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The functional central limit theorem for semimartingales taking values in a Hilbert space

V.V. Lavrent'ev

In [1] conditions are given for the validity of the theorem on weak convergence of the probability distributions of real semi martingales to the distribution of the Gaussian martingale (in particular, to the distribution of a Wiener process).

The aim of the present note is to indicate conditions for the validity of the functional central limit theorem for a sequence of semimartingales taking values in a real separable Hilbert space $\mathbf{H}$.

Following [2], we define $\mathbf{H}\hat{\otimes}_1 \mathbf{H}$ to be the completion of the tensor product $\mathbf{H} \otimes \mathbf{H}$ with respect to the projective norm $\|\cdot\|_1$.

The mapping $(x \otimes y,\ x' \otimes y') \to (x, x')(y, y'),\ x, x', y, y' \in \mathbf{H}$, extends uniquely to a continuous bilinear form $(\cdot,\cdot)$ on $(\mathbf{H}\hat{\otimes}_1\mathbf{H}) \times (\mathbf{H}\hat{\otimes}_1\mathbf{H})$. With every element $b \in \mathbf{H}\hat{\otimes}_1\mathbf{H}$ we can associate a continuous linear nuclear operator $\tilde{b}$, which is uniquely determined by $(\tilde{b}(h), g) = (b, h \otimes g)$. A symmetric positive nuclear operator is usually called an S-operator.

The process $[\![X]\!]_t$: $P - \lim_{n\to\infty} \sum_{k\geqslant 0} (X_{(k+1)2^{-n}\wedge t} - X_{k2^{-n}\wedge t})^{\otimes 2}$, $x^{\otimes 2} \equiv x \otimes x$, taking values in $\mathbf{H}\hat{\otimes}_1\mathbf{H}$, is called *the tensorial quadratic variation of* X (see [2]).

If M is a square integrable martingale, then by $\langle M\rangle$ we denote the predictable increasing process for which $\|M\|^2 - \langle M\rangle$ is a martingale. We define the $\mathbf{H}\hat{\otimes}_1\mathbf{H}$-valued process $\langle\langle M\rangle\rangle$ as the predictable process with finite variation for which $M \otimes M - \langle\langle M\rangle\rangle$ is a martingale. Let X be a semimartingale and for $\varepsilon > 0$ let $X_t^\varepsilon = \sum_{0<s\leqslant t} \Delta X_s I(\|\Delta X_s\| > \varepsilon)$. The process $X - X^\varepsilon$ is again a semimartingale, has bounded jumps, and admits a (unique) representation $X_t - X_t^\varepsilon = B_t^\varepsilon + M_t^\varepsilon$, where B^ε is a predictable process with locally integrable variation and M^ε a locally square integrable martingale. Denoting by $\mu = \mu(dt, dx)$ the integral random measure of the jumps of the semimartingale X, we obtain the following decomposition:

$$X_t = B_t^\varepsilon + M_t^\varepsilon + \int_0^t \int_{\|x\|>\varepsilon} x\mu\,(ds,\ dx). \tag{1}$$

Main Results. Let $X^n (n \geqslant 1)$ be semimartingales admitting an expansion (1). By $\{S_t^{n\varepsilon}\}_{t\geqslant 0}$ we denote the set of S-operators defined by $(S_t^{n\varepsilon}y,\ y) = M(M_t^{n\varepsilon},\ y \otimes y)$, $y \in \mathbf{H}$; $\nu^n = \nu^n(dt, dx)$ is the compensator of μ^n (see [1]); $\xrightarrow{D}$ $(\xrightarrow{D_f})$ is weak convergence of (finite-dimensional) distributions of processes; $\xrightarrow{p}$, $\xrightarrow{d}$ are convergence of random variables in probability and in distribution. Henceforth M denotes a continuous Gaussian martingale.

***Theorem* 1.1.** *Suppose that for any* $t > 0$ *and* $\varepsilon \in (0,\ 1]$ *the following conditions are satisfied*:

(A) $$\int_0^t \int_{\|x\|>\varepsilon} \nu^n\,(ds,\ dx) \xrightarrow{p} 0,$$ (B) $$B_t^{n\varepsilon} \xrightarrow{p} 0,$$

(C) $$((\langle\langle M^{n\varepsilon}\rangle\rangle_t,\ e_i \otimes e_j) \xrightarrow{p} ((\langle\langle M\rangle\rangle_t,\ e_i \otimes e_j),$$

(D) *the family* $\{S_t^{n1}\}$ *is compact, that is,*

$$\sup_n \sum_{i=1}^\infty (S_t^{n1}e_i,\ e_i) < \infty \quad and \quad \sup_n \sum_{i=r}^\infty (S_t^{n1}e_i,\ e_i) \to 0 \quad for\ r \to \infty.$$

Then $X^n \xrightarrow{D_f} M$ *(*$\{e_i\}$ *is an orthonormal basis for* $\mathbf{H}$*).*

1.2. *If* (A), (C), *and* (D) *hold, and if* $\sup_{0<s\leqslant t} \|B_s^{n\varepsilon}\| \xrightarrow{p} 0$ *for any* $t > 0$ *and* $\varepsilon \in (0, 1]$ *then* $X^n \xrightarrow{D} M$.

Theorem 2.1. (A) *is equivalent to the condition*

(A*) $$\sup_{0<s\leqslant t} \| \Delta X_s^n \| \xrightarrow{p} 0, \qquad t>0.$$

2.2. *If* (A) (*or* (A*)) *is assumed, then* (C) *is equivalent to the condition*

(C*) $$([\![M^{n1}]\!]_t, e_i \otimes e_j) \xrightarrow{p} (\langle\langle M\rangle\rangle_t, e_i \otimes e_j), \qquad i, j \geqslant 1,$$

for all $t > 0$.

Corollary 1. *Let* X^n $(n \geqslant 1)$ *be a locally square integrable martingale, and suppose that the functional Lindeberg condition is satisfied*:

(L_2) $$\int_0^t \int_{\|x\|>\varepsilon} \| x \|^2 \nu^n (ds, dx) \xrightarrow{p} 0, \qquad \varepsilon \in (0, 1].$$

Then (C) *and* (C*) *are equivalent to the conditions*

(C_2) $$(\langle\langle X^n\rangle\rangle_t, e_i \otimes e_j) \xrightarrow{p} (\langle\langle M\rangle\rangle_t, e_i \otimes e_j),$$

(C_2^*) $$([\![X^n]\!]_t, e_i \otimes e_j) \xrightarrow{p} (\langle\langle M\rangle\rangle_t, e_i \cong e_j).$$

If the family of S-operators $\{S_t^n\}$, $((S_t^n y, y) = M(\langle\langle X^n\rangle\rangle_t, y \otimes y))$ *for any* $t > 0$ *is compact and if* (L_2), (C_2) *or* (L_2), (C_2^*) *hold, then* $X^n \xrightarrow{D} M$.

Corollary 2. *Let* X^n $(n \geqslant 1)$ *be locally square integrable martingales, suppose that* (L_2) *holds, and also* $M\, |(\langle\langle X^n\rangle\rangle_t - \langle\langle M\rangle\rangle_t, e_i \otimes e_j)\,| \to 0$, $i, j \geqslant 1$, $M\langle X^n\rangle_t \to \langle M\rangle_t$, $t > 0$. *Then* $X^n \xrightarrow{D} M$.

Remark 1. Theorems 1 and 2 and the corollaries imply, by a simple reformulation, sufficient conditions for the weak convergence to a Wiener process of the sum of random variables (which always form a semimartingale) in the form of series. Corollary 2 for this particular case is considered in [3].

Remark 2. If (A), (B), (C), and (D) hold for some fixed $t = T$, then so does the central limit theorem $X_T^n \xrightarrow{d} \mathcal{N}(0, S_T)$, where $\mathcal{N}(0, S_T)$ is a random variable having normal distribution with the zero mean covariational operator $S_T((S_T y, y) = M(\langle\langle M\rangle\rangle_T, y \otimes y))$.

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Moscow State University

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