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ON SOME CLASSES OF RANDOM VARIABLES ON CYCLES OF PERMUTATIONS

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ABSTRACT. In this paper the case of a nonuniform distribution on the set S_n of all permutations of degree n is investigated.
Bibliography: 7 titles.

§1. Introduction

Limit distributions of random variables depending on the number of cycles of a certain length in a permutation of degree n as $n \to \infty$ have been studied by various authors. Gončarov [1] showed that the general number of cycles of a random permutation of degree n is asymptotically normal if all permutations are equiprobable. In later work (see, for example, [2]–[5]), on the one hand, more general random variables were considered (arbitrary linear combinations of numbers of cycles of a specific length), and on the other hand, the requirement that all permutations be equiprobable was removed.

In the present work we also study the case of a nonuniform distribution on the set S_n of all permutations of degree n. Suppose $\{m_1, \dots, m_a\}$ is a finite set of natural numbers, each of which is greater than one and is not divisible by any of the others. For $a > 3$ we assume that these numbers are relatively prime. From S_n we extract the following sets: $S_n^{(1)}$ is the collection of all permutations of degree n whose cycle lengths are not multiples of any of the numbers $m_1, \dots, m_a$; $S_n^{(2)}$ is the collection of all permutations of degree n for which, on the contrary, the length of every cycle is a multiple of one of the numbers $m_1, \dots, m_a$; $S_n(m, \beta)$ is the collection of permutations of degree n whose cycle lengths are equal to β modulo m. Below we assume that on each of these sets a permutation is assigned a uniform distribution. Suppose $\alpha_k^{(1)}$, $\alpha_k^{(2)}$ and $\alpha_k^{(3)}$ are the numbers of cycles of length k in a random permutation from $S_n^{(1)}$, $S_n^{(2)}$ and $S_n(m, \beta)$ respectively.

Let

$$\xi_s^{(1)} = \sum_{k=1}^{s} c_k \alpha_k^{(1)}, \quad \xi_s^{(2)} = \sum_{k=1}^{s} c_k \alpha_k^{(2)}, \tag{1}$$

where c_k is a real number and $s = s(n)$ is a function taking on positive integer values. We assume that also $s \to \infty$ as $n \to \infty$. For $c_1 = c_2 = \cdots = c_s = 1$ the variable $\xi_s^{(1)}$ ($\xi_s^{(2)}$, respectively) is equal to the number of all cycles in a random permutation from $S_n^{(1)}$ ($S_n^{(2)}$) whose lengths do not exceed s.

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Further, let

$$\xi_n(m,\beta)=\sum_{k=1}^{n}\alpha_k^{(3)}.$$

In §2 we provide expressions for the generating functions of the variables $\xi_s^{(1)}$ and $\xi_s^{(2)}$. In §3 we prove asymptotic normality of $\xi_s^{(1)}$ and $\xi_s^{(2)}$ for $s = O(n^{1-\alpha})$, $\alpha > 0$, and under certain restrictions on the sequence $\{c_k\}$. In §4 limit distributions are found for the variable $\xi_n(m,\beta)$ with $\beta = m$ and $\beta = m/2$ and various relations between n and m.

§2. Generating functions of the variables $\xi_s^{(1)}$ and $\xi_s^{(2)}$

Suppose that on the sets $S_n^{(i)}$, $i = 1, 2$, we are given a uniform probability distribution. The number of permutations from $S_n^{(i)}$ having k_1 unit cycles, $\dots$, k_s cycles of length s is denoted by $M_n^{(i)}(k_1,\dots,k_s)$. The number of permutations from S_n with cyclic structure $(k_1,\dots,k_n)$ is denoted by $M_n(k_1,\dots,k_n)$. The exponential generating function of the numbers $M_n(k_1,\dots,k_n)$ has the form (see, for example, [6], Chapter 4, §2, formula (3a))

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\widetilde{\sum}M_n(k_1,k_2,\dots,k_n)\,x_1^{k_1}x_2^{k_2}\dots x_n^{k_n}=\exp\left\{\sum_{k=1}^{\infty}x_k\frac{z^k}{k}\right\},\tag{2}$$

where $\tilde{\Sigma}$ denotes summation over all possible collections of nonnegative numbers $k_1,\dots,k_n$ for which $k_1+2k_2+\cdots+nk_n=n$. Let

$$f_{n,s}^{(i)}(x_1,x_2,\dots,x_s)=\widetilde{\sum}M_n^{(i)}(k_1,k_2,\dots,k_s)\cdot x_1^{k_1}x_2^{k_2}\dots x_s^{k_s},\tag{3}$$

$$F_s^{(i)}(x_1,x_2,\dots,x_s,z)=\sum_{n=0}^{\infty}f_{n,s}^{(i)}(x_1,x_2,\dots,x_s)\cdot\frac{z^n}{n!},\quad i=1,2.\tag{4}$$

The function $F_s^{(1)}(x_1,\dots,x_s,z)$ can be obtained from (2) if we set $x_{km_i}=0$ ($k = 1, 2, \dots$; $i = 1,\dots,a$) and let $x_j = 1$ for $j > s$, $j \neq km_i$. The function $F_s^{(2)}(x_1,\dots,x_s,z)$ can be obtained from (2) if we set $x_j = 0$ ($j \neq km_i$, $k = 1, 2, \dots$; $i = 1,\dots,a$) and $x_{km_i} = 1$ ($km_i > s$).

For simplicity of presentation we carry out a detailed derivation of the generating functions for $a = 2$. In this case

$$F_s^{(1)}(x_1,x_2,\dots,x_s,z)=\exp\left\{\sum_{k=1}^{s}{}^{*}x_k\frac{z^k}{k}+\sum_{k=s+1}^{\infty}{}^{*}\frac{z^k}{k}\right\},$$

$$F_s^{(2)}(x_1,x_2,\dots,x_s,z)=\exp\left\{\sum_{k=1}^{s}{}^{**}x_k\frac{z^k}{k}+\sum_{k=s+1}^{\infty}{}^{**}\frac{z^k}{k}\right\},$$

where $\sum_{k=a}^{b}{}^{*}$ denotes summation over all values of k in the interval $[a, b]$ which are not multiples of either m_1 or m_2 (in the general case, none of the numbers $m_1,\dots,m_a$); $\sum_{k=a}^{b}{}^{**}$, on the contrary, denotes summation over all values of k from $[a, b]$ which are multiples of either m_1 or m_2 (in the general case, at least one of $m_1,\dots,m_a$). We form an expression for $F_s^{(2)}(x_1,\dots,x_s,z)$:

$$F_s^{(2)}(x_1,x_2,\dots,x_s,z)=\exp\left\{\sum_{k=1}^{s}{}^{**}(x_k-1)\frac{z^k}{k}+\sum_{k=1}^{\infty}{}^{**}\frac{z^k}{k}\right\}.$$

Suppose m is the greatest common divisor of m_1 and m_2, $m_1 = m\bar{m}_1$, $m_2 = m\bar{m}_2$ and $M = m\bar{m}_1\bar{m}_2$. Then

$$\sum_{k=1}^{\infty}{}^{**} \frac{z^k}{k} = \sum_{r=1}^{\infty} \frac{z^{rm_1}}{rm_1} + \sum_{r=1}^{\infty} \frac{z^{rm_2}}{rm_2} - \sum_{r=1}^{\infty} \frac{z^{rM}}{rM}$$
$$= -\frac{1}{m_1} \ln(1 - z^{m_1}) - \frac{1}{m_2} \ln(1 - z^{m_2}) + \frac{1}{M} \ln(1 - z^M).$$

Finally we obtain

$$F_s^{(2)}(x_1, x_2, \ldots, x_s, z) = \sqrt[M]{\frac{1 - z^M}{(1 - z^{m_1})^{\bar{m}_2}(1 - z^{m_2})^{\bar{m}_1}}} \exp\left\{\sum_{k=1}^{s}{}^{**} (x_k - 1) \frac{z^k}{k}\right\}. \tag{5}$$

Analogously

$$F_s^{(1)}(x_1, x_2, \ldots, x_s, z) = \frac{1}{1 - z} \sqrt[M]{\frac{(1 - z^{m_1})^{\bar{m}_2}(1 - z^{m_2})^{\bar{m}_1}}{1 - z^M}} \exp\left\{\sum_{k=1}^{s}{}^{*} (x_k - 1) \frac{z^k}{k}\right\}. \tag{6}$$

Now it is easy to obtain the generating functions for the variables $\xi_s^{(1)}$ and $\xi_s^{(2)}$ defined in (1). We denote these generating functions by $g_{n,s}^{(i)}(x)$, $i = 1, 2$. It is clear that

$$g_{n,s}^{(i)}(x) = \frac{1}{M_n^{(i)}} f_{n,s}^{(i)}(x^{c_1}, x^{c_2}, \ldots, x^{c_s}),$$

where $M_n^{(i)} = f_{n,s}^{(i)}(1, 1, \ldots, 1)$ is the cardinality of the set $S_n^{(i)}$. From (4) it is evident that $g_{n,s}^{(i)}(x)$ is equal to the ratio of the coefficients of z^n in the expansion in powers of z of the function $G_s^{(i)}(x, z) = F_s^{(i)}(x^{c_1}, \ldots, x^{c_s}, z)$ and $G_s^{(i)}(1, z)$, $i = 1, 2$. Taking account of (5) and (6), we have

$$G_s^{(1)}(x, z) = \frac{1}{1 - z} \sqrt[M]{\frac{(1 - z^{m_1})^{\bar{m}_2}(1 - z^{m_2})^{\bar{m}_1}}{1 - z^M}} \exp\left\{\sum_{k=1}^{s}{}^{*} (x^{c_k} - 1) \frac{z^k}{k}\right\}, \tag{7}$$

$$G_s^{(2)}(x, z) = \sqrt[M]{\frac{1 - z^M}{(1 - z^{m_1})^{\bar{m}_2}(1 - z^{m_2})^{\bar{m}_1}}} \exp\left\{\sum_{k=1}^{s}{}^{**} (x^{c_k} - 1) \frac{z^k}{k}\right\}. \tag{8}$$

For arbitrary a, quite analogously we obtain the following expressions for $G_s^{(1)}$ and $G_s^{(2)}$:

$$G_s^{(1)}(x, z; a)$$
$$= \frac{1}{1 - z}\left[\prod_{t=1}^{a} \prod_{i_1<i_2<\ldots<i_t} (1 - z^{m^{i_1 i_2 \ldots i_t}})^{(-1)^{t+1} M/m^{i_1 i_2 \ldots i_t}}\right]^{1/M} \exp\left\{\sum_{k=1}^{s}{}^{*} (x^{c_k} - 1) \frac{z^k}{k}\right\}, \tag{9}$$

$$G_s^{(2)}(x, z; a)$$
$$= \left[\prod_{t=1}^{a} \prod_{i_1<i_2<\ldots<i_t} (1 - z)^{m^{i_1 i_2 \ldots i_t}})^{(-1)^{t} M/m^{i_1 \ldots i_t}}\right]^{1/M} \exp\left\{\sum_{k=1}^{s}{}^{**} (x^{c_k} - 1) \frac{z^k}{k}\right\}, \tag{10}$$

where $m^{i_1 \cdots i_t}$ is the least common multiple of the numbers $m_{i_1}, \ldots, m_{i_t}$; $M = m^{12 \cdots a}$.

§3. Asymptotic normality of the variables $\xi_s^{(a)}$ and $\xi_s^{(2)}$

Let

$$a_s = {\sum_{k=1}^{s}}^{*} \frac{c_k}{k}, \quad d_s^2 = {\sum_{k=1}^{s}}^{*} \frac{c_k^2}{k}.$$

THEOREM 1. *Suppose the following conditions are satisfied as $n \to \infty$:*

1) $s \to \infty$, *where* $s \leqslant Cn^{1-\alpha}$, *$C$ a constant,* $\alpha > 0$.

2) $d_s^2 \to \infty$.

3) $c_k/d_s \to 0$ *as* $s \to \infty$ *uniformly with respect to k.*

Then the distribution of the random variable $\theta_s^{(1)} = (\xi_s^{(1)} - a_s)/d_s$ converges to a normal (0, 1) *distribution.*

PROOF. Suppose $A_s^{(1)}(t)$ is the characteristic function of the variable $\theta_s^{(1)}$. It can easily be expressed in terms of the generating function $g_{n,s}^{(1)}(x)$:

$$A_s^{(1)}(t) = e^{-ita_s/d_s} g_{n,s}^{(1)}(e^{it/d_s}). \tag{11}$$

As already noted, $g_{n,s}^{(1)}(x)$ is equal to the ratio of the coefficients of z^n in the expansion in powers of z of the functions $G_s^{(1)}(x, z; a)$ and $G_s^{(1)}(1, z; a)$. We proceed with the study of the asymptotics of these coefficients for $x = e^{it/d_s}$. We denote them by A_n and B_n respectively.

We show that as $n \to \infty$

$$A_n \sim \rho \frac{\varphi_s(1)}{\Gamma(\delta) n^{1-\delta}}, \tag{12}$$

where ρ is a constant, $\Gamma(\delta)$ is the gamma function,

$$\begin{gathered} \varphi_s(z) = \exp\{P_s(z)\}, \quad P_s(z) = {\sum_{k=1}^{s}}^{*} (x^{c_k} - 1)\frac{z^k}{k}, \\ \delta = 1 - \sum_{i=1}^{a} \frac{1}{m_i} + \sum_{i_1<i_2} \frac{1}{m^{i_1 i_2}} - \ldots + (-1)^a \frac{1}{M} \end{gathered} \tag{13}$$

(recall that $m^{i_1 \cdots i_k}$ and M denote the least common multiples of the numbers $m_{i_1}, \ldots, m_{i_k}$ and $m_1, \ldots, m_a$ respectively). It is easy to show that $0 < \delta < 1$. Indeed, δM is the number of integers in $(1, M)$ which cannot be divided by any of the numbers $m_1, \ldots, m_a$.

We write A_n according to the Cauchy formula:

$$A_n = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{G_s^{(1)}(x, z; a)}{z^{n+1}} dz.$$

The function $G_s^{(1)}(x, z; a)$ can be represented in the following form:

$$G_s^{(1)}(x, z; a) = \sqrt[M]{\prod_{k=1}^{M} (1 - z\zeta^{-k})^{\beta_k}} \cdot \varphi_s(z), \tag{14}$$

where ζ is a primitive Mth root of unity, and β_k are integers with $\beta_M = -\delta M$. Let

$$K(z) = \prod_{k=1}^{M-1} (1 - z\zeta^{-k})^{\beta_k};$$

it is clear that $R(z)$ is a rational function with integer coefficients. In this notation

$$A_n = \frac{1}{2\pi i}\int\limits_{-\gamma}(1-z)^{-\delta}\sqrt[M]{R(z)}\,\frac{\varphi_s(z)}{z^{n+1}}\,dz.$$

Over the contour of integration γ we take a circumference of variable radius $r_n > 1$ with M cuts along the radial rays from $z = \zeta^k$ to $z = r_n\zeta^k$. The points ζ^k, $k = 1, \ldots, M$, can be included in the contour of integration, since the singularities of the integrand are determined by the powers of the binomials $(1 - z\zeta^{-k})^{\beta_k/M}$, and $\beta_k/M > -1$ for all k, which will be shown below. The circuit of the contour is taken counterclockwise. The function being integrated is M-valued. We choose that branch for which, when $0 < z < 1$, both $(1 - z)^{-\delta}$ and $[R(z)]^{1/M}$ take on positive values. We represent A_n in the form of a sum of $M + 1$ integrals:

$$A_n = I_0 + I_1 + \ldots + I_M, \tag{15}$$

where

$$I_0 = \frac{1}{2\pi i}\int\limits_{|z|=r_n}\frac{G_s^{(1)}(x, z; a)}{z^{n+1}}\,dz,$$

and I_k is the result of integration along both edges of the kth cut.

First consider I_M, the integral along the lower and upper edges of the cut on the real line (from 1 to r_n). Since $(1 - z)^{-\delta} > 0$ for $0 < z < 1$ (on the strength of the choice of the branch of the function under the integral), for $z > 1$ we have

$$(1-z)^{-\delta} = (z-1)^{-\delta}e^{-i\delta\arg(1-z)}.$$

In passing from the point $1 - \varepsilon$ to the point $1 + \varepsilon$ on the upper edge of a cut, the argument of the function under the integral decreases by π, while in passing from this point on the lower edge of the cut, also to the point $1 + \varepsilon$, the argument increases by π. Therefore

$$I_M = \frac{1}{2\pi i}\int\limits_{r_n}^{1}(z-1)^{-\delta}e^{-i\delta\pi}[R(z)]^{1/M}\frac{\varphi_s(z)}{z^{n+1}}\,dz$$

$$+\frac{1}{2\pi i}\int\limits_{1}^{r_n}(z-1)^{-\delta}e^{i\delta\pi}[R(z)]^{1/M}\frac{\varphi_s(z)}{z^{n+1}}\,dz$$

$$=\frac{\sin(-\delta\pi)}{\pi}\int\limits_{r_n}^{1}(y-1)^{-\delta}[R(y)]^{1/M}\frac{\varphi_s(y)}{y^{n+1}}\,dy.$$

Analogously it can be shown that

$$I_k = \frac{\sin(\delta_k\pi)}{\pi}\int\limits_{r_n\zeta^k}^{\zeta^k}|(1-z\zeta^{-k})^{\delta_k}|\,[R_k(z)]^{1/M}\frac{\varphi_s(z)}{z^{n+1}}\,dz,$$

where

$$\delta_k = \frac{\beta_k}{M}, \quad R_k(z) = \prod_{i=1, i\neq k}^{M}(1-z\zeta^{-i})^{\beta_i}, \quad k = 1, 2, \ldots; M-1.$$

We proceed with estimation of these integrals.

1. In I_M we set $r_n = 1 + \varepsilon_n$, $\varepsilon_n = (2^{\alpha-1} + 1)\ln n/n$, and make the change of variable $y = 1 + u/n$. Then

$$I_M = \frac{\sin \delta\pi}{\pi n^{1-\delta}} \int_0^{n\varepsilon_n} \frac{\left[R\left(1+\frac{u}{n}\right)\right]^{1/M} \varphi_s\left(1+\frac{u}{n}\right)}{u^{\delta}\left(1+\frac{u}{n}\right)^{n+1}}\, du.$$

It is easy to see that

$$\left[R\left(1+\frac{u}{n}\right)\right]^{1/M} = [R(1)]^{1/M}(1+o(1)).$$

The function $\varphi_s(1 + u/n)$ can be represented in the following form:

$$\varphi_s\left(1+\frac{u}{n}\right) = \varphi_s(1) + \varphi_s'(1+\omega)\frac{u}{n} = \varphi_s(1)\,[1+q_n],$$

where

$$q_n = \frac{\varphi_s'(1+\omega)}{\varphi_s(1)}\,\frac{u}{n}, \qquad 0 < \omega < \frac{u}{n} \leqslant \varepsilon_n.$$

Since by the hypothesis of the theorem the variables c_k/d_s vanish uniformly with respect to k and $s = o(n)$, for $x = e^{it/d_s}$ we have

$$(1+\varepsilon_n)^s \sim 1, \qquad |x^{c_k} - 1| \sim t\left|\frac{c_k}{d_s}\right| \leqslant t\lambda_s, \qquad k = 1, 2, \ldots,$$

where $\lambda_s \to 0$ as $s \to \infty$. Therefore

$$|q_n| \leqslant \varepsilon_n \left|\frac{\exp\{P_s(1+\omega)\}\cdot P_s'(1+\omega)}{\exp\{P_s^{\tau}(1)\}}\right|$$

$$\leqslant \varepsilon_n \exp\{|P_s(1+\omega)| + |P_s(1)|\}\cdot\left|\sum_{k=1}^{s}{}^{*}(x^{c_k}-1)(1+\omega)^{k-1}\right|$$

$$\leqslant \varepsilon_n \exp\{t\lambda_s(1+\varepsilon_n)^s \ln s + t\lambda_s \ln s\}\cdot t\lambda_s s(1+\varepsilon_n)^s$$

$$\sim ts\varepsilon_n\lambda_s \exp\{2t\lambda_s \ln s\} = t\varepsilon_n\lambda_s s^{1+2t\lambda_s} = o\left(n^{-\frac{\alpha}{2}}\right), \qquad \alpha > 0.$$

Consequently, as $n \to \infty$

$$\begin{aligned} I_M &\sim \frac{\sin\delta\pi}{\pi n^{1-\delta}}[R(1)]^{1/M}\varphi_s(1)\int_0^{n\varepsilon_n} u^{-\delta}e^{-u}\,du \\ &\sim [R(1)]^{1/M}\frac{\sin\delta\pi}{\pi n^{1-\delta}}\varphi_s(1)\Gamma(1-\delta) = \frac{[R(1)]^{1/M}}{\Gamma(\delta)}\,\frac{\varphi_s(1)}{n^{1-\delta}}. \end{aligned} \tag{16}$$

2. In each of the I_k, $k = 1, \ldots, M-1$, we make the change of variable $z = \zeta^k(1 + v/n)$. Then

$$|I_k| \leqslant \frac{1}{\pi n}\int_0^{n\varepsilon_n}\left(\frac{v}{n}\right)^{\delta_k}\left|\left[R_k\left(\zeta^k+\zeta^k\frac{v}{n}\right)\right]^{1/M}\right|\frac{\left|\varphi_s\left(\zeta^k+\zeta^k\frac{v}{n}\right)\right|}{\left(1+\frac{v}{n}\right)^{n+1}}\,dv.$$

Clearly $|[R_k(\zeta^k + \zeta^k v/n)]^{1/M}|$ is a bounded quantity. As in step 1, it can be shown that as $n \to \infty$

$$\varphi_s\left(\zeta^k + \zeta^k \frac{v}{n}\right) = \varphi_s(\zeta^k)[1 + o(1)].$$

Thus we obtain

$$|I_k| \leqslant \frac{T}{n^{1+\delta_k}} |\varphi_s(\zeta^k)| \int_0^{n\varepsilon_n} v^{\delta_k} e^{-v}\, dv \sim T \cdot \frac{\Gamma(1+\delta_k)\,|\varphi_s(\zeta^k)|}{n^{1+\delta_k}},$$

where T is a constant. Hence

$$|I_k| \leqslant |I_M| \cdot \frac{|\varphi_s(\zeta^k)|}{|\varphi_s(1)|} \frac{T^*}{n^{\delta+\delta_k}},$$

where T^* is a constant.

As in the derivation of (15), we have

$$\frac{|\varphi_s(\zeta^k)|}{|\varphi_s(1)|} \leqslant \exp\{|P_s(\zeta^k)| + |P_s(1)|\} \leqslant \exp\{2t\lambda_s \ln s\} < n^{2t\lambda_s},$$

and therefore the asymptotics of I_k/I_M depends on the quantity $\delta + \delta_k$. When $a = 2$ it is evident from (7) that the variables $\beta_k = \delta_k M$ can take on only the following values: $\overline{m}_1 + \overline{m}_2 + 1$, $\overline{m}_1 - 1$, $\overline{m}_2 - 1$, -1, whence $\delta M + \beta_k > 0$ and $\delta + \delta_k > 0$. In the case $a = 3$ it is also easy to establish that $\delta + \delta_k > 0$ for $k = 1, \ldots, M - 1$. For arbitrary a and $m_1, \ldots, m_a$ the proof presents certain difficulties. However, in the case where $m_1, \ldots, m_a$ are relatively prime, it is easy to see that β_k $(k \neq M)$ can take on only the values

$$\pm (m_{i_1}-1)(m_{i_2}-1)\ldots(m_{i_r}-1), \qquad r<a,$$

while at the same time

$$\delta M = (m_1-1)(m_2-1)\ldots(m_a-1),$$

i.e., $\delta M + \beta_k > 0$ and $\delta + \delta_k > 0$. Thus, as $n \to \infty$,

$$I_k = o(|I_M|), \qquad k = 1, 2, \ldots, M-1. \tag{17}$$

3. Finally, we estimate I_0, the integral over the circumference. For $|z| = 1 + \varepsilon_n$ we have

$$|(1-z)^{-\delta} \sqrt[M]{R(z)}| \leqslant (2+\varepsilon_n)^{2^{a-1}} \varepsilon_n^{-2^{a-1}}, \qquad |\varphi_s(z)| \leqslant s^{2t\lambda_s}$$

(as in the derivation of (16)). Therefore

$$|I_0| \leqslant (2+\varepsilon_n)^{2^{a-1}} \varepsilon_n^{-2^{a-1}} (1+\varepsilon_n)^{-n} s^{2t\lambda_s}$$

$$< 3^{2^{a-1}} \left[\frac{(2^{a-1}+1)\ln n}{n}\right]^{-2^{a-1}} n^{-2^{a-1}-1+2t\lambda_s} = O\left(\frac{1}{n^{1-2t\lambda_s}(\ln n)^{2^{a-1}}}\right) = o\left(\frac{1}{n^{1-\delta'}}\right),$$

where δ' is a positive number. Hence as $n \to \infty$

$$I_0 = o(|I_M|). \tag{18}$$

From (15)–(18) we obtain

$$A_n \sim \frac{[R(1)]^{1/M} \varphi_s(1)}{\Gamma(\delta)\, n^{1-\delta}},$$

i.e., (12) with $\rho = \sqrt[M]{R(1)}$.

Quite analogously it can be proved that as $n \to \infty$

$$B_n \sim \frac{[R(1)]^{1/M}}{\Gamma(\delta)\, n^{1-\delta}}. \tag{19}$$

Now we return to the characteristic function (11):

$$A_n^{(1)}(t) = e^{-ita_s/d_s} \frac{A_n}{B_n} \sim \exp\left\{-it\frac{a_s}{d_s} + \sum_{k=1}^{s}{}^{*} \frac{1}{k}\left(e^{itc_k/d_s} - 1\right)\right\}$$

$$= \exp\left\{-it\frac{a_s}{d_s} + \sum_{k=1}^{s}{}^{*} \frac{itc_k}{d_s} - \sum_{k=1}^{s}{}^{*} \frac{t^2c_k^2}{2d_s^2} + T(s)\right\} = \exp\left\{-\frac{t^2}{2} + T(s)\right\},$$

where

$$T(s) = \sum_{k=1}^{s}{}^{*} \frac{1}{k} \sum_{m=3}^{\infty} \left(it\frac{c_k}{d_s}\right)^m \frac{1}{m!}.$$

Under the assumptions of the theorem, we have

$$|T(s)| < \sum_{k=1}^{s}{}^{*} \frac{1}{k} \sum_{m=3}^{\infty} \left|t\frac{c_k}{d_s}\right|^m < \sum_{k=1}^{s}{}^{*} \frac{1}{k} 2 \left|t\frac{c_k}{d_s}\right|^3 \leqslant 2t^3 \sum_{k=1}^{s}{}^{*} \frac{1}{k} \frac{c_k^2}{d_s^2} \lambda_s = 2t^3\lambda_s,$$

since $|c_k/d_s| \leqslant \lambda_s$, where $\lambda_s \to 0$ as $s \to \infty$. Consequently, as $n \to \infty$

$$A_n^{(1)}(t) \to e^{-t^2/2}.$$

The theorem is proved.

We go on to investigate the random variable $\xi_s^{(2)}$, given on $S_n^{(2)}$ (see (1)). Let

$$b_s = \sum_{k=1}^{s}{}^{**} \frac{c_k}{k}, \quad h_s^2 = \sum_{k=1}^{s}{}^{**} \frac{c_k^2}{k}.$$

THEOREM 2. *Suppose that as $n \to \infty$:*

1) $s \to \infty$, *with* $s \leqslant Cn^{1-\alpha}$, *$C$ a constant*, $\alpha > 0$,

2) $h_s^2 \to \infty$,

3) $c_k/h_s \to 0$ *as* $s \to \infty$ *uniformly with respect to k.*

Then the distribution of the random variable $\theta_s^{(2)} = (\xi_s^{(2)} - b_s)/k_s$ *is asymptotically normal with parameters* (0, 1).

PROOF. The characteristic function of the variable $\theta_s^{(2)}$ is

$$A_n^{(2)}(t) = e^{-itb_s/h_s} g_n^{(2)}\left(e^{it/h_s}\right).$$

The function $g_n^{(2)}(x)$ is equal to the ratio of the coefficients of z^n in the expansion in powers of z of the functions $G_s^{(2)}(x, z; a)$ and $G_s^{(2)}(1, z; a)$ defined in (10).

We note that if the numbers $m_1, \dots, m_a$ have a greatest common divisor $m > 1$, then $G_s^{(2)}(x, z; a)$ is in fact a function of x and $v = z^m$. Thus the problem reduces to an investigation of the coefficients of v^p, $p = n/m$, in the expansion in powers of v of the functions

$$H_s(x, v; a) = \left[\prod_{t=1}^{a} \prod_{i_1<i_2<\dots<i_t} \left(1 - v^{\overline{m}^{i_1 i_2 \dots i_t}}\right)^{(-1)^t M/\overline{m}^{i_1 \dots i_t}}\right]^{1/M}$$
$$\times \exp\left\{\sum_{l=1}^{s/m}{}'' (x^{c_{ml}} - 1)\frac{v^l}{ml}\right\}$$

and $H_s(1, v; a)$, where $m^{-i_1 \cdots i_t} = \text{l.c.m.}(m_{i_1}/m, \dots, m_{i_t}/m)$, $M = \text{l.c.m.}(m_1, \dots, m_a)$, and Σ'' denotes summation only over numbers which are multiples of at least one of $m_1/m, \dots, m_a/m$.

We denote these coefficients by C_p and D_p respectively. Exactly as in the proof of (12) and (19), it can be established that for $x = e^{it/h_s}$ we have

$$C_p \sim \frac{R^* \psi_s(1)}{\Gamma(\delta^*)\, p^{1-\delta^*}}, \quad D_p \sim \frac{R^*}{\Gamma(\delta^*)\, p^{1-\delta^*}},$$

where R^* is a constant, $0 < \delta^* < 1$ and

$$\psi_s(v) = \exp\left\{\sum_{l=1}^{s/m}{}'' (x^{c_{ml}} - 1)\frac{v^l}{ml}\right\}.$$

Consequently,

$$A_n^{(2)}(t) = e^{-itb_s/h_s}\frac{C_p}{D_p} \sim \exp\left\{-it\frac{b_s}{h_s} + \sum_{l=1}^{s/m}{}'' \frac{1}{ml}\left(e^{itc_{ml}/h_s} - 1\right)\right\}.$$

It is easy to see that $A_n^{(2)}(t) \to e^{-t^2/2}$ as $n \to \infty$. The theorem is proved.

§4. Limit distributions of $\xi_n(m, \beta)$

Consider $S_n(m, \beta)$, the collection of permutations of degree n which have cycle lengths equal to β modulo m. If $C_{nk}(m, \beta)$ is the number of permutations in $S_n(m, \beta)$ with k cycles and

$$C_n(x; m, \beta) = \sum_{k=0}^{n} C_{nk}(m, \beta)\, x^k,$$

then

$$\sum_{n=0}^{\infty} C_n(x; m, \beta)\frac{t^n}{n!} = e^{xa(t; m, \beta)}, \tag{20}$$

where

$$a(t; m, \beta) = \sum_{j=0}^{\infty} \frac{t^{mj+\beta}}{mj+\beta}.$$

With the help of the mth root of unity we can write

$$a(t; m, \beta) = -\frac{1}{m}\sum_{k=1}^{m} e^{-\frac{2\pi\beta k i}{m}} \ln\left(1 - te^{\frac{2\pi k i}{m}}\right); \tag{21}$$

here we have in mind the principal value of the logarithm. The total number of permutations in $S_n(m, \beta)$ can be expressed as follows:

$$C_n(m, \beta) = C_n(1; m, \beta). \tag{22}$$

On $S_n(m, \beta)$ we assign a uniform probability distribution and consider the random variable $\xi_n(m, \beta)$, equal to the number of cycles in a randomly chosen permutation. The generating function of this variable can be written as follows:

$$P_n(x; m, \beta) = C_n(x; m, \beta)/C_n(m, \beta). \tag{23}$$

We will consider two cases: 1) $\beta = m$; 2) $\beta = m/2$, m even. The case $m = 2$ was investigated in [5].

From (20) and (21) we have

$$C_n(x; m, m) = n! \binom{\frac{x+n}{m} - 1}{n/m}, \quad m \mid n, \tag{24}$$

$$C_n\left(x; m, \frac{m}{2}\right) = n! \sum_{j=0}^{2n/m} \binom{x/m}{2n/m - j} \binom{x/m + j - 1}{j}, \quad \frac{m}{2} \,\Big|\, n. \tag{25}$$

Taking (22) and (23) into account, from these formulas we can easily obtain expressions for the generating functions of $\xi_n(m, m)$ and $\xi_n(m, m/2)$. Below an important role will be played by the following

LEMMA. *The following asymptotic representations are valid for the functions* $C_n(x; m, m)$ *and* $C_n(x; m, m/2)$ *as* $n \to \infty$:

$$C_n(x; m, m) = \frac{n!}{\Gamma\left(\frac{x}{m}\right)\left(\frac{n}{m}\right)^{1-\frac{x}{m}}} (1 + o(1)), \quad m \mid n, \tag{26}$$

$$C_n\left(x; m, \frac{m}{2}\right) = \frac{n!\, 2^{\frac{2x}{m} - 1}}{\Gamma\left(\frac{x}{m}\right)\left(\frac{n}{m}\right)^{1-\frac{x}{m}}} (1 + o(1)), \quad \frac{m}{2} \,\Big|\, n, \tag{27}$$

where $\Gamma(y)$ *is the gamma-function and* $o(1)$ *is a variable which vanishes uniformly with respect to* x *in* $[1 - \delta, 1 + \delta]$, $\delta > 0$.

PROOF. Using Stirling's formula for the gamma-function, as $k \to \infty$ we can obtain the following estimates:

$$\binom{y + k - 1}{k} = \frac{1}{\Gamma(y)\, k^{1-y}} (1 + o(1)), \tag{28}$$

$$\binom{y}{k} = \frac{(-1)^k y (y - 1)}{\Gamma(2 - y)\, k^{1+y}} (1 + o(1)), \tag{29}$$

which are uniform with respect to y in $[1 - \delta, 1 + \delta]$, $\delta > 0$.

(26) follows directly from (28). Therefore it remains to prove (27). We choose a number ν, $0 < \nu < \frac{1}{2}$, and separate the sum in (25) into three parts:

$$\frac{1}{n!} C_n\left(x; m, \frac{m}{2}\right) = S_1 + S_2 + S_3, \tag{30}$$

where

$$S_1 = \sum_{j=0}^{[(n/m)^\nu]} \binom{x/m}{(2n/m)-j}\binom{(x/m)+j-1}{j},$$

and S_2 and S_3 are the corresponding sums from $[(n/m)^\nu]+1$ to $2n/m - [(n/m)^\nu] - 1$ and from $2n/m - [(n/m)^\nu]$ to $2n/m$.

Applying (28), we find that

$$S_3 = \frac{1}{\Gamma\left(\frac{x}{m}\right)\left(\frac{2n}{m}\right)^{1-(x/m)}} \sum_{k=0}^{[(n/m)^\nu]} \binom{x/m}{k}(1+o(1)),$$

whence it follows that

$$S_3 = \frac{2^{2x/m-1}}{\Gamma\left(\frac{x}{m}\right)\left(\frac{n}{m}\right)^{1-x/m}}(1+o(1)) \tag{31}$$

as $n \to \infty$ uniformly with respect to x.

Using (29), we obtain

$$S_1 = \frac{(-1)^{2n/m}(x/m)((x/m)-1)}{\Gamma\left(2-\frac{x}{m}\right)\left(\frac{2n}{m}\right)^{1+x/m}} \sum_{j=0}^{[(n/m)^\nu]} \binom{-x/m}{j}(1+o(1)).$$

From this we deduce the estimate

$$S_1 = S_3 \cdot O\left(\frac{1}{n^{2x/m}}\right), \tag{32}$$

which is uniform with respect to x.

In turn, we separate the sum S_2 into two sums $S_2^{(1)}$ and $S_2^{(2)}$ with limits of summation from $[(n/m)^\nu]+1$ to $n/m-1$ and from n/m to $2n/m - [(n/m)^\nu] - 1$ respectively.

With the help of (28) and (29) we obtain

$$|S_2^{(1)}| \leqslant \frac{\frac{x}{m}\left|\frac{x}{m}-1\right|}{\Gamma\left(\frac{x}{m}\right)\Gamma\left(2-\frac{x}{m}\right)} \frac{1+o(1)}{\left(\frac{2n}{m}\right)^{1+(x/m)}} \sum_{j=[(n/m)^\nu]+1}^{(n/m)-1} \frac{1}{j^{1-(x/m)}},$$

$$|S_2^{(2)}| \leqslant \frac{\frac{x}{m}\left|\frac{x}{m}-1\right|}{\Gamma\left(\frac{x}{m}\right)\Gamma\left(2-\frac{x}{m}\right)} \frac{1+o(1)}{\left(\frac{n}{m}\right)^{1-(x/m)}} \sum_{k=[(n/m)^\nu]+1}^{n/m} \frac{1}{k^{1+(x/m)}}.$$

From these inequalities it follows that $S_2^{(1)} = S_3 \cdot O((n/m)^{-x/m})$ and $S_2^{(2)} = S_3 \cdot O((n/m)^{-x\nu/m})$, i.e.

$$S_2 = S_3 \cdot O\left(\frac{1}{(n/m)^{x\nu/m}}\right), \quad n \to \infty, \tag{33}$$

uniformly with respect to x in the indicated region. The validity of (27) now follows from (30)–(33). The lemma is proved.

Setting $x = 1$ in (26) and (27) and taking account of (22), we obtain

COROLLARY 1. *As $n \to \infty$ the following asymptotic formulas hold*:

$$C_n(m, m) = \frac{n!}{\Gamma(1/m)\,(n/m)^{1-(1/m)}}(1 + o(1)), \quad m \mid n,$$

$$C_n(m, m/2) = \frac{n!\,2^{(2/m)-1}}{\Gamma(1/m)\,(n/m)^{1-(1/m)}}(1 + o(1)), \quad \frac{m}{2} \Big| n.$$

From the lemma, Corollary 1 and (23) we obtain

COROLLARY 2. *As $n \to \infty$, for the generating functions of the random variables $\xi_n(m, m)$ and $\xi_n(m, m/2)$ the asymptotic representations*

$$P_n(x; m, m) = \frac{\Gamma(1/m)}{\Gamma(x/m)}\left(\frac{n}{m}\right)^{(x-1)/m}(1 + o(1)), \quad m \mid n, \tag{34}$$

$$P_n(x; m, m/2) = \frac{\Gamma(1/m)}{\Gamma(x/m)}\left(\frac{4n}{m}\right)^{(x-1)/m}(1 + o(1)), \quad \frac{m}{2} \Big| n, \tag{35}$$

hold uniformly for all x, $1 - \delta \leqslant x \leqslant 1 + \delta$, $\delta > 0$.

We can now find limit distributions for the random variable $\xi_n(m, m/2)$.

THEOREM 3. *Suppose $m = m(n)$ is a function whose values are the even natural numbers, with $m/2|n$.*

a) *If $m^{-1}\ln(n/m) \to \lambda < \infty$ as $n \to \infty$, then the random variable $\xi_n(m, m/2)$ in the limit has a Poisson distribution with parameter λ.*

b) *If $m^{-1}\ln(n/m) \to \infty$ as $n \to \infty$, then the random variable*

$$\xi_n'(m, m/2) = \left(\xi_n(m, m/2) - \frac{1}{m}\ln\frac{n}{m}\right)\left(\frac{1}{m}\ln\frac{n}{m}\right)^{-1/2}$$

has an asymptotically normal distribution with parameters (0, 1).

Exactly the same assertions also hold for the random variable $\xi_n(m, m)$ under the condition that $m = m(n)$ takes on natural values and $m|n$.

PROOF. Consider $M_n(t; m, m/2)$, the moment-generating function of the random variable $\xi_n(m, m/2)$. From (35) it follows that as $n \to \infty$

$$M_n(t; m, m/2) = \frac{\Gamma(1/m)}{\Gamma\left(\frac{1}{m}e^t\right)}\left(\frac{4n}{m}\right)^{\frac{1}{m}(e^t - 1)}(1 + o(1)),$$

uniformly for all t, $-\delta' \leqslant t \leqslant \delta'$, $\delta' > 0$. Hence in case a) it follows that

$$\lim_{n \to \infty} M_n(t; m, m/2) = e^{\lambda(e^t - 1)}$$

for every fixed t in $[-\delta', \delta']$. Since the right side of the last equality contains the moment-generating function of a Poisson distribution with parameter λ, the validity of part a) of the theorem follows from Curtiss' theorem (see [7]).

From (35) it is easy to obtain an asymptotic representation for the moment-generating function of the variable $\xi_n'(m, m/2)$:

$$M_n'(t; m, m/2) = e^{-t\sigma}\frac{\Gamma(1/m)}{\Gamma\left(\frac{1}{m}e^{t/\sigma}\right)}\left(\frac{4n}{m}\right)^{\frac{1}{m}(e^{t/\sigma} - 1)}(1 + o(1)),$$

uniform with respect to $t \in [-\delta', \delta']$ as $n \to \infty$; here $\sigma^2 = m^{-1}\ln(n/m)$.

From this representation it follows that

$$\lim_{n\to\infty} M_n'(t; m, m/2) = e^{t^2/2}$$

for every fixed t in $[-\delta', \delta']$.

The validity of part b) of Theorem 3 now follows from the above-mentioned theorem of Curtiss.

The proof of the corresponding assertions of the theorem with regard to the random variable $\xi_n(m, m)$ is carried out in exactly the same way, using (34). Theorem 3 is proved.

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