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Ramification filtration and differential forms

V. A. Abrashkin

Abstract. Let L be a complete discrete valuation field of prime characteristic p with finite residue field. Denote by $\Gamma_L^{(v)}$ the ramification subgroups of $\Gamma_L = \mathrm{Gal}(L^{\mathrm{sep}}/L)$. We consider the category $\mathrm{M}\Gamma_L^{\mathrm{Lie}}$ of finite $\mathbb{Z}_p[\Gamma_L]$-modules H, satisfying some additional (Lie)-condition on the image of Γ_L in $\mathrm{Aut}_{\mathbb{Z}_p} H$. In the paper it is proved that all information about the images of the groups $\Gamma_L^{(v)}$ in $\mathrm{Aut}_{\mathbb{Z}_p} H$ can be explicitly extracted from some differential forms $\widetilde{\Omega}[N]$ on the Fontaine etale ϕ-module $M(H)$ associated with H. The forms $\widetilde{\Omega}[N]$ are completely determined by a canonical connection ∇ on $M(H)$. In the case of fields L of mixed characteristic, which contain a primitive pth root of unity, we show that a similar problem for $\mathbb{F}_p[\Gamma_L]$-modules also admits a solution. In this case we use the field-of-norms functor to construct the corresponding ϕ-module together with the action of the Galois group of a cyclic extension L_1 of L of degree p. Then our solution involves the characteristic p part (provided by the field-of-norms functor) and the condition for a "good" lift of a generator of $\mathrm{Gal}(L_1/L)$. Apart from the above differential forms the statement of this condition uses the power series coming from the p-adic period of the formal group $\mathbb{G}_m$.

Keywords: local field, Galois group, ramification filtration.

Introduction

Let L be a complete discrete valuation field with finite residue field of characteristic p. Let Γ_L be the absolute Galois group of L. Let $\{\Gamma_L^{(v)}\}_{v\geqslant 0}$ be the filtration of Γ_L by the ramification subgroups, [1]. This filtration provides Γ_L with additional structure and allows us to introduce various classes of infinite field extensions (arithmetically profinite, deeply ramified etc.), which play an important role in modern arithmetic algebraic geometry. For Γ_L-modules H, the evaluation of $v_0(H)\in\mathbb{Q}$ such that $\Gamma_L^{(v)}$ act trivially on H for $v > v_0(H)$, provides us with good estimates for discriminants of the fields of definition of $h\in H$. Such estimates are used very often to answer various number theoretic questions.

However, an explicit description of the structure of the ramification filtration for a very long time was known only at the level of the Galois groups of abelian field extensions. At the time when the structure of the Galois group Γ_L was completely described (the case of the maximal p-extensions – Shafarevich [2], Demushkin [3], and the general case – Janssen–Wingberg [4]) it became clear that Γ_L is a very

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weak invariant of the field L. The situation cardinally changed later when it was established (Mochizuki [5], author [6]) that taking $\{\Gamma_L^{(v)}\}_{v\geqslant 0}$ into account gives us absolute invariant of the field L. However, in order to work with this invariant we still need to know an explicit description of the filtration by the groups $\Gamma_L^{(v)}$.

I. R. Shafarevich always paid attention to this problem, for example, cf. Introduction to [7]. His motivation was the following: for every prime number p there is only one such filtration and we know almost nothing about its structure. In 1990's the author developed a nilpotent version of the Artin–Schreier theory and obtained an explicit description of the ramification filtration modulo the subgroup of pth commutators of Γ_L. Such description was obtained, first, in the characteristic p case, [8]–[10], and then developed in the mixed characteristic case, [11]–[13]. These results play a crucial role in this paper, where we study the images of the ramification subgroups $\Gamma_L^{(v)}$ in the group of automorphisms $\mathrm{Aut}_{\mathbb{Z}_p} H$ of finite $\mathbb{Z}_p[\Gamma_L]$-modules H. Our main result states that this arithmetic structure can be completely described (under some additional condition) in purely geometric properties of Fontaine's etale ϕ-modules $M(H)$.

More precisely, if $\mathrm{char}\, L = p$ we construct differential forms $\widetilde{\Omega}[N]$, $N\in\mathbb{N}$, on an extension of scalars of $M(H)$, and specify the way how the image of the ramification filtration in $\mathrm{Aut}_{\mathbb{Z}_p} H$ can be recovered from these forms. Note that the definition of $\widetilde{\Omega}[N]$ depends only on a natural connection constructed on $M(H)$. If $\mathrm{char}\, L = 0$ we assume that L contains a pth root of unity $\zeta_1\neq 1$ and restrict ourselves to the case of Galois $\mathbb{F}_p$-modules. Then we apply the field-of-norms functor to reduce the situation to the characteristic p case and use a characterization of "good" lifts of automorphisms of our cyclic field extension of L from [11], [12]. This characterization uses again the differential forms $\widetilde{\Omega}[N]$ and a power series coming from the p-adic period of the formal group $\mathbb{G}_m$.

By our opinion this result establishes quite interesting link between the Galois theory of local fields and very popular area of D-modules, lifts of Frobenius, Higgs vector bundles etc.

§ 1. Statement of the main result

1.1. General notation. Everywhere in the paper p is a fixed prime number. If E_0 is a field then E_0^{sep} is its separable closure in some algebraic closure E_0^{alg} of E_0. If E is a field such that $E_0\subset E\subset E_0^{\mathrm{sep}}$ set $\Gamma_E=\mathrm{Gal}(E_0^{\mathrm{sep}}/E)$. The field E_0^{sep} will be considered as a left Γ_{E_0}-module, i. e., for any $\tau_1,\tau_2\in\Gamma_{E_0}$ and $o\in E_0^{\mathrm{sep}}$, $(\tau_1\tau_2)o=\tau_1(\tau_2 o)$. If $\mathrm{char}\, E_0=p$ we set for any $a\in E_0^{\mathrm{sep}}$, $\sigma(a)=a^p$.

If V is a module over a ring R then $\mathrm{End}_R V$ is the R-algebra of R-linear endomorphisms of V. We always consider V as a left $\mathrm{End}_R V$-module, i. e., if $l_1,l_2\in\mathrm{End}_R V$ and $v\in V$ then $(l_1l_2)v=l_1(l_2(v))$. We also consider $\mathrm{End}_R V$ as a Lie R-algebra with the Lie bracket $[l_1,l_2]=l_1l_2-l_2l_1$. If S is an R-module then we often denote by V_S the extension of scalars $V\otimes_R S$.

1.2. Functorial system of lifts to characteristic 0. Suppose $K_0=k_0((t_0))$ is the field of formal Laurent series in a (fixed) variable t_0 with coefficients in a finite field k_0 of characteristic p. The uniformiser t_0 provides a p-basis for any field extension E of K_0 in K_0^{sep}, i. e., the set $\{1,t_0,\dots,t_0^{p-1}\}$ is a E^p-basis of E. We use this p-basis to construct a compatible system of lifts $O(E)$ of the fields E to characteristic 0. This is a special case of the construction of lifts from [14]; it can be explained as follows.

For all $M\in\mathbb{N}$, set $O_M(E)=W_M(\sigma^{M-1}E)[\bar t_0]$, where $\bar t_0=[t_0]$ is the Teichmuller representative of t_0 in the ring of Witt vectors $W_M(E)$. The rings $O_M(E)$ are the lifts of E modulo p^M, i. e., they are flat $\mathbb{Z}/p^M$-algebras such that $O_M(E)\otimes_{\mathbb{Z}/p^M}\mathbb{Z}/p=E$. Note that the system

$$\{O_M(E)\mid M\in\mathbb{N},\, K_0\subset E\subset K_0^{\mathrm{sep}}\}$$

is functorial in M and E. In particular, if E/K_0 is Galois then there is a natural action of $\mathrm{Gal}(E/K_0)$ on $O_M(E)$ and $O_M(E)^{\mathrm{Gal}(E/K_0)}=O_M(K_0)$. The morphisms $W_M(\sigma)$ induce σ-linear morphisms on $O_M(E)$ which will be denoted again by σ. In particular, $\sigma(\bar t_0)=\bar t_0^{\,p}$.

Introduce the lifts of the above fields E to characteristic 0 by setting $O(E)=\varprojlim_M O_M(E)$. Then $O_M(E)=O(E)/p^M$ and $O(E)[1/p]$ is a complete discrete valuation field with uniformiser p and the residue field E. Clearly, we have the induced morphism σ on each $O(E)$. Also, if E/K_0 is Galois then $\mathrm{Gal}(E/K_0)$ acts on $O(E)$ and $O(E)^{\mathrm{Gal}(E/K_0)}=O(K_0)$. Notice that $O(K_0)=\varprojlim_M W_M(k_0)((\bar t_0))$ is the completion of the ring of formal Laurent series $W(k)((\bar t_0))$.

Set $O_{\mathrm{sep}}=O(K_0^{\mathrm{sep}})$.

The system of lifts $O(E)$, $E\subset K_0^{\mathrm{sep}}$, can be extended to the system of lifts of all extensions of K_0 in K_0^{alg}. Indeed, note that $K_0^{\mathrm{alg}}=\bigcup_{n\geqslant 0}K_0(t_n)^{\mathrm{sep}}$, where $t_n^{p^n}=t_0$. Then t_n gives the p-basis $\{1,t_n,\dots,t_n^{p-1}\}$ for all separable extensions E of $K_0(t_n)$ in K_0^{alg} and we obtain (as earlier) the corresponding lifts $O(E)$. The system of lifts $O(E)$ is functorial in $E\subset K_0^{\mathrm{alg}}$ (use that any separable extension E_n of $K_0(t_n)$ appears uniquely as the composite $E_0K_0(t_n)$, where E_0/K_0 is separable). In particular, consider $K_0^{\mathrm{rad}}=\bigcup_{N\in\mathbb{N}}K_0^{\mathrm{ur}}(t_0^{1/N})$. Then for the above defined lift of K_0^{rad} we have $O(K_0^{\mathrm{rad}})=\bigcup_{N\in\mathbb{N}}O(K_0^{\mathrm{ur}})[\bar t_0^{1/N}]$, where $K_0^{\mathrm{ur}}=\bar k_0((t_0))$ is the maximal unramified extension of K_0.

1.3. Equivalence of the categories of p-groups and Lie algebras [15]. Let L be a finitely generated Lie $\mathbb{Z}_p$-algebra of nilpotent class $<p$, i. e., the ideal of pth commutators $C_p(L)$ of L is equal to zero. Let A be an enveloping algebra of L. Then the elements of $L\subset A$ generate the augmentation ideal J of A. There is a morphism of $\mathbb{Z}_p$-algebras $\Delta\colon A\to A\otimes A$ uniquely determined by the condition $\Delta(l)=l\otimes 1+1\otimes l$ for all $l\in L$. Then the set $\exp(L)\mod J^p$ is identified with the group of all "diagonal elements modulo degree p" consisting of $a\in 1+J\mod J^p$ such that $\Delta(a)\equiv a\otimes a\mod (J\otimes 1+1\otimes J)^p$.

In particular, there is a natural embedding $L\subset A/J^p$ and the identity

$$\exp(l_1)\cdot\exp(l_2)\equiv\exp(l_1\circ l_2)\mod J^p$$

induces the Campbell–Hausdorff composition law

$$(l_1,l_2)\mapsto l_1\circ l_2=l_1+l_2+\frac{1}{2}[l_1,l_2]+\cdots,\qquad l_1,l_2\in L.$$

This composition law provides the set L with a group structure. We denote this group by $G(L)$. Clearly $G(L)\simeq \exp(L) \mod J^p$.

With the above notation the functor $L\mapsto G(L)$ determines the equivalence of the categories of finitely generated $\mathbb{Z}_p$-Lie algebras and profinite p-groups of nilpotence class $< p$. Note that a subset $I\subset L$ is an ideal in L if and only if $G(I)$ is a normal subgroup in $G(L)$.

1.4. Lie-condition. For any finite field extension K of K_0 in K_0^{sep}, let $\mathrm{M}\Gamma_K$ be the category of finitely generated $\mathbb{Z}_p$-modules H with continuous left action of Γ_K. Each element $h\in H$ is defined over some finite extension $K(h)$ of K. In some sense the family of these fields determines "arithmetic" properties of H. More detailed information about the fields $K(h)$ can be obtained from the knowledge of the images of the ramification subgroups in upper numbering $\Gamma_K^{(v)}$, $v>0$, in $\mathrm{Aut}_{\mathbb{Z}_p} H$. For example, the minimal number $v_0(H)\in\mathbb{Q}$ such that all $\Gamma_K^{(v)}$ with $v>v_0(H)$ act trivially on H provides us with upper estimates for the discriminants of the fields of definition of $h\in H$ (cf. [6]).

Let $H_0\in \mathrm{M}\Gamma_{K_0}$ and let $\pi_{H_0}\colon \Gamma_{K_0}\to \mathrm{Aut}_{\mathbb{Z}_p} H_0$ be the group homomorphism which determines the Γ_{K_0}-module structure on H_0. Consider the full subcategory $\mathrm{M}\Gamma_{K_0}^{\mathrm{Lie}}$ in $\mathrm{M}\Gamma_{K_0}$ which consists of modules H_0 satisfying the following condition.

Condition (Lie). The image $I(H_0):=\pi_{H_0}(\mathcal{I})\subset \mathrm{Aut}_{\mathbb{Z}_p}(H_0)$ of the wild inertia subgroup $\mathcal{I}\subset\Gamma_{K_0}$ appears in the form $\exp(L(H_0))$, where $L(H_0)\subset \mathrm{End}_{\mathbb{Z}_p} H_0$ is a Lie subalgebra such that $L(H_0)^p=0$.

The condition $L(H_0)^p=\{l_1\cdots l_p \mid l_1,\dots,l_p\in L(H_0)\}=0$ (the product is taken in $\mathrm{End}_{\mathbb{Z}_p} H_0\supset L(H_0)$) implies that $L(H_0)$ is a finitely generated nilpotent $\mathbb{Z}_p$-algebra Lie of nilpotence class $<p$. This gives the group isomorphism $\exp\colon G(L(H_0))\simeq I(H_0)$. Note that any normal subgroup of $I(H_0)$ appears in the form $\exp G(J)$, where J is a Lie ideal of $L(H_0)$.

Remark 1.1. If $pH_0=0$ and $\dim_{\mathbb{F}_p} H_0\leqslant p$ then H_0 is of the Lie type.

1.5. The first main result: the characteristic p case. Suppose $H_0\in\mathrm{M}\Gamma_{K_0}^{\mathrm{Lie}}$. Our target is to determine for all $v>0$, the images $\pi_{H_0}(\Gamma_{K_0}^{(v)})$ of the ramification subgroups $\Gamma_{K_0}^{(v)}$ via an explicit construction of the ideals $L(H_0)^{(v)}\subset L(H_0)$ such that $\exp(L(H_0)^{(v)})=\pi_{H_0}(\Gamma_{K_0}^{(v)})$.

Our approach uses Fontaine's "analytical" description of the Galois modules $H_0\in \mathrm{M}\Gamma_{K_0}^{\mathrm{Lie}}$ in terms of etale $(\phi, O(K_0))$-modules $M(H_0)$. A geometric nature of $M(H_0)$ is supported by the existence of an analogue of the classical connection $\nabla\colon M(H_0)\to M(H_0)\otimes_{O(K_0)}\Omega^1_{O(K_0)}$ (cf. [16]). (This map is uniquely characterized by the condition $\nabla\phi=(\phi\otimes\phi)\cdot\nabla$.) The required information about the behaviour of ramification subgroups can be then extracted from some differential forms

$$\widetilde{\Omega}[N]\in M(H_0)_{O(K_0^{\mathrm{rad}})}\otimes_{O(K_0)}\Omega^1_{O(K_0)}.$$

The construction of these differential forms is given completely in terms of the above connection ∇ and can be explained as follows.

Let $K\subset K_0^{\mathrm{sep}}$ be a fixed tamely ramified finite extension of K_0 such that $\pi_{H_0}(\Gamma_K)=I(H_0)$. Then $H:=H_0|_{\Gamma_K}$ can be described via an etale $(\phi, O(K))$-module $M(H)=M(H_0)\otimes_{O(K_0)} O(K)$. Recall that $M(H)=(H\otimes_{\mathbb{Z}_p} O_{\mathrm{sep}})^{\Gamma_K}$ is a finitely generated $O(K)$-module with a σ-linear morphism $\phi\colon M(H)\to M(H)$ such that the image $\phi(M(H))$ generates $M(H)$ over $O(K)$. This allows us to identify the elements of H with a set of O_{sep}-solutions of a suitable system of equations with coefficients in $O(K)$.

We establish below the construction of $M(H)$ by introducing a $\mathbb{Z}_p$-linear embedding $\mathcal{F}\colon H\to M(H)$ which induces by extension of scalars the identification $H_{O(K)}\simeq M(H)$. (We denote this identification by the same symbol $\mathcal{F}$.)

Now let $\widetilde{B}$ be (a inique) $O(K)$-linear operator on $M(H)$ such that for any $m\in\mathcal{F}(H)$, $\nabla(m)=\widetilde{B}(m)\,d\bar{t}/\bar{t}$. Then for every $N\in\mathbb{Z}_{\geqslant 0}$ we introduce the differential forms

$$\widetilde{\Omega}[N]=\phi^N\widetilde{B}\phi^{-N}\,\frac{d\bar{t}}{\bar{t}}\in \operatorname{End}M(H)_{O(K^{\mathrm{rad}})}\otimes_{O(K)}\Omega^1_{O(K)}.$$

Now we can use the identification $\mathcal{F}\colon H_{O(K)}\simeq M(H)$ to obtain the corresponding differential forms $\Omega[N]$ on $\operatorname{End}(H)_{O(K^{\mathrm{rad}})}$ and to verify that

$$\Omega[N]\in L(H)_{O(K^{\mathrm{rad}})}\otimes_{O(K)}\Omega^1_{O(K)}=L(H_0)_{O(K_0^{\mathrm{rad}})}\otimes_{O(K_0)}\Omega^1_{O(K_0)}.$$

Remark 1.2. Our differential forms will usually appear in the form $\Omega=F\cdot d\bar{t}_0/\bar{t}_0$, where $F\in L(H)_{K_0^{\mathrm{rad}}}$. Then we set by definition

$$(\mathrm{id}_{L(H)}\otimes\sigma)\Omega=(\mathrm{id}_{L(H)}\otimes\sigma)F\cdot\frac{d\bar{t}_0}{\bar{t}_0}.$$

Our first main result can be stated as follows.

Theorem 1.1. *Suppose $H_0\in\mathrm{M}\Gamma^{\mathrm{Lie}}_{K_0}$ is finite. Then there is $N_0(H_0)\in\mathbb{N}$ such that for any (fixed) $N\geqslant N_0(H_0)$ the following property holds*:

if $(\mathrm{id}_{L(H)}\otimes\sigma^{-N})\Omega[N]=\sum_{r\in\mathbb{Q}}\bar{t}_0^{\,-r}l_r\,d\bar{t}_0/\bar{t}_0$, where all $l_r\in L(H_0)_{W(\bar{k}_0)}$, then the ideal $L(H_0)^{(v)}$ is the minimal ideal in $L(H_0)$ such that for all $r\geqslant v$, $l_r\in L(H_0)^{(v)}_{W(\bar{k}_0)}$.

Corollary 1.1. *If $v_0(H_0)=\max\{r\mid l_r\neq 0\}$ then the ramification subgroups $\Gamma^{(v)}_{K_0}$ act trivially on H_0 if and only if $v>v_0(H_0)$.*

Remark 1.3. The construction of $\Omega[N]$ almost does not depend on the choice of the tamely ramified finite field extension K of K_0. It depends essentially on the choice of the uniformising element t_0 in K_0 and a compatible system of $\alpha(k)\in W(k)$, where $[k:k_0]<\infty$, such that the trace of $\alpha(k)$ in the field extension $W(k)[1/p]/K_0$ equals 1.

Remark 1.4. If H_0 is not p-torsion Theorem 1.1 can be applied to the factors H_0/p^M and our result describes the structure of the images of $L(H_0)^{(v)}$ in all $L(H_0)/p^M$.

1.6. The second main result: the mixed characteristic case. Let E_0 be a finite field extension of $\mathbb{Q}_p$ with residue field k_0 and a uniformizing element π_0. Assume that E_0 contains a pth primitive root of unity ζ_1. Consider the category $\mathrm{M}\Gamma^{\mathrm{Lie}}_{E_0,1}$ of finitely generated $\mathbb{F}_p[\Gamma_{E_0}]$-modules which satisfy a direct analog of the Lie condition from § 1.4.

Take the infinite arithmetically profinite field extension $\widetilde{E}_0$ obtained from E_0 by joining all p-power roots of π_0. Then the theory of the field-of-norms functor X provides us with the complete discrete valuation field of characteristic p, $X(\widetilde{E}_0)=K_0$, which has the same residue field and the uniformizing element t_0 obtained from the p-power roots of π_0. The functor X also provides us with the identification of Galois groups $\Gamma_{K_0}=\Gamma_{\widetilde{E}_0}\subset\Gamma_{E_0}$.

If $H_{E_0}\in \mathrm{M}\Gamma^{\mathrm{Lie}}_{E_0,1}$ then we obtain $H_0:=H_{E_0}|_{\Gamma_{K_0}}\in \mathrm{M}\Gamma^{\mathrm{Lie}}_{K_0}$. As earlier, take a finite tamely ramified extension K of K_0 (it corresponds to a unique tamely ramified extension E of E_0 with uniformizer π such that $\pi^{e_0}=\pi_0$, where e_0 is the ramification index of E/E_0), and construct $(\phi,O(K))$-module $M(H)$. This module inherits the action of $\mathrm{Gal}(E(\sqrt[p]{\pi})/E)=\langle\tau_0\rangle^{\mathbb{Z}/p}$. (Here τ_0 is such that $\tau_0(\sqrt[p]{\pi})=\zeta_1\sqrt[p]{\pi}$.) This situation was considered in all details in the papers [11], [12]. In particular, in those papers we gave a characterization of the "good" lifts $\widehat{\tau}_0$ of τ_0. By definition, $\widehat{\tau}_0\in\Gamma_E$ is "good" if its restriction to H_{E_0} belongs to the image of the ramification subgroup $\Gamma_E^{(e^*)}$. Here $e^*:=pe/(p-1)$ and $e=e(E/\mathbb{Q}_p)$ is the ramification index of $E/\mathbb{Q}_p$. (This makes sense because $\tau_0\in\mathrm{Gal}(E(\sqrt[p]{\pi}/E)^{(e^*)})$.) Note that the field-of-norms functor is compatible with ramification filtrations on Γ_{E_0} and Γ_{K_0}. Therefore, the knowledge of "good lifts" $\widehat{\tau}_0$ together with Theorem 1.1 gives a complete description of the image of the ramification filtration of Γ_{E_0} in $\mathrm{Aut}\,H_{E_0}$.

In [12] we proved that the action of $\widehat{\tau}_0$ appears from an action of a formal group scheme of order p. As a result, the lift $\widehat{\tau}_0$ is completely determined by the value $d\widehat{\tau}_0(0)\in L(H)$ of its differential at 0, and we can use the characterization of differentials of "good" lifts from [12], Theorem 5.1.

Namely, let us first specify our pth root of unity

$$\zeta_1=1+\sum_{j\geqslant 0}[\beta_j]\pi^{(e^*/p)+j}\mod p$$

(here all $[\beta_j]$ are the Teichmuller representatives of elements from the residue field of E). Then we introduce the power series $\omega(t)\in O(K)$ such that

$$1+\sum_{j\geqslant 0}\beta_j^p t^{e^*+pj}=\widetilde{\exp}(\omega(t)^p)$$

(here $\widetilde{\exp}$ is the truncated exponential). The series $\omega(t)^p$ is a kind of approximation of the p-adic period of the formal multiplicative group, which appears usually in explicit formulas for the Hilbert symbol (e.g., [17]). In other words, we obtain another geometric condition characterizing "arithmetic" of the Γ_{E_0}-module H_{E_0}.

Theorem 1.2. *The lift $\widehat{\tau}_0$ is "good" if and only if*

$$d\widehat{\tau}_0(0)\equiv\sum_{m\geqslant 0}\mathrm{Res}\big(\omega(t)^{p^{m+1}}\Omega[m]\big)\mod L(H)_k^{(e^*)}.$$

Remark 1.5. Notice that the power series $\omega(t)^p$ has non-zero coefficients only for the powers t^{e^*+pj} and all these exponents $\geqslant e^*$. Therefore, the differential forms $\Omega[m]$ contribute to the right-hand side only via the images of $\mathcal{F}^0_{e^*+pj,-m}t^{-(e^*+pj)}$

in $L(H)_k$. But for $m \gg 0$, these images belong to the images of the ramification ideals $\mathcal{L}_k^{(e^*)}$ and, therefore, disappear modulo $L(H)_k^{(e^*)}$, and the sum in the right-hand side is, as a matter of fact, finite.

§ 2. ϕ-module $M(H)$

2.1. Specification of $\log \pi_H : \Gamma_K \to G(L(H))$. As earlier, $H_0 \in \mathrm{M}\Gamma_{K_0}^{\mathrm{Lie}}$, K is a finite tamely ramified extension of K_0 in K_0^{sep} such that $\pi_{H_0}(\Gamma_K) = I(H_0)$, $H = H_0|_{\Gamma_K}$. Set $L(H) = L(H_0)$, $\pi_H = \pi_{H_0}|_{\Gamma_K}$.

Consider the continuous group epimorphism $l_H := \log(\pi_H) \colon \Gamma_K \to G(L(H))$. Since the p-group $G(L(H))$ has nilpotence class $< p$ this epimorphism can be described in terms of the covariant version of the nilpotent Artin–Schreier theory from [9]. Namely, there are $e \in L(H)_{O(K)}$ and $f \in L(H)_{O_{\mathrm{sep}}}$ such that $(\mathrm{id}_{L(H)} \otimes \sigma)(f) = e \circ f$ and for any $\tau \in \Gamma_K$, $l_H(\tau) = (-f) \circ (\mathrm{id}_{L(H)} \otimes \tau) f$.

It could be easily verified that l_H is a group homomorphism. Indeed,

$$\begin{aligned} l_H(\tau_1\tau_2) &= (-f) \circ (\mathrm{id}_{L(H)} \otimes \tau_1\tau_2) f \\ &= (-f) \circ (\mathrm{id}_{L(H)} \otimes \tau_1) f \circ (-f) \circ (\mathrm{id}_{L(H)} \otimes \tau_2) f = l_H(\tau_1) \circ l_H(\tau_2), \end{aligned}$$

because $(\mathrm{id}_{L(H)} \otimes \tau_1) l_H(\tau_2) = l_H(\tau_2)$.

Notation. We will use below the following notation: $\sigma_H = \mathrm{id}_H \otimes \sigma$ and $\sigma_{L(H)} = \mathrm{id}_{L(H)} \otimes \sigma$. For example, if $u = \sum_\alpha h_\alpha \otimes o_\alpha$, where all $h_\alpha \in H$ and $o_\alpha \in O(K)$ then $\sigma_H(u) = \sum_\alpha h_\alpha \otimes \sigma(o_\alpha)$. Or, if X is a linear operator on $L(H)_{O(K)}$ then $\sigma_{L(H)} X$ is also a linear operator such that

$$\sigma_{L(H)} X \left(\sum_\alpha h_\alpha \otimes o_\alpha \right) = \sum_\alpha \sigma_H(X(h_\alpha))(1 \otimes o_\alpha).$$

In addition, $\mathcal{X} := X \cdot \sigma_H$ is a unique σ-linear operator such that $\mathcal{X}|_H = X|_H$, and we have the following identity: $\sigma_H \cdot X = \sigma_{LH}(X) \cdot \sigma_H$. If there is no risk of confusion we will use just the notation σ.

Remark 2.1. Originally we developed in [9] the contravariant version of the nilpotent Artin–Schreier theory, cf. the discussion in [18]. The contravariant version uses similar relations $\sigma_{L(H)}(f) = f \circ e$ and the map l_H was defined via $\tau \mapsto (\mathrm{id}_{L(H)} \otimes \tau) f \circ (-f)$. In this case l_H determines the group homomorphism from Γ_K to the opposite group $G^0(L(H))$ (this group is isomorphic to $G(L(H))$ via the map $g \mapsto g^{-1}$). The results from the papers [8]–[10] were obtained in terms of the contravariant version, but the results from [11]–[13], [19] used the covariant version. We can easily switch from one theory to another via the automorphism $-\mathrm{id}_{L(H)}$.

We consider O_{sep} as a left Γ_K-module via the action $o \mapsto \tau(o)$ with $o \in O_{\mathrm{sep}}$ and $\tau \in \Gamma_K$. This corresponds to our earlier agreement about the left action of the elements of Γ_K as endomoprhisms of O_{sep}. As a result we obtain the left Γ_K-module structure on $H_{O_{\mathrm{sep}}}$ by the use of the (left) action of Γ_K on H via $h \mapsto l_H(\tau)(h)$.

Because $L(H)_{O_{\mathrm{sep}}} \subset \mathrm{End}_{O_{\mathrm{sep}}}(H_{O_{\mathrm{sep}}})$ we can introduce for any $h \in H$, $\mathcal{F}(h) := \exp(-f)(h) \in H_{O_{\mathrm{sep}}}$.

Proposition 2.1. *For any $h \in H$, $\mathcal{F}(h) \in (H_{O_{\mathrm{sep}}})^{\Gamma_K}$.*

Proof. Suppose $f=\sum_\alpha l_\alpha\otimes o_\alpha$, where all $l_\alpha\in L(H)$ and $o_\alpha\in O_{\mathrm{sep}}$.

If $\tau\in\Gamma_K$ then

$$\begin{aligned}\tau(\mathcal{F}(h))&=(\tau\otimes\mathrm{id}_{O_{\mathrm{sep}}})\Big(\exp\Big(-\sum_\alpha l_\alpha\otimes\tau(o_\alpha)\Big)(h)\Big)\\&=(\tau\otimes\mathrm{id}_{O_{\mathrm{sep}}})\big(\exp\big((\mathrm{id}_{L(H)}\otimes\tau)(-f)\big)(h)\big)\\&=(\tau\otimes\mathrm{id}_{O_{\mathrm{sep}}})\big(\exp\big((-l_H(\tau))\circ(-f)\big)(h)\big)\\&=(\tau\otimes\mathrm{id}_{O_{\mathrm{sep}}})\big(\big(\exp(-l_H(\tau))\cdot\exp(-f)\big)h\big)\\&=(\tau\otimes\mathrm{id}_{O_{\mathrm{sep}}})(\pi_H(\tau^{-1})\otimes\mathrm{id}_{O_{\mathrm{sep}}})\mathcal{F}(h)\\&=\big(\tau\cdot\pi_H(\tau^{-1})\otimes\mathrm{id}_{O_{\mathrm{sep}}}\big)\mathcal{F}(h)=\mathcal{F}(h).\end{aligned}$$

The proposition is proved.

The elements e and f are not determined uniquely by l_H. A pair $e'\in L(H)_{O(K)}$ and $f'\in L(H)_{O_{\mathrm{sep}}}$ give the same group epimorphism l_H if and only if there is $x\in L(H)_{O(K)}$ such that $e'=\sigma(x)\circ e\circ(-x)$ and $f'=x\circ f$.

2.2. Special choice of $e\in L(H)_{O(K)}$. We can always assume (by replacing, if necessary, K by its finite unramified extension) that a uniformiser t in K is such that $t^{e_0}=t_0$, where e_0 is the ramification index for K/K_0. Then $O(K)=\varprojlim_M W_M(k)((\bar t))$, where $\bar t^{\,e_0}=\bar t_0$ and $\bar t$ is the Teichmuller representative of t. We denote by k the residue field of K and fix a choice of $\alpha_0=\alpha(k)\in W(k)$ such that its trace in the field extension $W(k)[1/p]/\mathbb{Q}_p$ equals 1.

Let $\mathbb{Z}^+(p):=\{a\in\mathbb{N}\mid\gcd(a,p)=1\}$ and $\mathbb{Z}^0(p)=\mathbb{Z}^+(p)\cup\{0\}$.

Definition 2.1. An element $e\in L(H)_{O(K)}$ is *special* if $e=\sum_{a\in\mathbb{Z}^0(p)}\bar t^{\,-a}l_{a0}$, where $l_{00}\in\alpha_0L(H)$ and for all $a\in\mathbb{Z}^+(p)$, $l_{a0}\in L(H)_{W(k)}$.

Lemma 2.1. *Suppose $e\in L(H)_{O(K)}$. Then there is $x\in L(H)_{O(K)}$ such that $\sigma(x)\circ e\circ(-x)$ is special.*

Proof. Use induction on s to prove lemma modulo the ideals of sth commutators $C_s(L(H)_{O(K)})$.

If $s=1$ there is nothing to prove.

Suppose lemma is proved modulo $C_s(L(H)_{O(K)})$.

Then there is $x\in L(H)_{O(K)}$ such that

$$\sigma(x)\circ e\circ(-x)=\sum_{a\in\mathbb{Z}^0(p)}\bar t^{\,-a}l_{a0}+l,$$

where $l\in C_s(L(H)_{O(K)})$. Using that

$$O(K)=(\sigma-\mathrm{id}_{O(K)})O(K)\oplus(\mathbb{Z}_p\alpha_0)\oplus\Bigg(\sum_{a\in\mathbb{Z}^+(p)}W(k)\bar t^{\,-a}\Bigg),\tag{2.1}$$

we obtain the existence of $x_s\in C_s(L(H)_{O(K)})$ such that

$$l=\sigma(x_s)-x_s+\sum_{a\in\mathbb{Z}^0(p)}\bar t^{\,-a}l_a,$$

where $l_0 \in \alpha_0 L$ and all remaining $l_a \in L_{W(k)}$. Then we can take $x' = x - x_s$ to obtain our statement modulo $C_{s+1}(L(H)_{O(K)})$.

Lemma 2.2. *Suppose $e \in L_{O(K)}$ is special and $x \in L_{O(K)}$. Then the element $\sigma(x) \circ e \circ (-x)$ is special if and only if $x \in L$ (or, equivalently, if $\sigma x = x$).*

Proof. Use relation (cf. [20])

$$\mathrm{Ad}(\exp(X))\exp(Y) = \exp\Bigg(\sum_{n\geqslant 0} \frac{1}{n!}\,\mathrm{ad}^n(X)(Y)\Bigg) \ \mathrm{mod}(\deg p),$$

where $\mathrm{Ad}(U)(V) = UVU^{-1}$ and $\mathrm{ad}(U)(V) = [U,V]$. Indeed, if $X = x \in L(H)$ and $Y = e$ then $\sum_{n\geqslant 0} \mathrm{ad}^n(x)(e)/n!$ is also special.

When proving the inverse statement we can use induction modulo the ideals $C_s(L(H))_{O(K)}$, $s \geqslant 1$, as follows.

Assume the lemma is proved modulo $C_s(L(H)_{O(K)})$. Then using the *if* part we can assume that $x \in C_s(L(H)_{O(K)})$. Therefore, $e + \sigma(x) - x$ is special modulo $C_{s+1}(L(H))_{O(K)}$, i. e.,

$$\sigma(x) - x \in \alpha_0 C_s(L) + \sum_{a\in\mathbb{Z}^+(p)} t^{-a} C_s(L)_{W(k)}$$

modulo $C_{s+1}(L(H)_{O(K)})$. By (2.1) this implies the congruence

$$\sigma(x) \equiv x \mod C_{s+1}(L(H))_{O(K)},$$

i. e., $x \in C_s(L(H)) \mod C_{s+1}(L(H)_{O(K)})$.

The lemma is proved.

2.3. Construction of the ϕ-module $M(H)$. Note that $\pi_H = \exp(l_H)$, and therefore for all $\tau \in \Gamma_K$, it holds

$$\pi_H(\tau) = \exp(-f)\cdot\exp(\mathrm{id}_{L(H)} \otimes \tau)f,$$

where $f \in L(H)_{O_{\mathrm{sep}}} \subset \mathrm{End}_{O_{\mathrm{sep}}}(H_{O_{\mathrm{sep}}})$, $\sigma_{L(H)}(f) = e \circ f$ and $\exp f \in \mathrm{Aut}_{O_{\mathrm{sep}}}(H_{O_{\mathrm{sep}}})$.

Let $\mathrm{MF}_K^{\mathrm{et}}$ be the category of etale ϕ-modules over $O(K)$. Recall that its objects are $O(K)$-modules of finite type M together with a σ-linear morphism $\phi\colon M \to M$ such that its $O(K)$-linear extension $\phi_{O(K)}\colon M \otimes_{O(K),\sigma} O(K) \to M$ is isomorphism. The correspondence $H \mapsto M(H) := (H \otimes_{\mathbb{Z}_p} O_{\mathrm{sep}})^{\Gamma_K}$, where $\phi\colon M(H) \to M(H)$ comes from the action of σ on O_{sep}, determines the equivalence of the categories $\mathrm{M}\Gamma_K$ and $\mathrm{MF}_K^{\mathrm{et}}$.

Consider the $\mathbb{Z}_p$-linear embedding $\mathcal{F}\colon H \to H_{O_{\mathrm{sep}}}$ from § 2.1. Let $M(H) = \mathcal{F}(H)_{O(K)}$. By extension of scalars we obtain natural isomorphisms (use that $O(K)$ and O_{sep} are flat $\mathbb{Z}_p$-modules):

$$\mathcal{F} \otimes \mathrm{id}_{O_{\mathrm{sep}}}\colon H_{O_{\mathrm{sep}}} \simeq M(H)_{O_{\mathrm{sep}}}, \qquad \mathcal{F} \otimes \mathrm{id}_{O(K)}\colon H_{O(K)} \simeq M(H),$$

which will be denoted for simplicity just by $\mathcal{F}$.

Note that by Proposition 2.1, $M(H) = (H_{O_{\mathrm{sep}}})^{\Gamma_K}$.

The $O(K)$-module $M(H)$ is provided with the σ-linear morphism $\phi\colon M(H) \to M(H)$ uniquely determined for all $h \in H$ via

$$\phi(\mathcal{F}(h)) = \exp(-\sigma_{L(H)} f)(h) = (\exp(-f)\cdot\exp(-e))(h) = \mathcal{F}(\exp(-e)(h)).$$

Consider the $O(K)$-linear operator

$$A := \exp(-e) \in \exp(L(H)_{O(K)}) \subset \mathrm{Aut}_{O(K)} H_{O(K)}.$$

Then $\mathcal{A} := A \cdot \sigma_H$ will be a unique σ-linear operator on $L(H)_{O(K)}$ such that $\mathcal{A}|_H = A|_H$. Clearly, for any $u \in H_{O(K)}$,

$$\phi(\mathcal{F}(u)) = \mathcal{F}(\mathcal{A}(u)),$$

and $M(H)$ is etale ϕ-module associated with the $\mathbb{Z}_p[\Gamma_K]$-module H.

For example, suppose $pH = 0$ and $\{h_i \mid 1 \leqslant i \leqslant N\}$ is $\mathbb{F}_p$-basis of H. Then $\{\mathcal{F}(h_i) \mid 1 \leqslant i \leqslant N\}$ is a K-basis for $M(H)$. If $A(h_i) = \exp(-e)(h_i) = \sum_j a_{ij} h_j$ with all $a_{ij} \in K$ then $\phi(\mathcal{F}(h_i)) = \sum_j a_{ij}\mathcal{F}(h_j)$, and $((a_{ij}))$ appears as the corresponding "Frobenius matrix".

It can be easily seen also that if $\{h_i \mid 1 \leqslant i \leqslant N\}$ is a minimal system of $\mathbb{Z}_p$-generators in H then $\{\mathcal{F}(h_i) \mid 1 \leqslant i \leqslant N\}$ is a minimal system of $O(K)$-generators in $M(H)$.

2.4. The connection ∇ on $M(H)$. The $(\phi, O(K))$-module $M(H)$ can be provided with a connection $\nabla\colon M(H) \to M(H) \otimes_{O(K)} \Omega^1_{O(K)}$, [16]. This is an additive map uniquely determined by the properties:

a) for any $o \in O(K)$ and $m \in M(H)$, $\nabla(mo) = \nabla(m)o + m \otimes d(o)$;

b) $\nabla \cdot \phi = (\phi \otimes \phi) \cdot \nabla$.

By a), ∇ is uniquely determined by its restriction to $\mathcal{F}(H)$ (use that $M(H) = \mathcal{F}(H)_{O(K)}$). Let $\widetilde{B}$ be a unique $O(K)$-linear operator on $M(H)$ such that for any $m \in \mathcal{F}(H)$, $\nabla(m) = \widetilde{B}(m)\, d\bar{t}/\bar{t}$. Consider the $O(K)$-linear operator $B \in \mathrm{End}\, H_{O(K)}$ such that for all $u \in H_{O(K)}$, $\widetilde{B}(\mathcal{F}(u)) = \mathcal{F}(B(u))$. Obviously, $\widetilde{B}$ and B can be recovered one from another.

Note that for any $\mathbb{Z}_p$-module $\mathcal{C}$, the elements $c \in \mathcal{C}_{O(K)}$ appear uniquely in the form $c = \sum_n c_n \otimes \bar{t}^{\,n}$ with all $c_n \in \mathcal{C}_{W(k)}$. Therefore, the map $\mathrm{id}_{\mathcal{C}} \otimes \partial_{\bar{t}}\colon \mathcal{C}_{O(K)} \to \mathcal{C}_{O(K)}$ such that $c \mapsto \sum_n c_n \otimes n\bar{t}^{\,n}$ is well-defined. If $\mathcal{C} \subset \mathcal{C}_1$ is an embedding of $\mathbb{Z}_p$-modules then we have $(\mathrm{id}_{\mathcal{C}_1} \otimes \partial_{\bar{t}})|_{\mathcal{C}_{O(K)}} = \mathrm{id}_{\mathcal{C}} \otimes \partial_{\bar{t}}$.

With the above notation:

1) for all $m \in M(H)$, $\nabla(m) = (\widetilde{B} + \mathrm{id}_{\mathcal{F}(H)} \otimes \partial_{\bar{t}})(m)\, d\bar{t}/\bar{t}$;

2) for all $u \in H_{O(K)}$ and $X \in \mathrm{End}\, H_{O(K)}$,

$$(\mathrm{id}_H \otimes \partial_{\bar{t}})(X(u)) = (\mathrm{id}_{\mathrm{End}\, H} \otimes \partial_{\bar{t}})(X)(u) + X\big((\mathrm{id}_H \otimes \partial_{\bar{t}})u\big).$$

Proposition 2.2. *Let $C = -(\mathrm{id}_{\mathrm{End}\, H} \otimes \partial_{\bar{t}})AA^{-1}$ and let for any $n \geqslant 1$, $D^{(n)} = A \cdot \sigma_{\mathrm{End}\, H}(A) \cdots \sigma^{n-1}_{\mathrm{End}\, H}(A)$. Then*

$$B = \sum_{n \geqslant 0} p^n \,\mathrm{Ad}(D^{(n)}) \sigma^n_{\mathrm{End}\, H}(C).$$

Remark 2.2. In § 4 we will prove that $C, B \in L(H)_{O(K)} \subset \operatorname{End} H_{O(K)}$. In particular, $\sigma_{\operatorname{End} H} C = \sigma_{L(H)} C$ and the correspondence $u \mapsto (B + \mathrm{id}_{L(H)} \otimes \partial_{\bar t})(u)\, d\bar t/\bar t$ gives a connection on $L(H)_{O(K)}$.

Proof of Proposition 2.2. Indeed, for any $u \in H_{O(K)}$, it holds

$$
\begin{aligned}
(\nabla \cdot \phi)(\mathcal{F}(u)) = (\nabla \cdot \mathcal{F} \cdot \mathcal{A})u &= \big((\widetilde{B} + \mathrm{id}_{\mathcal{F}(H)} \otimes \partial_{\bar t}) \cdot \mathcal{F} \cdot \mathcal{A}\big)(u)\, \frac{d\bar t}{\bar t} \\
&= \big(\mathcal{F} \cdot (B + \mathrm{id}_H \otimes \partial_{\bar t}) \cdot \mathcal{A}\big)(u)\, \frac{d\bar t}{\bar t}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(\phi \otimes \phi)(\nabla(\mathcal{F}(u))) &= \phi\big(\mathcal{F}(B + \mathrm{id}_H \otimes \partial_{\bar t})u\big)\phi\left(\frac{d\bar t}{\bar t}\right) \\
&= p\big(\mathcal{F} \cdot \mathcal{A} \cdot (B + \mathrm{id}_H \otimes \partial_{\bar t})\big)(u)\, \frac{d\bar t}{\bar t}.
\end{aligned}
$$

Equivalently, we have the following identity on $H_{O(K)}$,

$$
(B \cdot A + (\mathrm{id}_H \otimes \partial_{\bar t}) \cdot A) \cdot \sigma_H = pA \cdot \sigma_H \cdot (B + \mathrm{id}_H \otimes \partial_{\bar t}).
$$

Rewrite this equality as follows

$$
(B \cdot A - pA \cdot \sigma_{\operatorname{End} H}(B)) \cdot \sigma_H = \big(-(\mathrm{id}_H \otimes \partial_{\bar t}) \cdot A + A \cdot \sigma_H \cdot (\mathrm{id}_H \otimes p\, \partial_{\bar t}) \cdot \sigma_H^{-1}\big) \cdot \sigma_H.
$$

Notice that on $H_{O(K)}$ we have $\sigma_H \cdot (\mathrm{id}_H \otimes p\, \partial_{\bar t}) \cdot \sigma_H^{-1} = \mathrm{id}_H \otimes \partial_{\bar t}$. As a result, the right-hand side equals $-(\mathrm{id}_{\operatorname{End} H} \otimes \partial_{\bar t})(A) \cdot \sigma_H$.

From $\sigma_H|_H = \mathrm{id}_H$ it follows that by restriction on H we have

$$
B \cdot A - pA \cdot \sigma_{\operatorname{End} H}(B) = -(\mathrm{id}_{\operatorname{End} H} \otimes \partial_{\bar t})A.
$$

By $O(K)$-linearity this identity holds on the whole $H_{O(K)}$.

As a result, our identity appears in the form

$$
\big(\mathrm{id}_H - p\operatorname{Ad}(A) \cdot \sigma_{\operatorname{End} H}\big)B = -(\mathrm{id}_H \otimes \partial_{\bar t})(A) \cdot A^{-1}.
$$

It remains to recover B using that

$$
(\mathrm{id}_H - p\operatorname{Ad} A \cdot \sigma_{\operatorname{End} H})^{-1} = \sum_{n \geqslant 0} p^n \operatorname{Ad}(D^{(n)}) \cdot \sigma^n_{\operatorname{End} H}.
$$

§ 3. Ramification filtration modulo pth commutators

Recall that $K = k((t)) \subset K_0^{\mathrm{tr}}$, $\pi_H(\Gamma_K) = \exp(L(H)) = I(H) \subset \operatorname{Aut}_{\mathbb{Z}_p}(H)$. Note also that $O(K) = W(k)((\bar t))$, where $\bar t^{\,e_0} = \bar t_0$ and e_0 is the ramification index of K/K_0.

3.1. Lie algebra $\mathcal{L}$ and identification $\eta_{<p}$. Let $K_{<p}$ be the maximal p-extension of K in K_0^{sep} with the Galois group of nilpotent class $< p$. Then $\mathcal{G}_{<p} := \mathrm{Gal}(K_{<p}/K) = \varprojlim_M \Gamma_K/\Gamma_K^{p^M} C_p(\Gamma_K)$.

Let $\widetilde{\mathcal{L}}_{W(k)}$ be a profinite free Lie $W(k)$-algebra with the set of topological generators $\{D_0\} \cup \{D_{an} \mid a \in \mathbb{Z}^+(p),\ n \in \mathbb{Z}/N_0\}$. Let $\mathcal{L}_{W(k)} = \widetilde{\mathcal{L}}_{W(k)}/C_p(\widetilde{\mathcal{L}}_{W(k)})$, where $C_p(\widetilde{\mathcal{L}}_{W(k)})$ is the ideal of pth commutators. Define the σ-linear action on $\mathcal{L}_{W(k)}$ via $D_{an} \mapsto D_{a,n+1}$ and $D_0 \mapsto D_0$, denote this action by the same symbol σ, and set $\mathcal{L} = \mathcal{L}_{W(k)}|_{\sigma=\mathrm{id}}$.

Fix $\alpha_0 \in W(k)$ such that the trace of α_0 in the field extension $W(k)[1/p]/\mathbb{Q}_p$ equals 1. For any $n \in \mathbb{Z}/N_0$, set $D_{0n} = (\sigma^n \alpha_0) D_0$.

We are going to apply the profinite version of the covariant nilpotent Artin–Schreier theory to the Lie algebra $\mathcal{L}$ and $e_{<p} = \sum_{a \in \mathbb{Z}^0(p)} \overline{t}^{\,-a} D_{a0} \in \mathcal{L} \mathbin{\widehat{\otimes}} O(K)$. In other words, if we fix

$$f_{<p} \in \{f \in \mathcal{L} \mathbin{\widehat{\otimes}} O_{\mathrm{sep}} \mid \sigma_{\mathcal{L}}(f) = e_{<p} \circ f\} \neq \varnothing,$$

then the map $\eta_{<p} := \pi_{f_{<p}}(e_{<p})$ given by $\tau \mapsto (-f_{<p}) \circ (\mathrm{id}_{\mathcal{L}} \otimes \tau) f_{<p}$ induces the group isomorphism $\overline{\eta}_{<p} \colon \Gamma_{<p} \simeq G(\mathcal{L})$.

The following property is an easy consequence of the above construction.

Proposition 3.1. *Suppose $e \in L(H)_{O(K)}$ is special and given with notation from the definition from § 2.2. Then the map $\log \pi_H \colon \mathcal{G}_{<p} \to G(L(H))$ is given via the correspondences $D_{a0} \mapsto l_{a0}$ (and $D_{an} \mapsto \sigma^n_{L(H)}(l_{a0})$) for all $a \in \mathbb{Z}^0(p)$.*

3.2. The ramification ideals $\mathcal{L}^{(v)}$. For $v \geqslant 0$, denote by $\mathcal{G}_{<p}^{(v)}$ the image of $\Gamma_K^{(v)}$ in $\mathcal{G}_{<p}$. Then $\overline{\eta}_{<p}(\mathcal{G}_{<p}^{(v)}) = G(\mathcal{L}^{(v)})$, where $\mathcal{L}^{(v)}$ is an ideal in $\mathcal{L}$. The images $\mathcal{L}^{(v)}(M)$ of the ideals $\mathcal{L}^{(v)}$ in the quotients $\mathcal{L}/p^M\mathcal{L}$ for all $M \in \mathbb{N}$ were explicitly described in [10]. By going to the projective limit on M this description can be presented as follows.

Definition 3.1. Let $\overline{n} = (n_1, \ldots, n_s)$ with $s \geqslant 1$. Suppose there is a partition $0 = i_0 < i_1 < \cdots < i_r = s$ such that for $i_j < u \leqslant i_{j+1}$, it holds $n_u = m_{j+1}$ and $m_1 > m_2 > \cdots > m_r$. Then set

$$\eta(\overline{n}) = \frac{1}{(i_1 - i_0)! \cdots (i_r - i_{r-1})!}.$$

If such a partition does not exist we set $\eta(\overline{n}) = 0$.

For $s \in \mathbb{N}$, $\overline{a} = (a_1, \ldots, a_s) \in \mathbb{Z}^0(p)^s$ and $\overline{n} = (n_1, \ldots, n_s) \in \mathbb{Z}^s$, set

$$[D_{\overline{a},\overline{n}}] = [\ldots, [D_{a_1 n_1}, D_{a_2 n_2}], \ldots, D_{a_s n_s}].$$

For $\alpha \geqslant 0$ and $N \in \mathbb{Z}_{\geqslant 0}$, introduce $\mathcal{F}^0_{\alpha,-N} \in L_{W(k)}$ such that

$$\mathcal{F}^0_{\alpha,-N} = \sum_{\substack{1 \leqslant s < p \\ \gamma(\overline{a},\overline{n}) = \alpha}} a_1 \eta(\overline{n}) p^{n_1} [D_{\overline{a},\overline{n}}].$$

Here $n_1 \geqslant 0$, all $n_i \geqslant -N$ and $\gamma(\overline{a},\overline{n}) = a_1 p^{n_1} + a_2 p^{n_2} + \cdots + a_s p^{n_s}$.

Note that the non-zero terms in the above expression for $\mathcal{F}^0_{\alpha,-N}$ can appear only if $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_s$ and α has at least one presentation in the form $\gamma(\overline{a},\overline{n})$.

Denote by $\mathcal{I}^{(v)}[N]$ the minimal closed ideal in $\mathcal{L}$ such that its extension of scalars $\mathcal{I}^{(v)}[N]_{W(k)}$ contain all $\mathcal{F}^0_{\alpha,-N}$ with $\alpha \geqslant v$.

Our result from [10] about explicit generators of the ideal $\mathcal{L}^{(v)}$ can be stated in the following form.

Theorem 3.1. *For any $v > 0$ and $M \in \mathbb{N}$, there is $\widetilde{N}(v,M) \in \mathbb{N}$ such that if $N \geqslant \widetilde{N}(v,M)$, then the images of the ideals $\mathcal{L}^{(v)}$ and $\mathcal{I}^{(v)}[N]$ in $\mathcal{L}/p^M$ coincide.*

3.3. Some relations. Let $A(\mathcal{L})$ be the enveloping algebra of $\mathcal{L}$ and $\widetilde{A}(\mathcal{L}) = A(\mathcal{L})/J(\mathcal{L})^p$, where $J(\mathcal{L})$ is the augmentation ideal in $A(\mathcal{L})$. Note that there is a natural embedding of $\mathbb{Z}_p$-modules $\mathcal{L} \subset \widetilde{A}(\mathcal{L})$.

Let $A_{<p} = \exp(-e_{<p}) \in \widetilde{A}(\mathcal{L})_{O(K)}$ and $C_{<p} = -(\mathrm{id}_{\widetilde{A}(\mathcal{L})} \otimes \partial_{\overline{t}})A_{<p} \cdot A_{<p}^{-1}$.

For $s \geqslant 1$, set $\overline{0}_s = (\underbrace{0,\dots,0}_{s\text{ times}})$.

Proposition 3.2. *Let $D^{(m)}_{<p} := A_{<p} \cdot \sigma_{\widetilde{A}(\mathcal{L})}(A_{<p}) \cdots \sigma^{m-1}_{\widetilde{A}(\mathcal{L})}(A_{<p})$, where $m \geqslant 1$. Then we have the following relations*:

$$C_{<p} = \sum_{s\geqslant 1,\overline{a}} a_1\eta(\overline{0}_s)[D_{\overline{a},\overline{0}_s}]\overline{t}^{\,-\gamma(\overline{a},\overline{0}_s)}, \tag{3.1}$$

$$B_{<p} := \sum_{n\geqslant 0} p^n \operatorname{Ad}(D^{(n)}_{<p})(\sigma^n_{\mathcal{L}}(C_{<p})) = \sum_{\alpha>0} \mathcal{F}^0_{\alpha,0}\overline{t}^{\,-\alpha}, \tag{3.2}$$

$$\operatorname{Ad}\sigma^{-m}_{\widetilde{A}(\mathcal{L})}(D^{(m)}_{<p})(B_{<p}) = \sum_{\alpha>0} \mathcal{F}^0_{\alpha,-m}\overline{t}^{\,-\alpha}. \tag{3.3}$$

Proof. For (3.1) use (cf. [20], Theorem 4.22), to obtain

$$d\exp(-e_{<p})\cdot\exp(e_{<p}) = \sum_{k\geqslant 1}\frac{1}{k!}(-\operatorname{ad} e_{<p})^{k-1}(-de_{<p})$$

and note that

$$(-\operatorname{ad} e_{<p})^{k-1}(de_{<p}) = (-1)^{k-1}\big[\underbrace{e_{<p},\dots,[e_{<p}}_{k-1\text{ times}}, de_{<p}],\dots\big]$$
$$= \big[\dots,[de_{<p},\underbrace{e_{<p}],\dots,e_{<p}}_{k-1\text{ times}}\big].$$

For (3.2) we need the following relation (cf. [20], § 4.4)

$$\exp(X)\cdot Y\cdot\exp(-X) = \sum_{n\geqslant 0}\frac{1}{n!}\operatorname{ad}^n(X)(Y).$$

After applying this relation to the summand with $m = 1$ we obtain

$$p\operatorname{Ad} D^{(1)}_{<p}(\sigma_{\mathcal{L}}C_{<p}) = \exp(-e_{<p})\cdot\sigma_{\mathcal{L}}(C_{<p})\cdot\exp(e_{<p})$$
$$= \sum_{s\geqslant 0}\eta(\overline{0}_s)(-1)^s\operatorname{ad}^s(e_{<p})(C_{<p}) = \sum_{\overline{n}\geqslant 0,\overline{a}}\ \sum_{n_1=0}^{n_1=1}\eta(\overline{n})p^{n_1}[D_{\overline{a},\overline{n}}]\overline{t}^{\,-\gamma(\overline{a},\overline{n})}.$$

Repeating this procedure we obtain relation (3.2).

Similar calculations prove the remaining item (3.3). Proposition 3.2 is proved.

§ 4. Proof of Theorem 1.1

Recall briefly what we've already achieved.

The field $K = k((t))$ is tamely ramified extension of $K_0 = k_0((t_0))$, where $t^{e_0} = t_0$, $H := H_0|_{\Gamma_K}$ and the corresponding group epimorphism $\pi_H\colon \Gamma_K \to I(H) \subset \mathrm{Aut}_{\mathbb{Z}_p} H$ is such that $I(H) = \exp(L(H))$, where $L(H) \subset \mathrm{End}_{\mathbb{Z}_p} H$ and $L(H)^p = 0$.

Applying formalism of nilpotent Artin–Schreier theory we obtained a special $e = \sum_{a\in\mathbb{Z}^0(p)} \bar{t}^{-a} l_{a0} \in L(H)_{O(K)}$ and $f \in L(H)_{O_{\mathrm{sep}}}$ such that

$$\exp(\sigma_{L(H)} f) = \exp(e) \cdot \exp(f)$$

and for any $\tau \in \Gamma_K$, $\pi_H(\tau) = \exp(-f) \cdot \exp(\mathrm{id}_{L(H)} \otimes \tau) f$.

We used $O(K)$-linear operator $\mathcal{F} = \exp(-f)$ to introduce $O(K)$-module $M(H) := \mathcal{F}(H_{O(K)})$. Let $A = \exp(-e)$ and let $\mathcal{A}$ be a unique σ-linear operator on $H_{O(K)}$ such that for any $h \in H$, $\mathcal{A}(h) = A(h)$. Then the σ-linear $\phi\colon M(H) \to M(H)$ is such that for any $u \in L(H)_{O(K)}$, $\phi(\mathcal{F}(u)) = \mathcal{F}(\mathcal{A}(u))$. As a result, we obtain the structure of etale $(\phi, O(K))$-module on $M(H)$ related to the Γ_K-module H.

Let ∇ be the connection on $M(H)$ from § 2.4, and let $\widetilde{B}$ be the $O(K)$-linear operator on $M(H)$ uniquely determined by the condition: for any $m \in \mathcal{F}(H)$, $\nabla(m) = \widetilde{B}(m) d\bar{t}/\bar{t}$. Then for any $u \in M(H)$,

$$\nabla(u) = (\widetilde{B} + \mathrm{id}_{\mathcal{F}(H)} \otimes \partial_{\bar{t}})(u)\,\frac{d\bar{t}}{\bar{t}}$$

and we introduce the differential forms

$$\widetilde{\Omega}[N] = \phi^N \widetilde{B} \phi^{-N}\,\frac{d\bar{t}}{\bar{t}} \in \mathrm{End}\, M(H)_{O(K^{\mathrm{rad}})} \otimes \Omega^1_{O(K)}.$$

Finally, define the $O(K)$-linear operator B on H by setting for any $u \in H_{O(K)}$, $\mathcal{F}(B(u)) = \widetilde{B}(\mathcal{F}(u))$, and transfer $\widetilde{\Omega}[N]$ to $\mathrm{End}\, H_{O(K^{\mathrm{rad}})}$ in the following form

$$\Omega[N] = \mathrm{Ad}(\mathcal{A}^N)(B)\,\frac{d\bar{t}}{\bar{t}} \in \mathrm{End}(H)_{O(K^{\mathrm{rad}})} \otimes_{O(K)} \Omega^1_{O(K)}.$$

Remark 4.1. Obviously we have the following identification:

$$\mathrm{End}(H)_{O(K^{\mathrm{rad}})} \otimes_{O(K)} \Omega^1_{O(K)} = \mathrm{End}(H_0)_{O(K_0^{\mathrm{rad}})} \otimes_{O(K_0)} \Omega^1_{O(K_0)}.$$

Recall that $\mathcal{A} = A \cdot \sigma_H$, where $A = \exp(-e)$.

Lemma 4.1. *If* $\mathcal{D}^{(N)} = \sigma^{-N}_{\mathrm{End}\, H}(A) \cdots \sigma^{-1}_{\mathrm{End}\, H}(A)$, *then*

$$\bigl(\sigma^{-N}_{\mathrm{End}\, H} \cdot \mathrm{Ad}(\mathcal{A}^N)\bigr)(B) = \mathrm{Ad}\,\mathcal{D}^{(N)}(B).$$

Proof. Use induction on $N \geqslant 0$. If $N = 0$ there is nothing to prove. Suppose lemma is proved for $N \geqslant 0$. Then

$$\begin{aligned}
\bigl(\sigma^{-(N+1)}_{\mathrm{End}\, H} \cdot \mathrm{Ad}(\mathcal{A}^{N+1})\bigr)(B) &= \sigma^{-(N+1)}_{\mathrm{End}\, H}\bigl(\mathcal{A} \cdot \mathrm{Ad}(\mathcal{A}^N)(B) \cdot \mathcal{A}^{-1}\bigr)\\
&= \sigma^{-(N+1)}_{\mathrm{End}\, H}\bigl(\mathcal{A} \cdot (\sigma^N_{\mathrm{End}\, H} \cdot \mathrm{Ad}(\mathcal{D}^{(N)})(B)) \cdot \mathcal{A}^{-1}\bigr)\\
&= \sigma^{-(N+1)}_{\mathrm{End}\, H}\bigl(A \cdot \sigma_H \cdot \bigl(\sigma^N_{\mathrm{End}\, H}\, \mathrm{Ad}(\mathcal{D}^{(N)})(B)\bigr) \cdot \sigma_H^{-1} \cdot A^{-1}\bigr)\\
&= \sigma^{-(N+1)}_{\mathrm{End}\, H}(A \cdot \bigl(\sigma^{N+1}_{\mathrm{End}\, H}\, \mathrm{Ad}(\mathcal{D}^{(N)})(B)\bigr) \cdot A^{-1})\\
&= \sigma^{-(N+1)}_{\mathrm{End}\, H}(A) \cdot \mathrm{Ad}(\mathcal{D}^{(N)})(B) \cdot \sigma^{-(N+1)}_{\mathrm{End}\, H}(A^{-1}) = \mathrm{Ad}(\mathcal{D}^{(N+1)})(B).
\end{aligned}$$

The lemma is proved.

Under the projection $\log\overline{\pi}_H\colon \mathcal{G}_{<p}\to G(L(H))$ we have:

$$D_{an}\mapsto l_{an}=\sigma^n_{L(H)}l_{a0},\quad e_{<p}\mapsto e,\quad f_{<p}\mapsto f,\quad A_{<p}\mapsto A,\quad C_{<p}\mapsto C,$$
$$B_{<p}\mapsto B,\quad \text{and}\quad \sigma^{-m}_{\widetilde{\mathcal{A}}(\mathcal{L})}D^{(m)}_{<p}\mapsto \mathcal{D}^{(m)}.$$

Remark 4.2. Because $\log\overline{\pi}_H(\mathcal{L}_{<p})=L(H)$ we obtain the proof of the property stated in Remark 2.2.

As a result, our differential form appears as the image of $\sum\mathcal{F}^0_{\alpha,-N}\bar{t}^{\,-\alpha}\,d\bar{t}/\bar{t}$.

It remains to notice that when getting back to the field K_0, we have $d\bar{t}/\bar{t}=e_0^{-1}\,d\bar{t}_0/\bar{t}_0$, $\bar{t}^{\,-\alpha}=\bar{t}^{\,-\alpha/e_0}$, $\Gamma_K^{(\alpha)}=\Gamma_{K_0}^{(\alpha/e_0)}$ and $\pi_H|_{\mathcal{I}}=\pi_{H_0}|_{\mathcal{I}}$.

Theorem 1.1 is proved.

Remark 4.3. a) The conjugacy class of the differential form $\Omega[N]$ does not depend on a choice of a special form for e.

b) It would be very interesting to verify whether our results could be established in the case of Γ_K-modules which do not satisfy the Lie condition, e. g., for the Γ_K-module from [21] (the case $n=p$ in the notation of that paper).

§ 5. Mixed characteristic

Let E_0 be a complete discrete valuation field of characteristic 0 with finite residue field k_0 of characteristic p. Let $\overline{E}_0$ be an algebraic closure of E_0 and for any field E such that $E_0\subset E\subset\overline{E}_0$, set $\mathrm{Gal}(\overline{E}_0/E)=\Gamma_E$. Suppose that E_0 contains a primitive pth root of unity ζ_1.

We are going to develop an analog of the above characteristic p theory in the context of finite $\mathbb{F}_p[\Gamma_{E_0}]$-modules H_{E_0} satisfying an analogue of the Lie condition from § 2.1:

if $\pi_{H_{E_0}}\colon\Gamma_{E_0}\to\mathrm{Aut}_{\mathbb{F}_p}(H_{E_0})$ *determines a* Γ_{E_0}*-action on* H_{E_0} *then there is a Lie* $\mathbb{F}_p$*-subalgebra* $L(H_{E_0})\subset\mathrm{End}_{\mathbb{F}_p}(H_{E_0})$ *such that* $L(H_{E_0})^p=0$ *and* $\exp(L(H_{E_0}))=\pi_{H_{E_0}}(I)$, *where* I *is the wild ramification subgroup in* Γ_{E_0}.

Remark 5.1. Contrary to the characteristic p case we restrict ourselves to the Galois modules killed by p because the theory from [11], [12] is developed recently only under that assumption.

Fix a choice of a uniformising element π_0 in E_0.

Let $\widetilde{E}_0=E_0(\{\pi_0^{(n)}\}\mid n\in\mathbb{Z}_{\geqslant 0}\})\subset\overline{E}_0$, where $\pi_0^{(0)}=\pi_0$ and for all $n\in\mathbb{N}$, $\pi_0^{(n)p}=\pi_0^{(n-1)}$. The field-of-norms functor X provides us with:

– a complete discrete valuation field $X(\widetilde{E}_0)=K_0$ of characteristic p with residue field k_0 and a fixed uniformizer $t_0=\varprojlim\pi_n^{(0)}$;

– an identification of $\Gamma_{K_0}=\mathrm{Gal}(K_0^{\mathrm{sep}}/K_0)$ with $\Gamma_{\widetilde{E}_0}\subset\Gamma_{E_0}$.

Let E be a finite tamely ramified extension of E_0 in $\overline{E}_0$ such that $\pi_{H_{E_0}}(\Gamma_E)=I(H_0)$. By replacing E with a suitable finite unramified extension we can assume that E has uniformiser π such that $\pi^{e_0}=\pi_0$. Let k be the residue field of E.

It is easy to see that the field $\widetilde{E}:=E\widetilde{E}_0$ appears in the form $E(\{\pi^{(n)}\mid n\geqslant 0\})$, where $\pi^{(0)}=\pi$, $\pi^{(n)p}=\pi^{(n-1)}$ and for all n, $\pi^{(n)e_0}=\pi_0^{(n)}$. In particular, $K:=X(\widetilde{E})=k((t))$, where $t=\varprojlim\pi^{(n)}$ is uniformiser such that $t^{e_0}=t_0$.

Let $\mathcal{G}_{<p}=\Gamma_K/\Gamma_K^pC_p(\Gamma_K)$ and $\Gamma_{<p}=\Gamma_E/\Gamma_E^pC_p(\Gamma_E)$. According to [11], [12] we have the following natural exact sequence:

$$\mathcal{G}_{<p}\to\Gamma_{<p}\to\langle\tau_0\rangle^{\mathbb{Z}/p}\to 1,$$

where $\tau_0\in\mathrm{Gal}(E(\pi^{(1)})/E)$ is such that $\tau_0(\pi^{(1)})=\zeta_1\pi^{(1)}$. We can use the identification $\overline{\eta}_{<p}\colon\mathcal{G}_{<p}\simeq G(\mathcal{L})$ from § 3.1 obtained via the special element $e_{<p}$ and the corresponding $f_{<p}$ such that $\sigma_{L(H)}(f_{<p})=e_{<p}\circ f_{<p}$.

Then we can use the equivalence of categories from § 1.3 to identify $\Gamma_{<p}$ with $G(L)$ where L is a profinite Lie $\mathbb{F}_p$-algebra included into the following exact sequence:

$$\mathcal{L}\to L\to\mathbb{F}_p\tau_0\to 0. \tag{5.1}$$

When studying the structure of (5.1) in [12] we proved that τ_0 can be replaced by a suitable $h_0\in\mathrm{Aut}\,K$. More precisely, suppose

$$\zeta_1\equiv 1+\sum_{j\geqslant 0}[\beta_j]\pi_0^{(e_0^*/p)+j}\mod p$$

with Teichmüller representatives $[\beta_j]$ of $\beta_j\in k$ and $e^*=ep/(p-1)$, where e is the ramification index for $E/\mathbb{Q}_p$. Then h_0 can be defined as follows: $h_0|_k=\mathrm{id}_k$ and

$$h_0(t)=t\biggl(1+\sum_{j\geqslant 0}\beta_j^pt^{e^*+pj}\biggr)=t\,\widetilde{\exp}(\omega(t)^p),$$

where $\widetilde{\exp}$ is the truncated exponential and $\omega(t)\in t^{e^*/p}k[[t]]^*$.

This allowed us to apply formalism of the nilpotent Artin–Schreier theory to specify “good” lifts $\tau_{<p}$ of τ_0 to L (what is equivalent to specifying “good lifts” $h_{<p}$ of h_0).

In particular, we obtained in [11], [12] the following description of the image $\overline{L}\subset L$ of $\mathcal{L}$ from exact sequence (5.1). Introduce the weight function wt on $\mathcal{L}_k$ by setting $\mathrm{wt}(D_{an})=s\in\mathbb{N}$ if and only if $(s-1)e^*\leqslant a<se^*$. Then

$$\mathrm{Ker}(\mathcal{L}\to L)=\mathcal{L}(p)=\{l\in\mathcal{L}\mid\mathrm{wt}(l)\geqslant p\},$$

$\overline{L}=\mathcal{L}/\mathcal{L}(p)$ and $\mathcal{L}\to\overline{L}$ is the natural projection.

If $h_{<p}$ is a lift of h_0 to $K_{<p}$ then it is uniquely determined by $c=c(h_{<p})\in L_K$ such that

$$(\mathrm{id}_{\mathcal{L}_{<p}}\otimes h_{<p})f=c\circ\bigl(\mathrm{Ad}(h_{<p})\otimes\mathrm{id}_{K_{<p}}\bigr)f.$$

This allowed us to describe the corresponding action of the group $\langle h_{<p}\rangle^{\mathbb{Z}/p}$ on f as an action of an infinitesimal group scheme of order p. The differential of this action is given by the “linear part” $c_1\in\overline{L}_K$ of c which could be described by a suitable recurrent procedure. Finally, we proved that $c_1(0)\in\overline{L}_k$ (where $c_1=\sum_{n\in\mathbb{Z}}c_1(n)t^n$ with all $c_1(n)\in\overline{L}_k$) is an absolute invariant of the lift $h_{<p}$.

Consider the expansion $\omega(t)^p=\sum_{j\geqslant 0}A_jt^{e^*+pj}$, $A_j\in k$.

Denote by $\overline{L}^{(e^*)} \subset \overline{L}$ the image of the ramification subgroup $\mathcal{L}^{(e^*)}$ in L (cf. (5.1)). Then Theorem 5.1 from [12] states:

• the lift $\tau_{<p}$ of τ_0 is "good" if and only if the value $c_1(0) \in \mathcal{L}_k$ of the differential $d\tau_{<p}$ at 0 satisfies the following congruence:

$$c_1(0) \equiv \sum_{j\geqslant 0}\sum_{i\geqslant 0} \sigma^i(A_j\mathcal{F}^0_{e^*+pj,-i}) \mod \overline{L}_k^{(e^*)}.$$

It remains to note that for $i,j \gg 0$, $\mathcal{F}^0_{e^*+pj,-i} \in \overline{L}_k^{(e^*)}$ and the right-hand double sum contains only finitely many nonzero terms modulo $\overline{L}_k^{(e^*)}$. It can be rewritten also in the following form:

$$\sum_{i,j\geqslant 0} \operatorname{Res}\bigl(\sigma^i(A_jt^{e^*+pj}\cdot\sigma^{-i}\Omega_{<p}[i])\bigr),$$

and the image of this expression in $L(H)_k$ equals

$$\sum_{i\geqslant 0} \operatorname{Res}\bigl(\sigma^{i+1}\omega(t)\cdot\Omega[i]\bigr).$$

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Bibliography

[1] J.-P. Serre, *Local fields*, Transl. from the French, Grad. Texts in Math., vol. 67, Springer-Verlag, New York–Berlin 1979.

[2] I. R. Šafarevič, "On p-extensions", *Mat. Sb.* **20(62)**:2 (1947), 351–363; English transl., Amer. Math. Soc. Transl. Ser. 2, vol. 4, Amer. Math. Soc., Providence, RI 1956, pp. 59–72.

[3] S. P. Demushkin, "The group of a maximal p-extension of a local field", *Izv. Akad. Nauk SSSR Ser. Mat.* **25**:3 (1961), 329–346.

[4] U. Jannsen and K. Wingberg, "Die Struktur der absoluten Galoisgruppe $\mathfrak{p}$-adischer Zahlkörper", *Invent. Math.* **70**:1 (1982/83), 71–98.

[5] Sh. Mochizuki, "A version of the Grothendieck conjecture for p-adic local fields", *Internat. J. Math.* **8**:4 (1997), 499–506.

[6] V. A. Abrashkin, "On a local analogue of the Grothendieck conjecture", *Internat. J. Math.* **11**:2 (2000), 133–175.

[7] H. Koch, *Galois theory of p-extensions*, Springer Monogr. Math., Springer-Verlag, Berlin 2002.

[8] V. A. Abrashkin, "A ramification filtration of the Galois group of a local field", Tr. St. Peterbg. Mat. Obs., vol. 3, St. Petersburg Univ., St. Petersburg 1994; English transl., *Proceedings of the St. Petersburg Mathematical Society*, vol. III, Amer. Math. Soc. Transl. Ser. 2, vol. 166, Amer. Math. Soc., Providence, RI 1995, pp. 35–100.

[9] V. A. Abrashkin, "Ramification filtration of the Galois group of a local field. II", *Number theory, algebra and algebraic geometry*, Collection of articles. To seventieth birthday of Academician I. R. Shafarevich, Trudy Mat. Inst. Steklov., vol. 208, Nauka, Fizmatlit, Moscow 1995, pp. 18–69; English transl., *Proc. Steklov Inst. Math.* **208** (1995), 15–62.

[10] V. A. Abrashkin, "A ramification filtration of the Galois group of a local field. III", *Izv. Ross. Akad. Nauk Ser. Mat.* **62**:5 (1998), 3–48; English transl., *Izv. Math.* **62**:5 (1998), 857–900.

[11] V. Abrashkin, "Groups of automorphisms of local fields of period p and nilpotent class $< p$. I", *Internat. J. Math.* **28**:6 (2017), 1750043.

[12] V. Abrashkin, "Groups of automorphisms of local fields of period p and nilpotent class $< p$. II", *Internat. J. Math.* **28**:10 (2017), 1750066.

[13] V. Abrashkin, "Groups of automorphisms of local fields of period p^M and nilpotent class $< p$", *Ann. Inst. Fourier (Grenoble)* **67**:2 (2017), 605–635.

[14] P. Berthelot and W. Messing, "Théorie de Deudonné cristalline. III. Théorèmes d'équivalence et de pleine fidélité", *The Grothendieck festschrift*, vol. 1, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA 1990, pp. 173–247.

[15] M. Lazard, "Sur les groupes nilpotents et les anneaux de Lie", *Ann. Sci. Ecole Norm. Sup.* (3) **71**:2 (1954), 101–190.

[16] J.-M. Fontaine, "Représentations p-adiques des corps locaux. I", *The Grothendieck festschrift*, vol. 2, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA 1990, pp. 249–309.

[17] V. Abrashkin and R. Jenni, "The field-of-norms functor and the Hilbert symbol for higher local fields", *J. Théor. Nombres Bordeaux* **24**:1 (2012), 1–39.

[18] V. Abrashkin, "Galois groups of local fields, Lie algebras and ramification", *Arithmetic and geometry*, London Math. Soc. Lecture Note Ser., vol. 420, Cambridge Univ. Press, Cambridge 2015, pp. 1–23.

[19] V. A. Abrashkin, "Ramification filtration via deformations", *Mat. Sb.* **212**:2 (2021), 3–37; English transl., *Sb. Math.* **212**:2 (2021), 135–169.

[20] A. Bonfiglioli and R. Fulci, *Topics in noncommutative algebra*, Lecture Notes in Math., vol. 2034, Springer, Heidelberg 2012.

[21] K. Imai, *Ramification groups of some finite Galois extensions of maximal nilpotency class over local fields of positive characteristic*, arXiv: 2102.07928.

Victor A. Abrashkin
Department of Mathematical Sciences,
Durham University, United Kingdom;
Steklov Mathematical Institute
of Russian Academy of Sciences, Moscow, Russia
E-mail: victor.abrashkin@durham.ac.uk

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