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Boubaker Operational Matrix Method for Fractional Emden-Fowler Problem

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Abstract. In this paper the singular Emden-Fowler equation of fractional order is introduced and a computational method is proposed for its numerical solution. For the approximation of the solutions we have used Boubaker polynomials and defined the formulation for its fractional derivative operational matrix. However, the use of Boubaker polynomials is most recent, and has not been discussed in the literature, since most of application areas of these polynomials require orthogonal polynomials, and here we have introduced it for the first time. The operational matrixof the Caputo fractional derivative tool converts the Emden–Fowler equation to a system of algebraic equations whose solutions are easy to compute. Numerical examples are examined to prove the validity and the effectiveness of the proposed method.

Keywords: Boubaker polynomials, operational matrix of fractional derivatives, collocation method, fractional Emden–Fowler type equations.

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1. Introduction and preliminaries

In mathematical physics and nonlinear mechanics there exists sufficiently large number of particular singular fractional differential equations for which an exact analytic solution in terms of known functions does not exist [9,10,14,17,22]. One of these equations describing phenomena in mathematical physics and astrophysics such as, the thermal behaviour of a spherical cloud of gas isothermal gas sphere and theory of stellar structure, theory of thermionic currents among many others, is called the singular Emden–Fowler equation of fractional order, formulated as follows [12,18,20,21]

$$D^{2\alpha}u(x) + \frac{\lambda}{x^{\alpha}}D^{\alpha}u(x) + s(x)g(u(x)) = h(x), \ x \in (0,1), \ \ \lambda > 0, \ \ \frac{1}{2} < \alpha \leqslant 1. \quad (1)$$

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Subject to the conditions:

$$u(0) = a, \quad D^{\alpha} u(0) = b,$$

where a and b are constants, and D^{α} denotes the Caputo fractional derivative.
When $\alpha = 1, \lambda = 2$, and $h(x) = 1$, Eq.(1) becomes the Lane-Emden type equation.
It is also defined in a more general form as follows :

$$D^{\alpha} u(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{u^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \quad n-1 < \alpha < n, \ n \in \mathbb{N}, \ \alpha > 0. \tag{2}$$

For the Caputo derivative we have $D^{\alpha} C = 0$, where C is a constant, and

$$D^{\alpha} x^{\beta} = \begin{cases} 0, & \text{for } \beta \in \mathbb{N} \cup \{0\} \text{ and } \beta < \lceil \alpha \rceil, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha} & \text{for } \beta \in \mathbb{N} \cup \{0\} \text{ and } \beta \geqslant \lceil \alpha \rceil \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > \lceil \alpha \rceil. \end{cases}$$

The problem (1) has been studied by different methods, using the residual power series [20], homotopy analysis [12], reproducing kernel Hilbert space [21], the fractional differential transformation [18], polynomial least squares [6], the shifted Legendre operational matrix [25], the Chebyshev wavelets [13], the orthonormal Bernoulli's polynomials [23], and the orthonormal Bernstein polynomials [1]. There are many studies about solutions of Emden-Fowler equations ($\alpha = 1$), for e.g. in the monographs [2, 7, 8, 24, 26–28], where both analytical and numerical approaches are presented.

The purpose of this paper is to use the operational matrix of fractional derivative based on Boubaker polynomials for solving singular initial value problems of fractional Emden–Fowler type equations (1). To the best of our knowledge, this is the first time where the Boubaker operational matrices are used to obtain solutions for the singular Emden–Fowler equations of fractional order. First we present a new theorem aiming to reduce the fractional Emden–Fowler problem to a system of algebraic equations. The Boubaker polynomials were introduced for the first time by Boubaker in (2007). The first monomial definition of the Boubaker polynomials on the interval $x \in [0, 1]$, was introduced in [3–5, 11, 16, 19] in the following form

$$\mathbf{B}_0(x) = 1, \quad \mathbf{B}_n(x) = \sum_{p=0}^{\xi(n)} [\frac{(n-4p)}{(n-p)} C_{n-p}^{p}](-1)^p x^{n-2p}, \quad n \geqslant 1, \tag{3}$$

where $\xi(n) = \left\lfloor \frac{i}{2} \right\rfloor = \frac{2n + ((-1)^n - 1)}{4}$, and $C_{n-r}^{r} = \frac{(n-p)!}{r!(n-2p)!}$. The symbol $\lfloor . \rfloor$ denotes the floor function, i.e., the function which maps a real number to the greatest preceding integer.

The Boubaker polynomials could be computed by the following recursive formula

$$\mathbf{B}_m(x) = x\mathbf{B}_{m-1}(x) - \mathbf{B}_{m-2}(x), \quad m \geqslant 2. \tag{4}$$

We will construct operational matrix of Caputo fractional derivative $\mathbf{D}^{(\alpha)}$ for the Boubaker polynomials using the following relation

$$D^{\alpha}\mathbf{B}(x) \simeq \mathbf{D}^{(\alpha)}\mathbf{B}(x), \tag{5}$$

where $\mathbf{B}(x) = [B_0(x), B_1(x), \ldots, B_N(x)]^T$ denotes the Boubaker vector and $\mathbf{D}^{(\alpha)}$ is an $(N+1) \times (N+1)$ dimensional matrix. This work focuses on solving equation (1) by using Boubaker operational matrix of fractional derivative.

Consequently, the remainder of the paper is arranged as follows. In Sec. 2., we express the Boubaker polynomials in terms of Taylor basis, and function approximation. In Sec. 3. the operational matrix of Caputo fractional derivatives is constructed. In Sec. 4., we use Boubaker polynomials method for solving the fractional Emden–Fowler type equations. In Sec. 5. some numerical examples are given to show the accuracy of this method.

2. Boubaker's matrix and function approximation

By using the expression (3) and taking $n = 0, \dots, N$, we can express Boubaker polynomials in terms of Taylor basis [5,11,19] as follows

$$\mathbf{B}(x) = \mathbf{MT}(x), \quad x \in [0,1], \tag{6}$$

where

$$\mathbf{T}(x) = [1, x, \dots, x^N]^T, \tag{7}$$

and if N is odd,

$$\mathbf{M} = \begin{bmatrix} m_{0,0} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & m_{1,0} & 0 & 0 & \cdots & 0 & 0 \\ m_{2,1} & 0 & m_{2,0} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{N-1,\frac{N-1}{2}} & 0 & m_{N-1,\frac{N-3}{2}} & 0 & \cdots & m_{N-1,0} & 0 \\ 0 & m_{n,\frac{N-1}{2}} & 0 & m_{N,\frac{N-3}{2}} & \cdots & 0 & m_{N,0} \end{bmatrix}$$

if N is even,

$$\mathbf{M} = \begin{bmatrix} m_{0,0} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & m_{1,0} & 0 & 0 & \cdots & 0 & 0 \\ m_{2,1} & 0 & m_{2,0} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & m_{N-1,\frac{N-2}{2}} & 0 & m_{N-1,\frac{N-4}{2}} & \cdots & m_{N-1,0} & 0 \\ m_{N,\frac{N}{2}} & 0 & m_{N,\frac{N-2}{2}} & 0 & \cdots & 0 & m_{N,0} \end{bmatrix}$$

where

$$\mathbf{B}_n(x) = \sum_{p=0}^{\xi(n)} m_{n,p} x^{n-2p}, \quad n = 0,\ 1, \dots, N,\ p = 0, 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor, \tag{8}$$

$$m_{n,p} = \Big[\frac{(n-4p)}{(n-p)} C_{n-p}^{p}\Big](-1)^p. \tag{9}$$

It can be observed, that $\mathbf{M}$ is a lower triangular matrix, and $|\mathbf{M}| = \Pi_{i=0}^{N} m_{i,0} = 1$ thus it is non-singular and the inverse $\mathbf{M}^{-1}$ exists.

It is clear that $\mathbf{S}_N = span\{B_0(x), B_1(x), \dots, B_N(x)\}$ is a finite dimensional and closed subspace of the Hilbert space $L^2[0,1]$, therefore $\mathbf{S}_N$ is a complete subspace and there is a unique best approximation out of $\mathbf{S}_N$ such that $u_N \in \mathbf{S}_N$ for each $u \in L^2[0,1]$, (see [15,19]), i.e.,

$$\forall y \in \mathbf{S}_N \ \ \|u - u_N\| \leqslant \|u - y\|. \tag{10}$$

Since $u_N \in \mathbf{S}_N$, there exist unique coefficients c_i, $i = 0, 1, \ldots, N$ such that

$$u(x) \simeq u_N(x) = \sum_{i=0}^{n} c_i B_n(x) = \mathbf{C}^T \mathbf{B}(x), \tag{11}$$

where $\mathbf{C}$ is an $(N+1) \times (1)$ vector given by $\mathbf{C} = [c_0, c_1, \ldots, c_N]^T$, $\mathbf{B}(\mathbf{x})$ is the vector function defined in Eq. 5, and the coefficients of vector $\mathbf{C}$ can be computed by $\mathbf{C}^T \langle \mathbf{B}(\mathbf{x}), \mathbf{B}(\mathbf{x}) \rangle = \langle u(x), \mathbf{B}(\mathbf{x}) \rangle$, such that

$$\langle u(x), \mathbf{B}(\mathbf{x}) \rangle = \int_0^1 u(x) \mathbf{B}^T(x) dx, \tag{12}$$

where $\langle .,. \rangle$ denotes the standard inner product on $L^2[0,1]$. Thus, by the definition $\mathbf{Q} = \langle \mathbf{B}(x), \mathbf{B}(x) \rangle$, we get

$$\mathbf{C}^T = \left(\int_0^1 u(x) \mathbf{B}^T(x) dx \right) \mathbf{Q}^{-1}, \tag{13}$$

where $\mathbf{Q}$ is the following $(N+1) \times (N+1)$ matrix

$$\mathbf{Q} = \langle \mathbf{B}(x), \mathbf{B}(x) \rangle = \int_0^1 \mathbf{B}(x) \mathbf{B}^T(x) dx = \mathbf{M} \left(\int_0^1 \mathbf{T}(x) \mathbf{T}^T(x) dx \right) \mathbf{M}^T = \mathbf{M} \mathbf{H} \mathbf{M}^T,$$

where $\mathbf{H} = [h_{ij}]_{(N+1)\times(N+1)}$ is the well-known Hilbert matrix, the components of which can be computed as

$$h_{ij} = \frac{1}{i+j+1}, \quad i, j = 1, \ldots, N. \tag{14}$$

Theorem 2.1 (see [11, 15]). *The elements $B_0, B_1, \ldots, B_N$ of the Hilbert space $L^2[0,1]$ form a linearly independent set in $L^2[0,1]$ if and only if*

$$\boldsymbol{G}(B_0, B_1, \ldots, B_N) = \begin{vmatrix} \langle B_0, B_0 \rangle & \langle B_0, B_1 \rangle & \cdots & \langle B_0, B_N \rangle \\ \langle B_1, B_0 \rangle & \langle B_1, B_1 \rangle & \cdots & \langle B_1, B_N \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle B_N, B_0 \rangle & \langle B_N, B_1 \rangle & \cdots & \langle B_N, B_N \rangle \end{vmatrix} \neq 0.$$

Theorem 2.1 proves that $\mathbf{Q}$ is symmetric and non-singular, so $\mathbf{Q}^{-1}$ exists.

Lemma 1 (see [15, 19]). *Suppose that $u \in C^{N+1}[0,1]$ and*

$$\mathbf{S}_N = span \{ B_N(x), B_n(x), \ldots, B_N(x) \}.$$

Let u_0 be the best approximation for u in $\mathbf{S}_N$ then

$$||u(x) - u_0(x)||_{L^2[0,1]} \leqslant \frac{Max_{x\in[0,1]} |u^{(N+1)}(x)|}{(N+1)\sqrt{2N+3}}. \tag{15}$$

Theorem 2.2 (see [15, 19]). *Suppose that $u \in L_2[0,1]$ and $u(x)$ is approximated by $\sum_{i=0}^{N} c_i B_i(x)$, then we have*

$$\lim_{N\to\infty} \left|\left| u(x) - \sum_{i=0}^{N} c_i B_i(x) \right|\right|_{L_2[0,1]} = 0. \tag{16}$$

3. The Boubaker operational matrix of fractional derivative

The main objective of this section is to derive the operational matrix of Caputo fractional derivatives based on the Boubaker polynomials.

For the vector $\mathbf{B(x)}$, we can approximate the operational matrices of fractional order integration as (see [19])

$$D^{\alpha}\mathbf{B}\left(x\right)\simeq\mathbf{D}^{(\alpha)}\mathbf{B}\left(x\right), \tag{17}$$

where $\mathbf{D}^{(\alpha)}$ denotes the $(N+1)\times(N+1)$ Caputo fractional operational matrix of integration for Boubaker polynomials, which can be expressed as follows

$$D^{\alpha}\mathbf{B}\left(x\right) \quad \simeq \quad \mathbf{M}D^{\alpha}\mathbf{T}\left(x\right)=\mathbf{MZ\bar{X}}\left(x\right), \tag{18}$$

where $\mathbf{Z}$ is the matrix given by

$$\mathbf{Z}=(Z_{i,j})=\begin{cases}\dfrac{\Gamma\left(j+1\right)}{\Gamma\left(j+1-\alpha\right)}, & i=j=\lceil\alpha\rceil,\dots,N\\ 0, & \text{otherwise,}\end{cases} \tag{19}$$

and $\mathbf{\bar{X}}=\left[\bar{X}_{i+1}\right]_{(N+1)\times(1)}$, where

$$\bar{X}_{i+1}=\begin{cases}x^{i-\alpha}, & i=\lceil\alpha\rceil,\dots,N\\ 0, & i=0,1,\dots,\lceil\alpha\rceil-1.\end{cases} \tag{20}$$

Now, $\bar{X}$ is expanded in terms of Boubaker polynomials as

$$\bar{X}=\mathbf{E}^{T}\mathbf{B(x)}, \tag{21}$$

where $\mathbf{E}=[e_0,e_1,\dots,e_m]$ and $e_i=\mathbf{Q}^{-1}\hat{E}_i$ with $\hat{E}_i=[\hat{e}_{i,0},\hat{e}_{i,1},\dots,\hat{e}_{i,m}]^T$.
The entries of the vector $\hat{E}_i$ can be calculated as

$$\hat{e}_{i,j}=\left(\int_0^1 x^{i-\alpha}B_j(x)dx\right)\mathbf{Q}^{-1}. \tag{22}$$

We have then

$$D^{\alpha}\mathbf{B}\left(x\right)\simeq\mathbf{D}^{(\alpha)}\mathbf{B}\left(x\right),\quad \mathbf{D}^{(\alpha)}=\mathbf{MZE}^{T}. \tag{23}$$

$\mathbf{D}^{(\alpha)}$is the operational matrix of the Caputo fractional derivative.

4. Solution to singular fractional Emden-Fowler problem

This section presents the derivation of the method for solving a singular initial value problem of fractional Emden–Fowler type equations.

Let us consider the fractional Emden–Fowler equation of the form

$$D^{2\alpha}u(x)+\frac{\lambda}{x^{\alpha}}D^{\alpha}u(x)+s(x)g(u(x))=h(x),\quad x\in(0,1),\quad \lambda>0,\quad \frac{1}{2}<\alpha\leqslant 1, \tag{24}$$

with initial conditions

$$u\left(0\right)=a,\quad D^{\alpha}u\left(0\right)=b. \tag{25}$$

We use the approximations of $u(x)$, $s\left(x\right)$, and $g\left(u\left(x\right)\right)$ by the Boubaker polynomials as

$$u\left(x\right)=\sum_{i=0}^{m}c_iB_i\left(x\right)=\mathbf{C}^T\mathbf{B}\left(x\right),\quad s\left(x\right)g\left(u\left(x\right)\right)=s\left(x\right)g\left(\mathbf{C}^T\mathbf{B}\left(x\right)\right),\tag{26}$$

where the unknowns are $\mathbf{C}=\left[c_0,c_1,\ldots,c_m\right]^T$. Using operational matrix of fractional derivative, Eq. (23) can be written as

$$\mathbf{C}^T\mathbf{D}^{(2\alpha)}\mathbf{B}\left(x\right)+\frac{\lambda}{x^\alpha}\mathbf{C}^T\mathbf{D}^{(\alpha)}\mathbf{B}\left(x\right)+s\left(x\right)g\left(\mathbf{C}^T\mathbf{B}\left(x\right)\right)=h\left(x\right).\tag{27}$$

Collocating Eq. (27) at $m-1$ collocation points leads to

$$\mathbf{C}^T\mathbf{D}^{(2\alpha)}\mathbf{B}\left(x_i\right)+\frac{\lambda}{x^\alpha}\mathbf{C}^T\mathbf{D}^{(\alpha)}\mathbf{B}\left(x_i\right)+s\left(x_i\right)g\left(\mathbf{C}^T\mathbf{B}\left(x_i\right)\right)=h\left(x_i\right),\tag{28}$$

where a set of suitable collocation points is defined as follows

$$x_i=\frac{1}{2}\left(\cos\left(\frac{i\pi}{m}\right)+1\right),\ i=0,\ldots,m-1.\tag{29}$$

In addition, the initial conditions (25) provide two algebraic equations

$$\mathbf{C}^T\mathbf{B}\left(0\right)=a,\quad \mathbf{C}^T\mathbf{D}^{(\alpha)}\mathbf{B}\left(0\right)=0.\tag{30}$$

Finally, we can compute the values for the components of $\mathbf{C}$ by solving the system of Eq. (28) and Eq. (30). Hence, the approximate solution for $u(x)$ can be computed by using Eq. (11).

5. Numerical examples

In this section, we apply the method presented in Sec. 4. to solve fractional Emden–Fowler Equation. Numerical computations have been performed using **Matlab** programing language.

Example 1. We consider the following fractional Emden–Fowler equation :

$$D^{2\alpha}u(x)+\frac{2}{x^\alpha}D^\alpha u(x)+u(x)^n=0,\tag{31}$$

subject to the following conditions $u(0)=1,\quad D^\alpha u(0)=0$.

1. For the case $\alpha=1$, and $n=0$, Eq. (31) has the following exact solution $u(x)=1-\frac{1}{3!}x^2$. By applying the above method, and taking $m=2$, we find

$$\mathbf{D}^{(1)}=\begin{bmatrix}0&0&0\\1&0&0\\0&2&0\end{bmatrix},\quad \mathbf{D}^{(2)}=\begin{bmatrix}0&0&0\\0&0&0\\2&0&0\end{bmatrix},\quad \mathbf{C}=\begin{bmatrix}c_0\\c_1\\c_2\end{bmatrix}=\begin{bmatrix}\frac{4}{3}\\0\\-\frac{1}{6}\end{bmatrix}$$

   Hence, the solution is $u(x)=\mathbf{C}^T\mathbf{B}\left(x\right)=\left[\frac{4}{3},0,-\frac{1}{6}\right]\begin{bmatrix}1\\x\\x^2+2\end{bmatrix}=1-\frac{1}{3!}x^2$

   which is the exact solution found previously.

2. Case $\alpha = 1$, and $n = 1$, Eq. (31) has the exact solution [2] $u(t) = \dfrac{\sin(x)}{x}$.
Applying the technique described in Sec. 4., with $m = 3$, we approximate the solution by

$$u(x) = c_0 B_0(x) + c_1 B_1(x) + c_2 B_2(x) + c_3 B_3(x) = \mathbf{C}^T \mathbf{B}(x),$$

where,

$$\mathbf{D}^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ -5 & 0 & 3 & 0 \end{bmatrix}, \quad \mathbf{D}^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

then, we find the following system of equations

$$c_0 + 2c_2 = 1, \quad c_1 + c_3 = 0,$$
$$c_0 + \frac{41}{12}c_1 + \frac{137}{16}c_2 + \frac{2465}{192}c_3 = 0, \quad c_0 + \frac{33}{4}c_1 + \frac{129}{16}c_2 + \frac{721}{64}c_3 = 0,$$

which has the solution

$$c_0 = \frac{25\,673}{19\,113}, \quad c_1 = -\frac{256}{19\,113}, \quad c_2 = -\frac{3280}{19\,113}, \quad c_3 = \frac{256}{19\,113}$$

so, in this case the approximation of $u_3(\mathrm{x})$ is

$$\begin{aligned} u_3(x) &= \begin{bmatrix} \frac{25\,673}{19\,113} & -\frac{256}{19\,113} & -\frac{3280}{19\,113} & \frac{256}{19\,113} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2+2 \\ x^3+x \end{bmatrix} = \\ &= \frac{256}{19\,113}x^3 - \frac{3280}{19\,113}x^2 + 1. \end{aligned}$$

Tab. 1 shows the absolute error between the approximate solution obtained for the values of $m = 3$, and $m = 6$, using the operational matrix of Boubaker polynomials, and the exact solution. It can be seen from Tab. 1 that the solutions obtained by the proposed method is almost identical to the exact solutions. Clearly, increasing more higher the values of m leads to highly accurate results.

Table 1. Absolute error for different values of m for $\alpha = 1$

x	0.1	0.3	0.5	0.7	0.9
$m = 3$	3.6881E−5	1.5070E−4	7.9560E−5	1.9380E−4	3.9614E−4
$m = 6$	3. 389 1E−8	3. 051 2E−7	8. 479 1E−8	1. 671 4E−7	2. 961 1E−7

Example 2. We consider the following fractional Emden–Fowler equation [20]

$$D^{2\alpha}u(x) + \frac{1}{x^\alpha}D^\alpha u(x) + (1 + x^\alpha)(u(x)) = h(x),$$

subject to the conditions $u(0) = 3, \quad D^\alpha u(0) = 0,$ where

$$h(x) = \Gamma(1 + 2\alpha) + \frac{\Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha)} + (1 + x^\alpha)\left(3 + x^{2\alpha}\right).$$

The exact solution is given by $u(x) = 3 + x^{2\alpha}$.
Applying the method developed in Sec. 3. and Sec. 4. for $m = 2$, $\alpha = 1$, we have

$$u(x) = c_0 B_0(x) + c_1 B_1(x) + c_2 B_2(x) = C^T \mathbf{B}(x).$$

Therefore, using Eq. (28) we obtain : $1.5c_0 + 2.75c_1 + 7.375c_2 = 8.875$.
Now, applying again Eq. (28) we have $c_0 + 2c_2 = 3$ and $c_1 = 0$.
Finally, we get $c_0 = 1, c_1 = 0$ and $c_2 = 1$.
Thus, $u(x)$ can be written as

$$u(x) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 + 2 \end{bmatrix} = 3 + x^2,$$

which is none other than the exact solution.
Tab. 2 shows the absolute errors between the approximate solutions obtained for the respectives values of ($\alpha = 0.75$, $\alpha = 0.85$, $\alpha = 1$), and the exact solutions.

Table 2. Absolute error for different values of α for $m = 2$

x	0.1	0.3	0.5	0.7	0.9
$\alpha = 1$	0.000	0.000	0.000	0.000	0.000
$\alpha = 0.85$	6.7607×10^{-2}	1.0759×10^{-2}	8.3712×10^{-3}	6.9827×10^{-3}	1.1870×10^{-3}
$\alpha = 0.75$	9.7032×10^{-2}	1.0264×10^{-2}	5.1643×10^{-3}	4.9891×10^{-3}	1.9924×10^{-3}

Example 3. Consider the following fractional Emden–Fowler equation

$$D^{2\alpha} u(x) + \frac{\lambda}{x^\alpha} D^\alpha u(x) - 2(2x^2 + 3)u(x) = h(x), \tag{32}$$

subject to the conditions $u(0) = 1, \quad D^\alpha u(0) = 0$, with $\alpha = 1, \lambda = 2$, and $h(x) = 0$. Eq (32) has the exact solution (see [2]) $u(t) = \exp(x^2)$.

In Fig. 1 (a), are plotted the exact and the approximate solutions of $u(x)$ for $m = 4$, and $m = 6$. Definitely, by increasing the value of m, the approximate value of $u(x)$ will be closer to the exact value. Fig. 1 (b) represents the absolute error in this case.

Example 4. Consider the following fractional Emden–Fowler equation (see [20])

$$D^{2\alpha} u(x) + \frac{1}{x^\alpha} D^\alpha u(x) - 9u(x) = h(x), \quad x \in (0, 1), \quad \frac{1}{2} < \alpha \leqslant 1, \tag{33}$$

subject to the boundary conditions $u(0) = 2$, $D^\alpha u(0) = 0$, where

$$h(x) = -9 + \frac{\Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha)} + \Gamma(1 + 2\alpha) + \left(\frac{\Gamma(1 + 3\alpha)}{\Gamma(1 + \alpha)} + \frac{\Gamma(1 + 3\alpha)}{\Gamma(1 + 2\alpha)} \right) x^\alpha - 9x^{2\alpha} - 9x^{3\alpha}.$$

The exact solution is given by

$$u(x) = 1 + x^{2\alpha} + x^{3\alpha}.$$

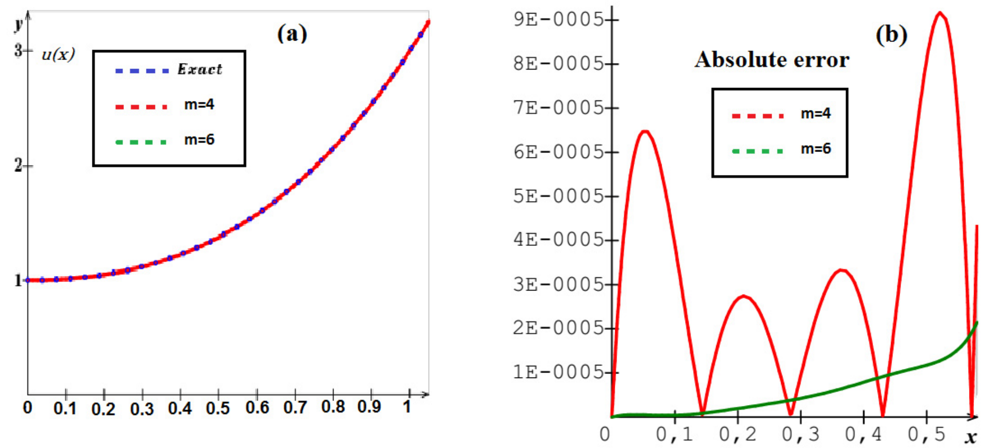

Fig. 1. (a) — graph of exact and approximate solution (for $m = 4$ and $m = 6$), (b) — graph of absolute error

For various values of $\alpha = 0.7$, 0.8, 1 and $m = 4$. The operational matrix for $\mathbf{D}^{(\alpha)}$ is given by

$$\mathbf{D}^{(0.7)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 7.5067 & -5.1084 & -7.0969 & 8.2458 & -3.4888 \\ -0.75854 & -2.489 & -1.8287 & 4.1182 & -2.1815 \\ 3.8518 & -5.5229 & -5.0826 & 8.4712 & -3.3057 \\ 1.8646 & -2.3313 & -0.31872 & 2.3752 & 0.61468 \end{bmatrix},$$

$$\mathbf{D}^{(1.4)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 4.7386 & 7.3096 & 0.56406 & -4.3120 & 2.8203 \\ -7.5859 & 0.98711 & 3.2963 & 0.21152 & -0.47593 \\ -3.8934 & -10.541 & -1.8169 & 11.281 & -3.7423 \end{bmatrix},$$

$$\mathbf{D}^{(0.8)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 6.1986 & -3.3934 & -5.2443 & 5.8093 & -2.3809 \\ -2.6616 & 3.8043 & 2.6493 & -2.9733 & 1.3151 \\ 1.7557 & -3.6652 & -2.8636 & 5.9132 & -2.2249 \\ 2.3933 & -4.6647 & -1.7933 & 5.0768 & -0.58478 \end{bmatrix},$$

$$\mathbf{D}^{(1.6)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 13.216 & -6.4671 & -11.403 & 12.352 & -4.9845 \\ -8.5829 & 6.6068 & 6.3194 & -5.0447 & 2.0389 \\ -10.082 & -5.9541 & 3.6179 & 5.9058 & -1.4201 \end{bmatrix},$$

$$\mathbf{D}^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ -5 & 0 & 3 & 0 & 0 \\ 0 & -4 & 0 & 4 & 0 \end{bmatrix}, \quad \mathbf{D}^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ -24 & 0 & 12 & 0 & 0 \end{bmatrix}.$$

Then, Eq. (33) with the initial condition has been solved with the proposed method and the values of the unknown matrix C^T are obtained and listed in Tab. 2. The values of absolute errors are shown in Tab. 3.

Table 3. Values of unknowns for $m = 4$, and for different values of α

Unknowns	c_0	c_1	c_2	c_3	c_4
$\alpha = 0.7$	-2.2706	0.57555	1.6168	-7.5937×10^{-2}	1.8568×10^{-2}
$\alpha = 0.8$	-2.2895	7.9481×10^{-2}	1.6352	0.14592	9.5450×10^{-3}
$\alpha = 1$	-1.0004	-0.99965	1.0002	0.99965	2.2216×10^{-5}

Table 4. Absolute errors for $m = 5$ and for different values of α

x	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 1$
0.1	5.5839×10^{-2}	2.8251×10^{-2}	3.8348×10^{-5}
0.2	4.9238×10^{-2}	2.3678×10^{-2}	3.4764×10^{-5}
0.3	4.5776×10^{-2}	2.0665×10^{-2}	3.1270×10^{-5}
0.4	4.3242×10^{-2}	1.8550×10^{-2}	2.9831×10^{-5}
0.5	4.0654×10^{-2}	1.7195×10^{-2}	3.2362×10^{-5}
0.6	3.7510×10^{-2}	1.6605×10^{-2}	4.0721×10^{-5}
0.7	3.3528×10^{-2}	1.6810×10^{-2}	5.6716×10^{-5}
0.8	2.8539×10^{-2}	1.7824×10^{-2}	8.2100×10^{-5}
0.9	2.2428×10^{-2}	1.9619×10^{-2}	1.1857×10^{-4}
1.0	1.5099×10^{-2}	2.2125×10^{-2}	1.6778×10^{-4}

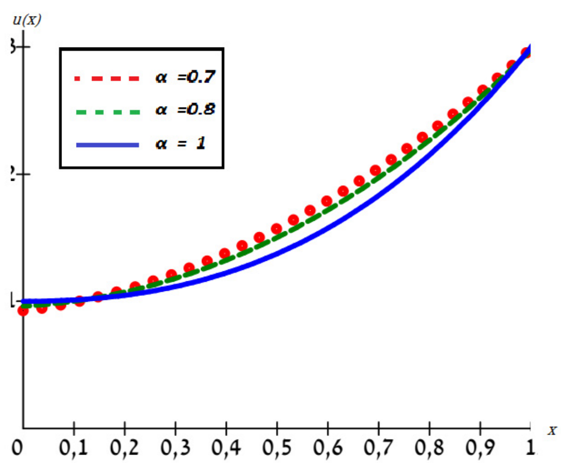

Fig. 2. The graph of $u(x)$ with $m = 4$ and $\alpha = 0.7, 0.8, 1$

Conclusions

In this paper, we introduced a new operational matrix of fractional derivative by using Boubaker polynomials. Then by using these matrices, we reduced the singular fractional Emden-Fowler type equations to a system of algebraic equations that can be solved easily. Numerical examples are included to demonstrate the validity and application of this method. The results revealed that the introduced method is very effective, straightforward, simple, and it can be applied to other related fractional problems, such as partial fractional differential and integro-differential equations. Further improvements involving Boubaker polynomials in the fractional case are possible and may be directions of the future research.

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Метод операционной матрицы Бубакера для дробной задачи Эмдена-Фаулера

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Аннотация. В работе введено сингулярное уравнение Эмдена–Фаулера дробного порядка и предложен вычислительный метод его численного решения. Для аппроксимации решений мы использовали полиномы Бубакера и определили формулировку его операционной матрицы дробной производной. Однако использование полиномов Бубакера появилось совсем недавно и в литературе не обсуждалось, поскольку в большинстве областей применения этих полиномов требуются ортогональные полиномы, и здесь мы ввели его впервые. Операционная матрица инструмента дробной производной Капуто преобразует уравнение Эмдена-Фаулера в систему алгебраических уравнений, решения которой легко вычислить. Рассмотрены численные примеры, подтверждающие обоснованность и эффективность предложенного метода.

Ключевые слова: многочлены Бубакера, операционная матрица дробных производных, метод коллокаций, дробные уравнения типа Эмдена–Фаулера.