, \sigma_k : \nabla \widehat{w} = \int_{\widehat{\Omega}} \mathbf{1}_{\overline{\Omega}}\, \sigma : \nabla \widehat{w}, \quad \int_{\widehat{\Omega}} f^*(\sigma_k)\, dx \longrightarrow \int_{\Omega} f^*(\sigma)\, dx$$

(observe that $0 \leqslant f^*(\sigma_k) \leqslant \eta_k f^*(\sigma) \leqslant f^*(\sigma)$, $\sigma_k \to \mathbf{1}_{\overline{\Omega}}\sigma$), we get

$$\int_{\widehat{\Omega}} \mathbf{1}_{\overline{\Omega}}\, \sigma : \nabla \widehat{w} - \int_{\Omega} f^*(\sigma)\, dx \leqslant \varepsilon + \int_{\widehat{\Omega}} \mathbf{1}_{\overline{\Omega}}\, \sigma_k : \nabla \widehat{w} - \int_{\widehat{\Omega}} f^*(\sigma_k)\, dx$$

for $k \gg 1$, and the suprema in (3.6) coincide. This leads to the final representation formula

$$\widetilde{J}[w] = \sup \left\{ \int_{\widehat{\Omega}} \mathbf{1}_{\overline{\Omega}}\, \sigma : \nabla \widehat{w} - \int_{\widehat{\Omega}} f^*(\sigma)\, dx : \ \sigma \in C_0^0(\widehat{\Omega}; \mathbb{R}^N), f^* \circ \sigma \in L^1(\widehat{\Omega}) \right\}, \quad (3.7)$$

which is valid for any $w \in BV(\Omega; \mathbb{R}^N)$. Now the right-hand side of (3.7) can be identified with the help of [DT, Proposition 1.2]: this right-hand side is equal to

$$\int_{\overline{\Omega}} f(\nabla^a \widehat{w})\, dx + \int_{\overline{\Omega}} f_\infty \left(\frac{\nabla^s \widehat{w}}{|\nabla^s \widehat{w}|} \right) d|\nabla^s \widehat{w}|,$$

where the first integral is equal to $\int_\Omega f(\nabla^a w)\, dx$. The second integral may be decomposed as $\int_\Omega \ldots + \int_{\partial\Omega} \ldots$, and here the boundary integral is given by

$$\int_{\partial\Omega} f_\infty(-(w - u_0) \otimes \nu)\, d\, \mathcal{H}^{n-1},$$

(see [AFP, Theorem 3.77, p. 171]). Thus, $\widetilde{J}[w] = K[w]$ for all $w \in BV(\Omega; \mathbb{R}^N)$.

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