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О КОЛИЧЕСТВЕ АЛГЕБРАИЧЕСКИХ ЧИСЕЛ В КОРОТКИХ ИНТЕРВАЛАХ, СОДЕРЖАЩИХ РАЦИОНАЛЬНЫЕ ТОЧКИ

В. И. БЕРНИК[1]***, Ф. ГЁТЦЕ***[2]***, Н. И. КАЛОША***[1]

[1]*Институт математики НАН Беларуси, ул. Сурганова, 11, 220072, г. Минск, Беларусь*
[2]*Университет Билефельда, Университетсштрассе, 25, D-33615, г. Билефельд, Германия*

В 2012 г. доказано, что действительные алгебраические числа распределены неравномерно, но регулярно согласно определениям Г. Вейля (1916) и А. Бейкера, В. Шмидта (1970). Особенно неравномерно они распределены в окрестностях рациональных чисел с малыми знаменателями. В данной статье впервые перечислены условия, которым должны удовлетворять короткие интервалы, чтобы им принадлежало много действительных алгебраических чисел. При выполнении таких условий распределение алгебраических чисел приобретает черты регулярности, что уже предполагает наличие законов приближения трансцендентных чисел алгебраическими числами. Это, в свою очередь, дает шансы на доказательство гипотезы Вирзинга о приближении действительных чисел алгебраическими и целыми алгебраическими числами.

Ключевые слова: алгебраическое число; диофантовы приближения; равномерное распределение; теорема Дирихле; теорема Хинчина.

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Авторы:
Василий Иванович Берник – доктор физико-математических наук, профессор; главный научный сотрудник отдела теории чисел.
Фридрих Гётце – доктор наук (математика), профессор математики.
Николай Иванович Калоша – кандидат физико-математических наук; научный сотрудник отдела теории чисел.

Authors:
Vasily I. Bernik, doctor of science (physics and mathematics), full professor; chief researcher at the department of number theory.
bernik@im.bas-net.by
Friedrich Götze, doctor of science (mathematics), full professor.
goetze@math.uni-bielefeld.de
Nikolai I. Kalosha, PhD (physics and mathematics); researcher at the department of number theory.
kalosha@im.bas-net.by

COUNTING ALGEBRAIC NUMBERS IN SHORT INTERVALS WITH RATIONAL POINTS

V. I. BERNIK[a], *F. GÖTZE*[b], *N. I. KALOSHA*[a]

[a]*Institute of Mathematics, National Academy of Sciences of Belarus, 11 Surhanava Street, Minsk 220072, Belarus*
[b]*Bielefeld University, 25 Universitätsstraße, Bielefeld D-33615, Germany*
Corresponding author: N. I. Kalosha (kalosha@im.bas-net.by)

In 2012 it was proved that real algebraic numbers follow a non-uniform but regular distribution, where the respective definitions go back to H. Weyl (1916) and A. Baker and W. Schmidt (1970). The largest deviations from the uniform distribution occur in neighborhoods of rational numbers with small denominators. In this article the authors are first to specify a general condition that guarantees the presence of a large quantity of real algebraic numbers in a small interval. Under this condition, the distribution of real algebraic numbers attains even stronger regularity properties, indicating that there is a chance of proving Wirsing's conjecture on approximation of real numbers by algebraic numbers and algebraic integers.

Key words: algebraic number; Diophantine approximation; uniform distribution; Dirichlet's theorem; Khinchine's theorem.

Introduction

A sequence of real numbers can satisfy a number of properties related to the evenness of its distribution. The simplest property of this type is everywhere density. A more restrictive property – uniform distribution – was defined by Weyl [1; 2], who proved the eponymous uniform distribution criterion. Weyl used this criterion to prove that the sequence $\{\alpha n\}$, $n = 1, 2, \ldots$, where curly braces denote the fractional parts, is uniformly distributed on the segment $[0, 1)$ if and only if α is irrational.

The uniform distribution property is often too restrictive, which motivated Baker and Schmidt to introduce the concept of a regular distribution [3]. A regular sequence of numbers has the following property: for an arbitrary interval, we can choose sufficiently many numbers that lie in that interval from the first N members of sequence, and these numbers also satisfy a natural lower bound on the minimal distance between them.

Baker and Schmidt proved the regularity of the set of real algebraic numbers and found an exact lower bound for the Hausdorff measure of the set of real numbers with a given order of approximation by algebraic numbers [3; 4].

Regularity was instrumental towards proving analogues of Khinchine's theorem [5] for polynomials [6; 7], as well as nondegenerate curves and surfaces [8–10].

Preliminaries and methodology

We are going to show that the regularity property can be generalized for small intervals. We are going to consider first K numbers of a sequence $\alpha_1, \alpha_2, \ldots, \alpha_K$ lying in the interval S of length $\mu S = K^{-l}$, $l > 0$. Our results are a natural generalization of the papers [11; 12].

Let $\deg P = n$ be the degree and $H = H(P) = \max_{0 \le j \le n} |a_j|$ be the height of a polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \in \mathbb{Z}[x], \ a_n \ne 0.$$

For a sufficiently large Q, we define the class of polynomials

$$\mathcal{P}_n(Q) = \{P \in \mathbb{Z}[x] : \deg P \le n, H(P) \le Q\}.$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of $P(x)$. The constants $c_1, c_2, \ldots$ are assumed to depend on n but not on H or Q; $\#A$ denotes the cardinality of a finite set A; μB is the Lebesque measure of a set $B \subset \mathbb{R}$. The set of all roots of the polynomials $P(x) \in \mathcal{P}_n(Q)$ will be denoted as $T_n(Q)$. Clearly, we have $\#\mathcal{P}_n(Q) \le (2Q+1)^{n+1}$, and $\#T_n(Q) \le n(2Q+1)^{n+1}$. A well-known result from [3], which is obvious for an odd degree n, states that $\#T_n(Q) \cap \mathbb{R} > c_1 Q^{n+1}$. The paper [12] proves that real algebraic numbers, ordered so that the height of the respective minimal polynomials is monotone nondecreasing, are uniformly distributed only if $n = 1$, i. e., if they are rational.

We are going to study the distribution of the numbers in the set $T_n(Q) \cap [0, 1)$ on short intervals $I \subset [0, 1)$, $\mu I = Q^{-\gamma}$, $\gamma > 1$. Any finite interval $[a, b)$ can be substituted for $[0, 1)$, but the formulas will become somewhat more complicated. Note that some of the theorems proved below can be generalized for the case where we replace polynomials with nondegenerate functions [8–10; 13].

Distribution of algebraic numbers in short intervals of the same length can vary dramatically. A recent article [11] proves that:

a) there exist intervals I_1 of length $\mu I_1 = 0.5Q^{-1}$ such that $T_n(Q) \cap I_1 = \varnothing$ for an arbitrary n;

b) if the constant c_2 is sufficiently large, then for any interval I_2, $\mu I_2 > c_2 Q^{-1}$, there exists a constant $c_3 > 0$ such that

$$\#T_n(Q) \cap I_2 > c_3 Q^{n+1} \mu I_2.$$

The statement a) can be proved by using rather basic properties of integer polynomials and their roots. However, the proof of the statement b) is more complex and is based on several recent theorems proved in metric Diophantine approximation [11].

It is easy to see that the number of intervals of type I_1 which do not contain algebraic numbers isn't large. Indeed, from [3] we know that

$$\#T_n(Q) \cap [0, 1) > c_5 Q^{n+1}.$$

Discussion and results, conclusion

In this paper we specify the condition on the intervals I of length $\mu I = Q^{-\gamma_1}$, where $\gamma_1 > 1$, which ensures that these intervals contain algebraic numbers from the class $T_n(Q)$. A similar condition for $\gamma_1 = \frac{3}{2}$ was proved in [14].

From Minkowski's theorem on linear forms [15] we have that for all $x \in [0, 1)$ and $Q > 1$ there exists an integer polynomial $P(x) \in \mathcal{P}_n(Q)$ such that

$$|P(x)| < 2(n+1)Q^{-n}.$$

The exponent $-n$ in the right-hand side of this inequality is optimal since for $x_1 = 2^{-\frac{1}{n+1}}$ we have $|P(x_1)| > c_5' Q^{-n}$. From Sprindžuk's theorem [16] we have that the inequality $|P(x)| < Q^{-w}$, $w > n$, can be satisfied only if $x \in B_1 \subset [0, 1)$, $\mu B_1 < \varepsilon_1$ for all $\varepsilon_1 > 0$. Therefore, if the set B_2 consists of points $x \in [0, 1)$ satisfying the condition $|P_k(x)| < c_5' Q^{-k-1}$ and Q is sufficiently large, then $\mu B_2 < \varepsilon_2$, $\varepsilon_2 > 0$. Moreover, there is a known upper bound for this measure: $\mu B_2 < c_6 Q^{-\frac{1}{n}}$.

Let us prove several upper bounds for the quantity $\#T_n(Q) \cap I$.

Theorem 1. *For* $\mu I_2 = Q^{-\gamma_2}$, $0 \le \gamma_2 < 1$ *we have*

$$\#T_n(Q) \cap I_2 < n^2 2^{n+5} Q^{n+1} \mu I_2.$$

P r o o f. Let us assume the opposite. Taking $c_7 = n^2 2^{n+5}$, we have

$$\#T_n(Q) \cap I_2 > c_7 Q^{n+1-\gamma_2}.$$

Take $\overline{b}_1 = (a_n, \ldots, a_1)$, i. e., a vector formed from the coefficients of $P(x)$. Since $\#\{\overline{b}_1\} = (2Q+1)^n < 2^{n+1}Q^n$ for $Q > Q_0$, we can choose at least $l_1 = c_7 2^{-n-1} Q^{1-\gamma_2}$ polynomials with the same vector $\overline{b}_1$ in the class of polynomials $P(x) \in \mathcal{P}_n(Q)$ with roots $\alpha_1 \in T_n(Q) \cap I_2$. The differences obtained by subtracting these polynomials from each other are nonzero integers. If α_{1i} is a root of $P_i(x) \in \mathcal{P}_n(Q)$ lying in the interval I_2, then a Tailor expansion of $P_i(x)$ on I_2 yields that

$$P_i(x) = P_i(\alpha_{1i}) + P_i'(\alpha_{1i})(x - \alpha_{1i}) + \frac{1}{2}P_i''(\alpha_{1i})(x - \alpha_{1i})^2 + \ldots,\ 1 \le i \le l_1, \tag{1}$$

$$P_i(\alpha_{1i}) = 0,\ |P_i'(\alpha_{1i})(x - \alpha_{1i})| < n^2 Q^{1-\gamma_2},$$

where $Q > Q_0$. The absolute values of the remaining terms of the expansion (1) are bounded from above by $nQ^{1-\gamma_2}$, allowing us to write

$$\left|P_i(x)\right| < 2n^2Q^{1-\gamma_2}.$$

Consider the differences

$$R_j(x) = P_{j+1}(x) - P_1(x),\ 1 \le j \le l_1 - 1.$$

The polynomials $R_j(x)$ are different nonzero integers satisfying the inequality

$$\left|R_j(x)\right| < 2n^2Q^{1-\gamma_2}.$$

For a sufficiently large l_1, one of them must be larger than $2n^2Q^{1-\gamma_2}$ in absolute value, which contradicts the previous inequality.

Sprindzhuk showed [16] that metric theorems on integer polynomials also hold if we go from $P(x)$ to the polynomials $P_1(x) = P(x-m)$, $m \in \mathbb{Z}$, or to $P_2(x) = x^nP\left(\frac{1}{x}\right)$. Using these substitutions, we can transition from arbitrary polynomials to ones satisfying the condition

$$|a_n| > c_8H(P). \tag{2}$$

The condition (2) makes it easy to prove that for all roots α_i, $1 \le i \le n$, of the polynomial $P(x)$ we have $\left|\alpha_j\right| < c_9$ [7; 16]. From now on, we can assume that the polynomials $P(x) \in \mathcal{P}_n(Q)$ satisfy the condition (2), and that all roots of these polynomials are bounded in absolute value by a constant that depends on n but doesn't depend on H or Q.

Let us introduce a classification of our intervals. An interval I of length $|I| = Q^{-\gamma_1}$ is called a (k, v)-interval if it contains a real algebraic number β_1 of degree $\deg\beta_1 = k < n$ and height $H(\beta_1) \le Q^v$, $0 \le v \le 1$.

Theorem 2. *For $\gamma_1 > k + nv$, (k, v)-intervals I contain no real algebraic points α_1, $\deg\alpha_1 = n$, $H(\alpha_1) \le Q$.*

P r o o f. Let $T_1(x) = b_kx^k + \ldots + b_1x + b_0$ denote the minimal polynomial of an algebraic number β_1, and $T_2(x) = a_nx^n + \ldots + a_1x + a_0$ be the minimal polynomial of α_1. The polynomials $T_1(x)$ and $T_2(x)$ have no common roots and their leading coefficients satisfy (2). Therefore, the roots $\beta_1, \ldots, \beta_k$ of the polynomials $T_1(x)$, as well as the roots $\alpha_1, \ldots, \alpha_n$ of $T_2(x)$, are bounded in absolute value by a certain constant c_9. This implies that their resultant is nonzero, $R(T_1, T_2) \ne 0$. The root $\beta_1 \in I$ defines the type of the interval I. Let $\alpha_1 \in I$, then

$$1 \le \left|R(T_1, T_2)\right| = a_n^kb_k^n \prod_{1\le i\le n, 1\le j\le k} \left(\alpha_i - \beta_j\right) \ll_n Q^{k+nv}\left|\alpha_1 - \beta_1\right| \ll_n Q^{k+nv-\gamma_1}. \tag{3}$$

By the conditions of the theorem, the exponent in the right-hand side of (3) is negative, and the inequality (3) is contradictory for a sufficiently large Q.

If $\gamma_1 < k + nv$, then the interval I_1 can contain algebraic numbers, however, there is an upper bound on their quantity.

Theorem 3. *For $\gamma_3 > 1 + v$, an interval I_3 of length $|I_3| = Q^{-\gamma_3}$ contains at most $2^{n+7}Q^{n+1-\gamma_3}$ algebraic numbers β_1, $\deg\beta_1 = n$, $H(\beta_1) \le Q$.*

P r o o f. Assume the opposite, i. e.,

$$\#\left\{\beta_1 \in I_3 : P(\beta_1) = 0,\ P(x) \in \mathcal{P}_n(Q)\right\} > 2^{n+7}Q^{n+1-\gamma_3}.$$

Let the vector $\overline{b}_2 = (a_n, \ldots, a_2)$ made up from the coefficients of $P(x) \in \mathcal{P}_n(Q)$ be fixed. Clearly, we have

$$\#\left\{\overline{b}_2\right\} = (2Q+1)^{n-1} < 2^nQ^{n-1},\ Q > Q_0(n).$$

Thus at least $l_2 = 2^7Q^{2-\gamma_3}$ polynomials have an identical vector $\overline{b}_2$. For all of these polynomials $P_j(x)$, $1 \le j \le l_2$, let us write Tailor expansions on the interval I_3 with respect to the root $\beta_1 = \beta_{1j}$, and let us estimate $P(x)$ from above.

$$P_j(x) = P(\beta_1) + P'(\beta_1)(x-\beta_1) + \frac{1}{2}P''(\beta_1)(x-\beta_1)^2 + \ldots + \frac{1}{n!}P^{(n)}(\beta_1)(x-\beta_1)^n,$$

$$\left|P_j(x)\right| \le n^2Q^{1-\gamma_3} + n\frac{1}{n}Q^{1-\gamma_3} < 2n^2Q^{1-\gamma_3},\ Q > Q_0.$$

Consider the differences

$$R_j(x) = P_{j+1}(x) - P_1(x),\ 1 \le j \le l_2 - 1 \ge \frac{l_2}{2} = 2^6 Q^{2-\gamma_3}. \tag{4}$$

The number of different polynomials in (4) is at least $2^6 Q^{2-\gamma_3}$. We also have $\deg R_j \le 1$, $H(R_j) \le 2Q$, and

$$|a_j x + b_j| = |R_j(x)| < 4n^2 Q^{1-\gamma_3},\ 1 \le j \le l_2 - 1 \ge \frac{l_2}{2} = 2^6 Q^{2-\gamma_3}. \tag{5}$$

From (5) we have

$$\left| x + \frac{b_j}{a_j} \right| < 4n^2 Q^{1-\gamma_3} |a_j|^{-1} \le 4n^2 Q^{1-\gamma_3}.$$

If among $2^6 Q^{2-\gamma_3}$ polynomials $R_j(x) = a_j x + b_j$ there exists at least one such that its root $-\frac{b_i}{a_i}$ is different from $\beta_1 = -\frac{b_0}{a_0}$, let us consider the resultant of this polynomial $a_i x + b_i$ and the polynomial $R_0 = a_0 x + b_0$ that defines the type of the interval I_3.

$$1 \le |R(R_0, R_j)| = \left| a_0 a_i \left(\frac{b_i}{a_i} - \frac{b_0}{a_0} \right) \right| \le 8n^2 Q^{1+v-\gamma_3}. \tag{6}$$

The inequality (6) is contradictory for $\gamma_3 > 1 + v$ and $Q > Q_0$. If all linear polynomials $R_j(x)$ are different but have the same root $-\frac{b_0}{a_0}$, they can be written as

$$k(a_0 x + b_0),\ k \in \mathbb{Z}.$$

Since $|a_0| > 0.5Q^v$, and we can take k to be larger than $k_0 = \frac{l_2}{8}$, we must also have

$$|a_0 k_0| > 8Q^{v+2-\gamma_3},$$

$$|a_0 k_0| < 4Q,$$

which is impossible for $\gamma_3 > 1 + v$. This concludes the proof.

Consider an interval I of length $|I| = Q^{-\gamma}$ which isn't a $(1, v)$-interval. Let $S(I)$ denote the number of algebraic numbers β of degree $\deg \beta = n$ and height $H(\beta) \le Q$ lying in the interval I.

Theorem 4. *The number $S(I)$ can be estimated from below as*

$$S(I) \ge cQ^{n+1-\gamma}$$

for some constant $c = c(n)$.

To prove theorem 4, we are going to use the following lemmas.

Lemma 1. *Let α_1 be the root of the polynomial $P(x)$ closest to x. Then for $P'(x) \ne 0$ and $P'(\alpha_1) \ne 0$ we have*

$$|x - \alpha_1| \le n|P(x)||P'(x)|^{-1},$$

$$|x - \alpha_1| \le 2^{n-1}|P(x)||P(\alpha_1)|^{-1}.$$

Lemma 1 is quite well-known [7; 16].

Lemma 2. *Let $P_1(x)$ and $P_2(x)$ be integer polynomials without common roots such that*

$$\deg P_1 \le n,\ \deg P_2 \le n,\ H(P_1) \le Q,\ H(P_2) \le Q,$$

and on an interval J, $\mu J = Q^{-\eta}$, $\eta > 0$, we have the inequalities

$$\max(|P_1(x)|,\ |P_2(x)|) < Q^{-\tau},$$

where $\tau > 0$. Then for any $\delta > 0$ and a sufficiently large $Q > Q_0(\delta)$ we have the inequality

$$\tau + 1 + 2\max(\tau + 1 - \eta, 0) < 2n + \delta.$$

A proof of lemma 2 can be found in [6].

Let $P_1(x), \ldots, P_m(x) \in \mathcal{P}_n(Q)$ denote polynomials with roots in the interval I. Let us order their roots as follows [16]:

$$|\alpha_1(P) - \alpha_2(P)| \le |\alpha_1(P) - \alpha_3(P)| \le \ldots \le |\alpha_1(P) - \alpha_n(P)|.$$

Let $\varepsilon > 0$ and $\varepsilon_1 = \dfrac{\varepsilon}{100}$. For $\varepsilon_1 > 0$ consider an ε_1-sieve for the quantities $|\alpha_1(P) - \alpha_j(P)|$, $j = 2, 3, \ldots, n$, denoting $T = \left[\varepsilon_1^{-1}\right] + 1$:

$$|\alpha_1(P) - \alpha_j(P)| = Q^{-\rho_j(P)}, \ \frac{l_j - 1}{N} < \rho_j \le \frac{l_j}{T}, \ l_j \in \mathbb{Z}, \ j = 2, 3, \ldots, n.$$

Clearly, the number of vectors $\overline{l} = (l_2, l_3, \ldots, l_n)$ is finite, depends only on ε_1 and doesn't depend on Q. Take $m_1 = c_4 m$ polynomials $P_j(x)$, and for each of them denote

$$\sigma(P) = \left\{x \in I : |P(x)| < Q^{-n}\right\}.$$

The set $\sigma(P)$ consists of at most n intervals of length

$$c_4 Q^{-n-1+p_1(P)}, \ p_1(P) = \frac{l_2 + l_3 + \ldots + l_n}{T}.$$

If

$$m_1 < c_5 Q^{n+1-p_1-\gamma},$$

then the measure of the set

$$B = I \cap \bigcup_{P_j, 1 \le j \le n} \sigma(P)$$

is larger than $0.5|I|$. Since we have

$$|P(x)| < Q^{-n}, \ p_1 < \varepsilon_1,$$

the lemma above and the inequality $|x - \alpha_1| < c_6 Q^{-n-1+\varepsilon_1}$ imply the existence of an algebraic number $\alpha_1 = \alpha_1(P)$. We can construct at least $c_7 Q^{n+1-\gamma-\varepsilon_1}$ such algebraic numbers $\alpha_1(P_j)$ in the interval I.

Now let $m_1 > c_5 Q^{-n-1+\varepsilon_1}$. Assume that

$$0 \le \{m_1\} < 0.5\varepsilon_1,$$

where $\{m_1\}$ denotes the fractional part of the number m_1, and the integer part is denoted as $[m_1]$.

From Dirichlet's principle, if the constant c_5 is sufficiently large, then the interval I contains at least $m_2 = c_8 Q^{n+1-\gamma-p_1}$ polynomials $P_j(x) \in \mathcal{P}_n(Q)$ with identical leading coefficients $a_n, a_{n-1}, \ldots, a_{n-[m_1]}$. Consider Taylor expansions of the polynomials $P_j(x)$ in the interval I with respect to the roots α_{1j}:

$$P_j(x) = P_j(\alpha_{1j}) + P_j'(\alpha_{1j})(x - \alpha_{1j}) + \frac{1}{2}P''(\xi_j)(x - \alpha_{1j})^2, \ \xi_j \in (x, \alpha_{1j}).$$

Since

$$P_j(\alpha_{1j}) = 0, \ |P_j'(\alpha_{1j})(x - \alpha_{1j})| < c_9 Q^{1-\gamma-p_1+(n-1)\varepsilon_1},$$

$$\left|\frac{1}{2}P''(\xi_j)(x - \alpha_{1j})^2\right| < c_{10} Q^{1-2\gamma},$$

for all $x \in I$ we have

$$|P_j(x)| < c_{11} Q^{1-p_1-\gamma+(n-1)\varepsilon_1}.$$

Consider $m_1 - 1$ differences $R_j(x)$ formed by subtracting $P_1(x)$ from the rest of the polynomials $P_j(x)$. Then $R_j(x)$ satisfy the inequalities:

$$|R_{j+1}(x)| = |P_{j+1}(x) - P_1(x)| < 2c_{11} Q^{1-p_1-\gamma+(n-1)\varepsilon_1},$$

$$\deg R_{j+1} \le n - [m_1] = p_1 + \gamma - 1.$$

There exist at least $Q^{\varepsilon/2}$ such polynomials $R_{j+1}(x)$. Consider the resultant of $R_{j+1}(x)$ and the polynomial $T(x)$:

$$1 \le \left|R\left(R_{j+1}, T\right)\right| < c_{12} Q^{\lambda(p_1 + \gamma - 1) + n_1 - \gamma - p_1}. \tag{7}$$

If

$$\gamma + p_1 > \lambda\left(p_1 + \gamma - 1\right) + n_1, \tag{8}$$

then the inequality (7) is contradictory.

For $n_1 = 1$, the inequality (8) can be written as

$$\gamma + p_1 > \lambda\left(p_1 + \gamma - 1\right) + 1. \tag{9}$$

Since we have $0 \le \lambda < 1$ and $p_1 \ge 0$, the inequality (9) follows from the inequality $\gamma \ge \lambda(\gamma - 1) + 1$, which holds if $1 \le \gamma < 2$. Thus the assumption $m_1 > cQ^{n+1-p_1-\gamma}$ is contradictory, which proves theorem 4.

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