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OPTIMAL STOPPING, RANDOMIZED STOPPING AND SINGULAR CONTROL WITH GENERAL INFORMATION FLOW[1)]

В этой статье мы преследуем двоякую цель. Во-первых, мы распространяем хорошо известное соотношение между оптимальной остановкой и рандомизированной остановкой заданного случайного процесса на ситуацию, когда доступный поток информации — это фильтрация, которая априори не предполагается как-либо связанной с фильтрацией случайного процесса. В этом случае мы говорим об оптимальной остановке и рандомизированной остановке при *общем* потоке информации. Во-вторых, следуя идее Н. В. Крылова (1977), мы вводим специальную задачу *сингулярного* стохастического управления с общей информацией и показываем, что она эквивалентна задачам оптимального управления и рандомизированного управления с частичной информацией. Далее мы показываем, что решение указанной задачи сингулярного управления может быть выражено в терминах вариационных неравенств для частичной информации.

Ключевые слова и фразы: оптимальная остановка, оптимальное управление, сингулярное управление, общий поток информации.

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1. Introduction. There are several classic papers in the literature on the relation between optimal stopping, randomized stopping and singular control of a given stochastic process with filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t \geqslant 0}$. See, e.g., [5] and

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the references therein. For other papers of related interest, see [1], [2], [6], [7], [9], [11], and [12]. Two fundamental references to the theory of optimal stopping are [4] and [13].

The purpose of this paper is to extend this relation to a situation where the admissible stopping times are required to be stopping times with respect to another given *information flow* $\mathbb{H} = \{\mathcal{H}_t\}_{t\geqslant 0}$. We make no assumptions a priori about the relation between $\mathbb{H}$ and $\mathbb{F}$.

If $\mathcal{H}_t \subseteq \mathcal{F}_t$ for all t, we call this a *partial information optimal stopping problem*.

If, on the other hand, $\mathcal{H}_t \supseteq \mathcal{F}_t$ for all t, we call this an *inside information optimal stopping problem*. Partial information optimal stopping problems are studied in [10], using a maximum principle for singular stochastic control of jump diffusions and associated reflected backward differential equations. A special inside information optimal stopping problem is studied (and solved) in [6], based on Malliavin calculus and forward integration theory.

In the current paper, the admissible controls, such as the singular and the optimal stopping controls, are required to be $\mathbb{H}$-adapted. This is a common situation in many applications, and one of our motivations for this paper is to be able to study such more realistic optimal stopping problems. In the current paper we extend the results of [5] and [10] to a more general setting. More precisely, we prove the equivalence of the following three problems:

- optimal stopping with general information flow;
- randomized stopping with general information flow;
- singular control with general information flow.

We then illustrate our result by finding explicit optimal stopping control under delayed information. Finally, we obtain variational inequalities for singular control under partial information. Our main idea is based on the following two key elements:

(i) we prove an extension of Lemma 2(a) in [8, Chap. 1, § 5] to a *general* information flow situation;

(ii) we extend the results in [5] to *general* information.

2. Framework and problem formulations. Let $(\Omega, \mathcal{F}, \mathbb{F}=\{\mathcal{F}_t\}_{t\geqslant 0}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions. Let $T \leqslant \infty$ be a fixed terminal time, and let $\mathbb{H} := \{\mathcal{H}_t\}_{t\geqslant 0}$ be another collection of complete σ-algebras $\mathcal{H}_t$, not necessarily satisfying the usual conditions.

We do not assume a priori that there is any relation between $\mathbb{H}$ and $\mathbb{F}$.

For example, we could have

- $\mathcal{H}_t = \mathcal{F}_{(t-\delta)^+}$, $t \geqslant 0$ (delayed/partial information case), or
- $\mathcal{H}_t = \mathcal{F}_{t+\delta}$, $t \geqslant 0$, with $\delta > 0$ (advanced information case).

Further, let $\mathcal{T}_{\mathbb{H}} = \mathcal{T}_{\mathbb{H}}^{(T)}$ denote the set of all $\mathbb{H}$-stopping times $\tau \leqslant T$, i.e., the set of all functions

$$\tau\colon \Omega \to [0,T]$$

such that $\{\omega\colon \tau(\omega) \leqslant t\} \in \mathcal{H}_t$ for all $t \in [0,T]$. In what follows we let $\{k(t)\}_{t\geqslant 0}$ be a given $\mathbb{F}$-predictable process which is continuous at $t = 0$. We assume that $t \mapsto k(t)$ is left-continuous with right-sided limits for all $t \in [0,T]$ (càglàd) and satisfies

$$\sup_{\tau\in\mathcal{T}_{\mathbb{H}}} \mathbf{E}[|k(\tau)|] =: \kappa < \infty. \tag{2.1}$$

We put $k(\tau(\omega)) = 0$ if $\tau(\omega) = \infty$.

Remark 2.1. If the filtration $\mathbb{H}$ satisfies the usual conditions, one can reduce the problem to the complete information case when $\mathbb{F} = \mathbb{H}$ by replacing the process $k(t)$ by its $\mathbb{H}$-optional conditional expectation $\widetilde{k}(t) := \mathbf{E}[k(t) \mid \mathcal{H}_t]$. However, if $\mathbb{H}$ is a strict subset of $\mathbb{F}$, we cannot go the other way. More precisely, given two arbitrary filtrations $\mathbb{F}$ and $\mathbb{H}$, there is more information in the statement

$$\sup_{\tau\in\mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau)] = \sup_{G\in\mathcal{G}_{\mathbb{H}}} \mathbf{E}\left[\int_0^T k(t)\,dG(t)\right] \quad \text{for any } \mathbb{F}\text{-adapted process } k(\,\cdot\,) \tag{2.2}$$

than in the statement

$$\sup_{\tau\in\mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau)] = \sup_{G\in\mathcal{G}_{\mathbb{H}}} \mathbf{E}\left[\int_0^T k(t)\,dG(t)\right] \quad \text{for all } \mathbb{H}\text{-adapted processes } k(\,\cdot\,). \tag{2.3}$$

Moreover, such a reduction may not be an advantage when it comes to solving the problem. (See Example 4.1.)

We also point out that several of our results do not need that the filtration $\mathbb{H}$ satisfies the so-called usual conditions, which would be needed for the reduction argument above.

Note. All integrals in this paper are interpreted in the Lebesgue–Stieltjes sense.

The purpose of the current paper is to study the relation between the following three problems in a general information flow context.

We first consider the following *general* information optimal stopping problem.

Problem 2.1 (optimal stopping). Find $\Phi \in \mathbf{R}$ and $\tau^* \in \mathcal{T}_{\mathbb{H}}$ such that

$$\Phi := \sup_{\tau\in\mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau)] = \mathbf{E}[k(\tau^*)]. \tag{2.4}$$

Next we formulate the corresponding *general* information randomized stopping problem.

Problem 2.2 (randomized stopping). Let $\mathcal{G}_{\mathbb{H}}$ be the set of $\mathbb{H}$-adapted, right-continuous, and nondecreasing processes $G(t)$, $t \in [0,T]$, such that

$$G(0) = 0 \quad \text{and} \quad G(T) \leqslant 1 \quad \text{a.s.}$$

Find $\Lambda \in \mathbf{R}$ and $G^* \in \mathcal{G}_{\mathbb{H}}$ such that

$$\Lambda := \sup_{G \in \mathcal{G}_{\mathbb{H}}} \mathbf{E}\biggl[\int_0^T k(t)\, dG(t)\biggr] = \mathbf{E}\biggl[\int_0^T k(t)\, dG^*(t)\biggr].$$

Finally, we introduce our corresponding *general* information singular control problem.

Problem 2.3 (singular control). Let $\mathcal{A}_{\mathbb{H}}$ denote the set of all $\mathbb{H}$-adapted nondecreasing right-continuous processes $\xi(t)\colon [0,T] \to [0,\infty)$ such that $\xi(0) = 0$ and

$$\int_{[0,T]} \exp(-\xi(s))\, d\xi(s) \leqslant 1.$$

Find $\Psi \in \mathbf{R}$ and $\xi^* \in \mathcal{A}_{\mathbb{H}}$ such that

$$\Psi := \sup_{\xi \in \mathcal{A}_{\mathbb{H}}} \mathbf{E}\biggl[\int_0^T k(t) \exp(-\xi(t))\, d\xi(t)\biggr] = \mathbf{E}\biggl[\int_0^T k(t) \exp(-\xi^*(t))\, d\xi^*(t)\biggr].$$

We will prove that all these three problems are equivalent, in the sense that

$$\Phi = \Lambda = \Psi,$$

and we will find explicit relations between the optimal τ^*, G^*, and ξ^*.

3. Randomized stopping and optimal stopping with general information flow. In this section we prove that Problem 2.1 and Problem 2.2 are equivalent. The following result may be regarded as an extension of Theorem 2.1 in [5] to general information.

Theorem 3.1. *We have*

$$\Lambda := \sup_{G \in \mathcal{G}_{\mathbb{H}}} \mathbf{E}\biggl[\int_0^T k(t)\, dG(t)\biggr] = \sup_{\tau \in \mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau)] =: \Phi.$$

Proof. Choose $\tau \in \mathcal{T}_{\mathbb{H}}$ and define, for $n = 1, 2, \dots,$

$$G^{(n)}(t) = \mathbf{1}_{\{t \geqslant \tau > 0\}} + (1 - \exp(-nt))\mathbf{1}_{\{\tau = 0\}} \quad \text{for } t < T; \qquad G^{(n)}(T) = 1. \tag{3.1}$$

Then $G^{(n)}(\,\cdot\,) \in \mathcal{G}_{\mathbb{H}}$ and we see that

$$\mathbf{E}[k(\tau)] = \lim_{n \to \infty} \mathbf{E}\biggl[\int_0^T k(t)\, dG^{(n)}(t)\biggr] \leqslant \sup_{G \in \mathcal{G}_{\mathbb{H}}} \mathbf{E}\biggl[\int_0^T k(t)\, dG(t)\biggr].$$

Since $\tau \in \mathcal{T}_{\mathcal{H}}$ was arbitrary, this proves that

$$\sup_{\tau \in \mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau)] \leqslant \sup_{G \in \mathcal{G}_{\mathbb{H}}} \mathbf{E}\biggl[\int_0^T k(t)\, dG(t)\biggr].$$

To get the opposite inequality, we define for each $G \in \mathcal{G}_{\mathbb{H}}$ and $r \in [0, G(T)) = [0, 1)$, the time change $\alpha(r)$ by

$$\alpha(r) = \inf\{s \geqslant 0 \colon G(s) \geqslant r\}.$$

Then $\{\omega \colon \alpha(r) \leqslant t\} = \{\omega \colon G(t) \geqslant r\} \in \mathcal{H}_t$, so $\alpha(r) \in \mathcal{T}_{\mathbb{H}}$ for all r. Moreover, $G(\alpha(t)) = t$ for a.a. t and hence

$$\begin{aligned}\mathbf{E}\biggl[\int_0^T k(t)\,dG(t)\biggr] &= \mathbf{E}\biggl[\int_0^{G(T)} k(\alpha(r))\,dr\biggr] \\ &\leqslant \int_0^1 \sup_{\tau \in \mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau)]\,dr = \sup_{\tau \in \mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau)].\end{aligned}$$

Theorem 3.1 is proved.

4. Singular control and optimal stopping with general information. In this section we prove that Problem 2.1 and Problem 2.3 are equivalent.

Theorem 4.1. *Define* $\mathcal{A}^c_{\mathbb{H}} = \{\xi \in \mathcal{A}_{\mathbb{H}} \colon \xi$ *is continuous*$\}$, $\overline{\mathbb{H}} = \{\overline{\mathcal{H}}_t\}_{0 \leqslant t \leqslant T} = \{\mathcal{H}_t \cap \mathcal{F}_{t-}\}_{0 \leqslant t \leqslant T}$ $(\mathcal{F}_{0-} := \mathcal{F}_0)$, $\mathcal{G}^*_{\overline{\mathbb{H}}} = \{G \in \mathcal{G}_{\overline{\mathbb{H}}} \colon G$ *is* $\overline{\mathbb{H}}$*-predictable*$\}$, *and* $\mathcal{T}^*_{\overline{\mathbb{H}}} = \{\tau \in \mathcal{T}_{\overline{\mathbb{H}}} \colon \tau$ *is* $\overline{\mathbb{H}}$*-predictable*$\}$. *Further, we assume that the information flow* $\overline{\mathbb{H}}$ *is right-continuous and* $\mathbf{E}[\sup_{0 \leqslant t \leqslant T} |k(t)|] < \infty$. *Then*

$$\begin{aligned}\sup_{\xi \in \mathcal{A}^c_{\mathbb{H}}} \mathbf{E}\biggl[\int_0^T k(t)\exp(-\xi(t))\,d\xi(t)\biggr] &= \Psi := \sup_{\xi \in \mathcal{A}_{\mathbb{H}}} \mathbf{E}\biggl[\int_0^T k(t)\exp(-\xi(t))\,d\xi(t)\biggr] \\ &= \sup_{G \in \mathcal{G}_{\mathbb{H}}} \mathbf{E}\biggl[\int_0^T k(t)\,dG(t)\biggr] = \sup_{\tau \in \mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau)] \\ &=: \Phi = \sup_{G \in \mathcal{G}^*_{\overline{\mathbb{H}}}} \mathbf{E}\biggl[\int_0^T k(t)\,dG(t)\biggr] = \sup_{\tau \in \mathcal{T}^*_{\overline{\mathbb{H}}}} \mathbf{E}[k(\tau)].\end{aligned}$$

Proof. Let $\xi \in \mathcal{A}_{\mathbb{H}}$. Then $w(t) := \int_{[0,t]} \exp(-\xi(s))\,d\xi(s) \in \mathcal{G}_{\mathbb{H}}$ and hence, by Theorem 3.1,

$$\begin{aligned}\mathbf{E}\biggl[\int_0^T k(t)\exp(-\xi(t))\,d\xi(t)\biggr] &= \mathbf{E}\biggl[\int_0^T k(t)\,dw(t)\biggr] \leqslant \sup_{G \in \mathcal{G}_{\mathbb{H}}} \mathbf{E}\biggl[\int_0^T k(t)\,dG(t)\biggr] \\ &= \sup_{\tau \in \mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau)].\end{aligned}$$

Therefore,

$$\sup_{\xi \in \mathcal{A}_{\mathbb{H}}} \mathbf{E}\biggl[\int_0^T k(t)\exp(-\xi(t))\,d\xi(t)\biggr] \leqslant \sup_{\tau \in \mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau)]. \tag{4.1}$$

To get the opposite inequality, choose $\delta > 0$ and $\tau \in \mathcal{T}_{\mathbb{H}}$. By left-continuity of k we may assume that $\tau < T$ a.s. Define, for $n = 1, 2, \ldots$,

$$u^{(n)}(t) = \begin{cases} 0 & \text{for } t < \tau, \\ n & \text{for } t \geqslant \tau, \end{cases} \qquad G^{(n)}(t) = \begin{cases} 0 & \text{for } t < \tau, \\ 1 - e^{-n(t-\tau)} & \text{for } t \geqslant \tau. \end{cases} \tag{4.2}$$

Then $\xi^{(n)}(t) := \int_0^t u^{(n)}(s)\,ds \in \mathcal{A}^c_{\mathbb{H}}$, $G^{(n)}(t) \in \mathcal{G}_{\mathbb{H}}$ and for any $\delta > 0$, we have

$$\begin{aligned} &\int_0^T k(t)u^{(n)}(t)\exp\Big(-\int_0^t u^{(n)}(s)\,ds\Big)\,dt \\ &\quad = \int_0^T k(t^+)u^{(n)}(t)\exp\Big(-\int_0^t u^{(n)}(s)\,ds\Big)\,dt \\ &\quad = \int_0^T k(t^+)\,dG^{(n)}(t) = \int_\tau^T k(t^+)\,dG^{(n)}(t) = I_n + J_n + K_n, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} I_n &= \int_\tau^{(\tau+\delta)\wedge T} k(\tau^+)dG^{(n)}(t), \\ J_n &= \int_\tau^{(\tau+\delta)\wedge T} (k(t^+) - k(\tau^+))\,dG^{(n)}(t), \\ K_n &= \int_{(\tau+\delta)\wedge T}^T k(t^+)dG^{(n)}(t). \end{aligned} \tag{4.4}$$

We see that if $\tau + \delta < T$, then

$$I_n = k(\tau^+)(1 - \exp(-n\delta)) \to k(\tau^+) \quad \text{as } n \to \infty,$$

and if $\tau + \delta > T$, we have

$$I_n = k(\tau^+)(1 - \exp(-n(T - \tau))) \to k(\tau^+) \quad \text{as } n \to \infty.$$

By right-continuity,

$$|J_n| \leqslant \sup_{t\in[\tau,\tau+\delta]} |k(t^+) - k(\tau^+)| \to 0 \quad \text{when } \delta \to 0.$$

Moreover,

$$|K_n| \leqslant \sup_{t\in[0,T]} |k(t^+)| \big[\exp(-n((\tau+\delta)\wedge T - \tau)) - \exp(-n(T-\tau))\big] \to 0$$

as $n \to \infty$. Combining the above in connection with $\mathbf{E}[\sup_{0\leqslant t\leqslant T} |k(t)|] < \infty$, it follows from the dominated convergence theorem that

$$\lim_{\delta\to 0}\lim_{n\to\infty} \mathbf{E}\Big[\int_0^T k(t)u^{(n)}(t)\exp\Big(-\int_0^t u^{(n)}(s)\,ds\Big)\,dt\Big] = \mathbf{E}[k(\tau^+)].$$

Therefore,

$$\sup_{\xi\in\mathcal{A}^c_{\mathbb{H}}} \mathbf{E}\left[\int_0^T k(t)\exp(-\xi(t))\,d\xi(t)\right] \geqslant \mathbf{E}[k(\tau^+)].$$

Since $\tau \in \mathcal{T}_{\mathbb{H}}$ was arbitrary, this proves that

$$\sup_{\xi\in\mathcal{A}^c_{\mathbb{H}}} \mathbf{E}\left[\int_0^T k(t)\exp(-\xi(t))\,d\xi(t)\right] \geqslant \sup_{\tau\in\mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau^+)]. \tag{4.5}$$

Let $\tau \in \mathcal{T}^*_{\overline{\mathbb{H}}}$. Then, using the fact that

$$\lim_{t\uparrow t_0} k(t^+) = k(t_0), \qquad t_0 \in (0,T],$$

we can find by assumption an announcing sequence $(\tau_n)_{n\geqslant 1} \subset \mathcal{T}_{\overline{\mathbb{H}}}$ of stopping times such that $\tau_n \leqslant \tau$ increases to τ and $\tau_n < \tau$, whenever $\tau > 0$. So $k(\tau_n^+) \to k(\tau)$ a.s. as $n \to \infty$. Then the dominated convergence theorem yields

$$\mathbf{E}[k(\tau_n^+)] \to \mathbf{E}[k(\tau)] \quad \text{as } n \to \infty.$$

Thus

$$\sup_{\tau\in\mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau^+)] \geqslant \mathbf{E}[k(\tau^*)], \qquad \tau^* \in \mathcal{T}^*_{\overline{\mathbb{H}}},$$

which implies

$$\sup_{\tau\in\mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau^+)] \geqslant \sup_{\tau\in\mathcal{T}^*_{\overline{\mathbb{H}}}} \mathbf{E}[k(\tau)]. \tag{4.6}$$

If $G \in \mathcal{G}^*_{\overline{\mathbb{H}}}$, we see from the proof of Theorem 3.1 that

$$\alpha(r) \in \mathcal{T}^*_{\overline{\mathbb{H}}}, \qquad r \in [0,1).$$

So

$$\sup_{G\in\mathcal{G}^*_{\overline{\mathbb{H}}}} \mathbf{E}\left[\int_0^T k(t)\,dG(t)\right] \leqslant \sup_{\tau\in\mathcal{T}^*_{\overline{\mathbb{H}}}} \mathbf{E}[k(\tau)] \tag{4.7}$$

by the proof of Theorem 3.1. Further, we observe that

$$\sup_{\xi\in\mathcal{A}^c_{\mathbb{H}}} \mathbf{E}\left[\int_0^T k(t)\exp(-\xi(t))\,d\xi(t)\right] \leqslant \sup_{G\in\mathcal{G}_{\mathbb{H}}} \mathbf{E}\left[\int_0^T k(t)\,dG(t)\right]. \tag{4.8}$$

On the other hand, we know in connection with our assumptions that

$$\mathbf{E}\left[\int_0^T k(t)\,dG(t)\right] = \mathbf{E}\left[\int_0^T {}^p(k)(t)\,dG(t)\right] = \mathbf{E}\left[\int_0^T k(t)\,d(G)^p(t)\right],$$

where ${}^p(\,\cdot\,)$ and $(\,\cdot\,)^p$ denote the predictable and the dual predictable projection with respect to the filtration $\mathbb{F}$, respectively. Under our assumptions on G it is known that $(G)^p(t)$, $0 \leqslant t \leqslant T$, is right-continuous and nondecreasing (see [3]). Further, we see from the definition of $(\,\cdot\,)^p$ that $(G)^p(t)$, $0 \leqslant t \leqslant T$, is $\overline{\mathbb{H}}$-adapted with $(G)^p(0) = 0$ and $(G)^p(T) \leqslant 1$ a.s. On the other hand, since $\overline{\mathbb{H}}$ is right-continuous, we find that elementary $\mathbb{F}$-predictable processes, which are $\overline{\mathbb{H}}$-adapted, are $\overline{\mathbb{H}}$-predictable. Using the latter fact combined with the monotone class theorem, it follows that $(G)^p(t)$, $0 \leqslant t \leqslant T$, belongs to $\mathcal{G}^*_{\overline{\mathbb{H}}}$. So

$$\begin{aligned}\sup_{G\in\mathcal{G}_{\mathbb{H}}} \mathbf{E}\biggl[\int_0^T k(t)\,dG(t)\biggr] &= \sup_{G\in\mathcal{G}_{\mathbb{H}}} \mathbf{E}\biggl[\int_0^T k(t)\,d(G)^p(t)\biggr]\\ &= \sup_{G\in\mathcal{G}^*_{\overline{\mathbb{H}}}} \mathbf{E}\biggl[\int_0^T k(t)\,dG(t)\biggr]. \end{aligned}\tag{4.9}$$

Therefore, (4.7) and (4.8) entail that

$$\begin{aligned}\sup_{\xi\in\mathcal{A}^c_{\mathbb{H}}} \mathbf{E}\biggl[\int_0^T k(t)\exp(-\xi(t))\,d\xi(t)\biggr] &\leqslant \sup_{G\in\mathcal{G}^*_{\overline{\mathbb{H}}}} \mathbf{E}\biggl[\int_0^T k(t)\,dG(t)\biggr]\\ &\leqslant \sup_{\tau\in\mathcal{T}^*_{\overline{\mathbb{H}}}} \mathbf{E}[k(\tau)]. \end{aligned}\tag{4.10}$$

So we conclude from (4.5), (4.1), Theorem 3.1, (4.9), (4.10) and (4.6) (in that order) that

$$\begin{aligned}\sup_{\tau\in\mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau^+)] &\overset{(4.5)}{\leqslant} \sup_{\xi\in\mathcal{A}^c_{\mathbb{H}}} \mathbf{E}\biggl[\int_0^T k(t)\exp(-\xi(t))\,d\xi(t)\biggr]\\ &\overset{(*)}{\leqslant} \sup_{\xi\in\mathcal{A}_{\mathbb{H}}} \mathbf{E}\biggl[\int_0^T k(t)\exp(-\xi(t))\,d\xi(t)\biggr] \overset{(4.1)}{\leqslant} \sup_{\tau\in\mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau)]\\ &\overset{\text{Th. 3.1}}{=} \sup_{G\in\mathcal{G}_{\mathbb{H}}} \mathbf{E}\biggl[\int_0^T k(t)\,dG(t)\biggr] \overset{(4.9)}{=} \sup_{G\in\mathcal{G}^*_{\overline{\mathbb{H}}}} \mathbf{E}\biggl[\int_0^T k(t)\,dG(t)\biggr]\\ &\overset{(4.10)}{\leqslant} \sup_{\tau\in\mathcal{T}^*_{\overline{\mathbb{H}}}} \mathbf{E}[k(\tau)] \overset{(4.6)}{\leqslant} \sup_{\tau\in\mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau^+)] \end{aligned}$$

(the inequality $\overset{(*)}{\leqslant}$ holds because $\mathcal{A}^c_{\mathbb{H}} \subset \mathcal{A}_{\mathbb{H}}$). Since the first term in this chain of inequalities/equations is the same as the last term, we conclude that all the terms are the same. Theorem 4.1 is proved.

It is of interest to find the connection between an optimal stopping time $\tau^* \in \mathcal{T}_{\mathbb{H}}$ for Problem 2.1 and the corresponding optimal singular controls G^* and ξ^* for Problems 2.2 and 2.3, respectively. The connection is given by the following result.

Theorem 4.2. (a) *Suppose $\tau^* \in \mathcal{T}_{\mathbb{H}}$ is an optimal stopping time for Problem 2.1. Define*

$$G^*(t) := \mathbf{1}_{\{t \geqslant \tau^* > 0\}} + \mathbf{1}_{\{\tau^*=0\}}. \tag{4.11}$$

Then $G^* \in \mathcal{G}_{\mathbb{H}}$ is an optimal singular control for Problem 2.2.

(b) *Conversely, suppose $G^* \in \mathcal{G}_{\mathbb{H}}$ is an optimal singular control for Problem 2.2. Define*

$$\alpha^*(r) := \inf\{s \geqslant 0 \colon G^*(s) \geqslant r\} \quad \text{for } r \in [0,1). \tag{4.12}$$

Then $\alpha^*(r) \in \mathcal{T}_{\mathbb{H}}$ and $\alpha^*(r)$ is an optimal stopping time for Problem 2.1, for all $r \in [0,1)$.

(c) *Suppose $\xi^* \in \mathcal{A}_{\mathbb{H}}$ is an optimal control for Problem 2.3. Then the process*

$$G^*(t) := \int_{[0,t]} \exp(-\xi^*(s))\, d\xi^*(s)$$

is an optimal control for Problem 2.2.

(d) *Conversely, suppose $G^*(t)$ is an optimal control for Problem 2.2. Define $\xi^*(t)$ to be a solution of the differential equation*

$$d\xi^*(t) = \exp(\xi^*(t))\, dG^*(t), \quad \xi^*(0^-) = 0.$$

Then $\xi^*(t)$ is an optimal control for Problem 2.3.

Proof. (a) Suppose $\tau^* \in \mathcal{T}_{\mathbb{H}}$ is optimal for Problem 2.1, and let G^* be as in (4.11). Then by Theorem 3.1

$$\begin{aligned}\sup_{\tau \in \mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau)] = \mathbf{E}[k(\tau^*)] = \mathbf{E}\left[\int_0^T k(t)\, dG^*(t)\right] \\ \leqslant \sup_{G \in \mathcal{G}_{\mathbb{H}}} \mathbf{E}\left[\int_0^T k(t)\, dG(t)\right] = \sup_{\tau \in \mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau)].\end{aligned}$$

Hence we have equality in the above, and therefore

$$\mathbf{E}\left[\int_0^T k(t)\, dG^*(t)\right] = \sup_{G \in \mathcal{G}_{\mathbb{H}}} \mathbf{E}\left[\int_0^T k(t)\, dG(t)\right],$$

which proves that G^* is optimal for Problem 2.2.

(b) Conversely, suppose $G^* \in \mathcal{G}_{\mathbb{H}}$ is optimal for Problem 2.2. Let $\alpha^*(r)$ be as in (4.12). Then $\alpha^*(r) \in \mathcal{T}_{\mathbb{H}}$ for all r and, by Theorem 3.1,

$$\begin{aligned}\sup_{G \in \mathcal{G}_{\mathbb{H}}} \mathbf{E}\left[\int_0^T k(t)\, dG(t)\right] &= \mathbf{E}\left[\int_0^T k(t)\, dG^*(t)\right] = \mathbf{E}\left[\int_0^{G^*(T)} k(\alpha^*(r))\, dr\right] \\ &= \int_0^1 \mathbf{E}[k(\alpha^*(r))]\, dr \leqslant \int_0^1 \sup_{\tau \in \mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau)]\, dr = \sup_{\tau \in \mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau)] \\ &\leqslant \sup_{G \in \mathcal{G}_{\mathbb{H}}} \mathbf{E}\left[\int_0^T k(t)\, dG(t)\right].\end{aligned}$$

We conclude that we have equality everywhere in the above, and therefore

$$\int_0^1 \mathbf{E}[k(\alpha^*(r))]\,dr = \sup_{\tau\in\mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau)]. \tag{4.13}$$

Since $\alpha^*(r)$ is a stopping time for all $r \in [0,1)$ we have

$$\mathbf{E}[k(\alpha^*(r))] \leqslant \sup_{\tau\in\mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau)] \quad \text{for all } r. \tag{4.14}$$

Therefore (4.13) is only possible if

$$\mathbf{E}[k(\alpha^*(r))] = \sup_{\tau\in\mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau)] \quad \text{for a.a. } r \in [0,1). \tag{4.15}$$

Choose arbitrary $\overline{r} \in (0,1]$. Since $\alpha^*(r)$ is left-continuous we can find $r_n \in (0,1)$ such that $\alpha^*(r_n)$ is optimal for all n and $\alpha^*(r_n) \to \alpha^*(\overline{r})$ as $n\to\infty$. This gives

$$\mathbf{E}[k(\alpha^*(\overline{r}))] = \lim_{n\to\infty} \mathbf{E}[k(\alpha^*(r_n))] = \sup_{\tau\in\mathcal{T}_{\mathbb{H}}} \mathbf{E}[k(\tau)].$$

Hence $\alpha^*(r)$ is an optimal stopping time for all $r\in(0,1]$.

(c), (d) If $G^*(t)$ and $\xi^*(t)$ are chosen as given in (c) and (d), respectively, then we see in either case that

$$\mathbf{E}\biggl[\int_0^T k(s)\,dG^*(s)\biggr] = \mathbf{E}\biggl[\int_0^T k(s)\exp(-\xi^*(s))\,d\xi^*(s)\biggr].$$

The two statements (c) and (d) follow from this.

Theorem 4.2 is proved.

Remark 4.1. In the case when $k(t)\geqslant 0$ for all $t\in[0,T]$, the optimal $G^*\in\mathcal{G}_{\mathbb{H}}$ satisfies

$$G^*(T) = 1,$$

and the optimal $\xi\in\mathcal{A}_{\mathbb{H}}$ satisfies

$$\int_{[0,T]} \exp(-\xi^*(s))\,d\xi^*(t) = 1.$$

Example 4.1 (the optimal time to sell when there is delayed information). To illustrate our results, we follow Example 3.1 in [9].

Let $\mathbb{F}$ be the filtration of one-dimensional Brownian motion $B(\cdot)$, and let $\mathbb{H}=\{\mathcal{H}_t\}$ be the delayed information flow given by $\mathcal{H}_t = \mathcal{F}_{(t-\delta)^+}$ for some constant $\delta>0$. Define

$$k(t) = \exp(-\rho t)(X(t)-a), \qquad t\in[0,T],$$

where $\rho > 0$, $a > 0$ are given constants, and the process X is a geometric Brownian motion of the form

$$dX(t) = X(t)[\mu(t)\,dt + \sigma(t)\,dB(t)], \qquad X(0) > 0,$$

where $\mu(t)$ and $\sigma(t) > 0$ are bounded $\mathbb{F}$-adapted processes. Then it follows from Theorem 3.1 in [9] that the optimal stopping time τ^* for Problem 2.1 has the form

$$\tau^* = \alpha + \delta,$$

where α is the optimal stopping time (which in some cases can be found explicitly) for a related optimal stopping problem with nondelayed information. By Theorem 4.2(a) the optimal G^* for the corresponding randomized stopping problem (Problem 2.2) is

$$G^*(t) = \begin{cases} 0, & t < \tau^*, \\ 1, & t \geqslant \tau^*. \end{cases}$$

And by Theorem 4.2(d) the optimal ξ^* for the corresponding singular control problem (Problem 2.3) is given by

$$\exp(-\xi^*(t))\,d\xi^*(t) = \delta_{\tau^*}(t), \qquad t \in [0, T],$$

where $\delta_{\tau^*}(t)$ is the Dirac point mass at $t = \tau^*$.

5. Singular control with partial information flow. In this section we assume that $\mathbb{H}$ satisfies the usual conditions and that we are in a partial information setting, i.e., that

$$\mathcal{H}_t \subseteq \mathcal{F}_t \quad \text{for all } t. \tag{5.1}$$

We also suppose that the terminal time horizon is finite: $T < \infty$.

We now turn to the partial information singular control problem (Problem 2.3).

Problem 5.1. Find $\Psi \in \mathbf{R}$ and $\widehat{\xi} \in \mathcal{A}_{\mathbb{H}}$ such that

$$\Psi = \sup_{\xi \in \mathcal{A}_{\mathbb{H}}} J(\xi) = J(\widehat{\xi}), \tag{5.2}$$

where

$$J(\xi) = \mathbf{E}\left[\int_0^T k(t)\exp(-\xi(t))\,d\xi(t)\right]. \tag{5.3}$$

Problem 5.1 can be considered as a generalization of the singular control problem discussed in [10, section 2], where a singular control version of the maximum principle is used. However, since the singular control ξ appears both in the integrand and as an integrator, the problem (5.2), (5.3) is not covered by the results in [10]. Here we give a direct approach based on a variational argument.

For $\xi \in \mathcal{A}_{\mathbb{H}}$, we define $\mathcal{V}(\xi)$ to be the set of càdlàg processes $\zeta(t)\colon [0,T] \to [0,\infty]$ of finite variation such that there exists $\delta = \delta(\xi) > 0$ such that

$$\xi + y\zeta \in \mathcal{A}_{\mathbb{H}} \quad \text{for all } y \in [0,\delta].$$

For $\xi \in \mathcal{A}_{\mathbb{H}}$ and $\zeta \in \mathcal{V}(\xi)$, we define $D(\xi,\zeta) \in \mathbf{R}$ by

$$\begin{aligned} D(\xi,\zeta) &:= \limsup_{y\to 0^+} \frac{1}{y}(J(\xi+y\zeta) - J(\xi)) \\ &= \limsup_{y\to 0^+} \frac{1}{y}\,\mathbf{E}\bigg[\int_0^T k(s)\big\{\exp\big(-(\xi(s)+y\zeta(s))\big)\,(d\xi(s)+y\,d\zeta(s)) \\ &\qquad - \exp(-\xi(s))\,d\xi(s)\big\}\bigg] \\ &= \limsup_{y\to 0^+} \frac{1}{y}\,\mathbf{E}\bigg[\int_0^T k(s)\big\{\exp(-\xi(s))\big(\exp(-y\zeta(s))-1\big)\,d\xi(s) \\ &\qquad + y\exp(-\xi(s))\exp(-y\zeta(s))\,d\zeta(s)\big\}\bigg] \\ &= \mathbf{E}\bigg[\int_0^T k(s)\exp(-\xi(s))\{-\zeta(s)\,d\xi(s)+d\zeta(s)\}\bigg]. \end{aligned} \tag{5.4}$$

Now suppose $\xi = \widehat{\xi}$ maximizes $J(\xi)$. Then by (5.4)

$$\mathbf{E}\bigg[\int_0^T k(s)\exp\big(-\widehat{\xi}(s)\big)\big\{-\zeta(s)\,d\widehat{\xi}(s)+d\zeta(s)\big\}\bigg] = D(\widehat{\xi},\zeta) \leqslant 0 \tag{5.5}$$

for all $\zeta \in \mathcal{V}(\widehat{\xi})$. In particular, if, for $\delta > 0$, we choose

$$\zeta_0(s) = \begin{cases} 0, & s < t, \\ \dfrac{(s-t)a}{\delta}, & t \leqslant s \leqslant t+\delta, \\ a, & s \geqslant t+\delta, \end{cases}$$

for some $t \in [0,T]$ and some bounded $\mathcal{H}_t$-measurable random variable $a \geqslant 0$, then $\zeta_0 \in \mathcal{V}(\widehat{\xi})$ and (5.5) gives

$$\begin{aligned} \mathbf{E}\bigg[\int_t^{t+\delta} k(s)\exp\big(-\widehat{\xi}(s)\big)\frac{(s-t)a}{\delta}\,d\widehat{\xi}(s) + \int_{t+\delta}^T k(s)\exp\big(-\widehat{\xi}(s)\big)a\,d\widehat{\xi}(s) \\ - \int_t^{t+\delta} k(s)\exp\big(-\widehat{\xi}(s)\big)\frac{a}{\delta}\,ds\bigg] \geqslant 0. \end{aligned}$$

Since this holds for all such a and all $\delta > 0$, we conclude that

$$\mathbf{E}\Big[\int_t^T k(s)\exp\big(-\widehat{\xi}(s)\big)\,d\widehat{\xi}(s) - k(t)\exp\big(-\widehat{\xi}(t)\big)\;\Big|\;\mathcal{H}_t\Big] \geqslant 0, \qquad t \in [0,T].$$

Next, let us choose

$$d\zeta_1(s) = d\widehat{\xi}(s) \quad \text{and} \quad d\zeta_2(s) = -d\widehat{\xi}(s).$$

Then $\zeta_i \in \mathcal{V}(\widehat{\xi})$ for $i = 1, 2$ and (5.5) gives

$$\mathbf{E}\Big[\int_0^T k(s)\exp\big(-\widehat{\xi}(s)\big)\big\{-\widehat{\xi}(s)\,d\widehat{\xi}(s) + d\widehat{\xi}(s)\big\}\Big] = 0. \tag{5.6}$$

Note that by the Fubini theorem we have

$$\begin{aligned}
&\int_0^T \Big(\int_t^T k(s)\exp\big(-\widehat{\xi}(s)\big)\,d\widehat{\xi}(s)\Big)\,d\widehat{\xi}(t) \qquad (5.7)\\
&\qquad = \int_0^T \Big(\int_0^s d\widehat{\xi}(t)\Big) k(s)\exp\big(-\widehat{\xi}(s)\big)\,d\widehat{\xi}(s)\\
&\qquad = \int_0^T k(s)\exp\big(-\widehat{\xi}(s)\big)\widehat{\xi}(s)\,d\widehat{\xi}(s).
\end{aligned}$$

Substituting (5.7) into (5.6) we get

$$\mathbf{E}\Big[\int_0^T \Big\{\int_t^T k(s)\exp\big(-\widehat{\xi}(s)\big)\,d\widehat{\xi}(s) - k(t)\exp\big(-\widehat{\xi}(t)\big)\Big\}\,d\widehat{\xi}(t)\Big] = 0.$$

This proves part (a) of the following theorem.

Theorem 5.1 (variational inequalities). (a) *Suppose $\widehat{\xi} \in \mathcal{A}_{\mathbb{H}}$ is optimal for the problem* (5.2), (5.3). *Then*

$$\mathbf{E}\Big[\int_t^T k(s)\exp\big(-\widehat{\xi}(s)\big)\,d\widehat{\xi}(s) - k(t)\exp\big(-\widehat{\xi}(t)\big)\;\Big|\;\mathcal{H}_t\Big] \geqslant 0, \qquad t \in [0,T], \tag{5.8}$$

and

$$\mathbf{E}\Big[\int_t^T k(s)\exp\big(-\widehat{\xi}(s)\big)\,d\widehat{\xi}(s) - k(t)\exp\big(-\widehat{\xi}(t)\big)\;\Big|\;\mathcal{H}_t\Big]\,d\widehat{\xi}(t) = 0, \qquad t \in [0,T]. \tag{5.9}$$

(b) *Conversely, suppose* (5.8), (5.9) *hold for some $\widehat{\xi} \in \mathcal{A}_{\mathbb{H}}$. Then*

$$D(\widehat{\xi}, \zeta) \leqslant 0 \quad \textit{for all } \zeta \in \mathcal{V}(\widehat{\xi}). \tag{5.10}$$

Proof. Statement (b) of Theorem 5.1 is proved by reversing the argument used to prove that (5.10) implies (5.8) and (5.9). We omit the details.

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