Natalia Gherstega, Functional bases of centro-affine invariants for the three-dimensional quadratic differential systems, *Bul. Acad. Ştiinţe Repub. Mold. Mat.*, 2006, номер 1, 57–64

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(50), 2006, Pages 57–64
ISSN 1024–7696

Functional bases of centro-affine invariants for the three-dimensional quadratic differential systems

Natalia Gherstega

Abstract. Functional bases of centro-affine invariants are constructed for the three-dimensional differential systems with polynomial right-hand sides of order less than three.

Mathematics subject classification: 34C14.
Keywords and phrases: Differential system, Lie algebra of operators, functional basis of centro-affine invariants.

1 On the number of elements in a functional basis of invariants

It is known [1–3] that in the study of a polynomial differential system with the aid of Lie algebras and the orbit's theory an important role belongs to invariants and comitants of the systems [4–5]. Functional basis of invariants (comitants) should be especially mentioned. This can be explained by the fact that knowledge of Lie algebra of operators allows us to determine beforehand the exact number of elements in a minimal basis. In this article using the general theorem of algebraic invariants theory [4] functional bases of centro-affine invariants are studied for different three-dimensional differential systems with right-hand sides of order less than three.

Consider the three-dimensional differential system

$$\dot{x}^j = a^j + a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \quad (j, \alpha, \beta = \overline{1,3}), \tag{1}$$

where the coefficient tensor $a^j_{\alpha\beta}$ is symmetrical in lower indices, in which the complete convolution holds, and the group of centro-affine transformations $GL(3,\mathbb{R})$: $\overline{x}^j = q^j_r x^r \quad (\Delta = \det(q^j_r) \neq 0; \quad j, r = \overline{1,3})$.

The Lie algebra of operators [1] for linear representation of the group $GL(3,\mathbb{R})$ in the space of coefficients of system (1) is given by the following operators:

$$d_i = D_i^{(0)} + D_i^{(1)} + D_i^{(2)} \quad (i = \overline{1,9}), \tag{2}$$

where

$$D_1^{(0)} = a^1 \frac{\partial}{\partial a^1}, \quad D_2^{(0)} = a^2 \frac{\partial}{\partial a^2}, \quad D_3^{(0)} = a^3 \frac{\partial}{\partial a^3},$$

$$D_4^{(0)} = a^2 \frac{\partial}{\partial a^1}, \quad D_5^{(0)} = a^3 \frac{\partial}{\partial a^1}, \quad D_6^{(0)} = a^1 \frac{\partial}{\partial a^2},$$

$$D_7^{(0)} = a^3 \frac{\partial}{\partial a^2}, \quad D_8^{(0)} = a^1 \frac{\partial}{\partial a^3}, \quad D_9^{(0)} = a^2 \frac{\partial}{\partial a^3}; \tag{3}$$

$$\begin{aligned}
D_1^{(1)} &= a_2^1 \frac{\partial}{\partial a_2^1} + a_3^1 \frac{\partial}{\partial a_3^1} - a_1^2 \frac{\partial}{\partial a_1^2} - a_1^3 \frac{\partial}{\partial a_1^3},\\
D_2^{(1)} &= -a_2^1 \frac{\partial}{\partial a_2^1} + a_1^2 \frac{\partial}{\partial a_1^2} + a_3^2 \frac{\partial}{\partial a_3^2} - a_2^3 \frac{\partial}{\partial a_2^3},\\
D_3^{(1)} &= -a_3^1 \frac{\partial}{\partial a_3^1} - a_3^2 \frac{\partial}{\partial a_3^2} + a_1^3 \frac{\partial}{\partial a_1^3} + a_2^3 \frac{\partial}{\partial a_2^3},\\
D_4^{(1)} &= a_1^2 \frac{\partial}{\partial a_1^1} + (a_2^2 - a_1^1) \frac{\partial}{\partial a_2^1} + a_3^2 \frac{\partial}{\partial a_3^1} - a_1^2 \frac{\partial}{\partial a_2^2} - a_1^3 \frac{\partial}{\partial a_2^3},\\
D_5^{(1)} &= a_1^3 \frac{\partial}{\partial a_1^1} + a_2^3 \frac{\partial}{\partial a_2^1} + (a_3^3 - a_1^1) \frac{\partial}{\partial a_3^1} - a_1^2 \frac{\partial}{\partial a_3^2} - a_1^3 \frac{\partial}{\partial a_3^3},\\
D_6^{(1)} &= -a_2^1 \frac{\partial}{\partial a_1^1} + (a_1^1 - a_2^2) \frac{\partial}{\partial a_1^2} + a_2^1 \frac{\partial}{\partial a_2^2} + a_3^1 \frac{\partial}{\partial a_3^2} - a_2^3 \frac{\partial}{\partial a_1^3},\\
D_7^{(1)} &= -a_2^1 \frac{\partial}{\partial a_3^1} + a_1^3 \frac{\partial}{\partial a_1^2} + a_2^3 \frac{\partial}{\partial a_2^2} + (a_3^3 - a_2^2) \frac{\partial}{\partial a_3^2} - a_2^3 \frac{\partial}{\partial a_3^3},\\
D_8^{(1)} &= -a_3^1 \frac{\partial}{\partial a_1^1} - a_3^2 \frac{\partial}{\partial a_1^2} + (a_1^1 - a_3^3) \frac{\partial}{\partial a_1^3} + a_2^1 \frac{\partial}{\partial a_2^3} + a_3^1 \frac{\partial}{\partial a_3^3},\\
D_9^{(1)} &= -a_3^1 \frac{\partial}{\partial a_2^1} - a_3^2 \frac{\partial}{\partial a_2^2} + a_1^2 \frac{\partial}{\partial a_1^3} + (a_2^2 - a_3^3) \frac{\partial}{\partial a_2^3} + a_3^2 \frac{\partial}{\partial a_3^3};
\end{aligned} \tag{4}$$

$$\begin{aligned}
D_1^{(2)} = &-a_{11}^1 \frac{\partial}{\partial a_{11}^1} + a_{22}^1 \frac{\partial}{\partial a_{22}^1} + a_{23}^1 \frac{\partial}{\partial a_{23}^1} + a_{33}^1 \frac{\partial}{\partial a_{33}^1} - 2a_{11}^2 \frac{\partial}{\partial a_{11}^2} -\\
&-a_{12}^2 \frac{\partial}{\partial a_{12}^2} - a_{13}^2 \frac{\partial}{\partial a_{13}^2} - 2a_{11}^3 \frac{\partial}{\partial a_{11}^3} - a_{12}^3 \frac{\partial}{\partial a_{12}^3} - a_{13}^3 \frac{\partial}{\partial a_{13}^3},\\
D_2^{(2)} = &-a_{12}^1 \frac{\partial}{\partial a_{12}^1} - 2a_{22}^1 \frac{\partial}{\partial a_{22}^1} - a_{23}^1 \frac{\partial}{\partial a_{23}^1} + a_{11}^2 \frac{\partial}{\partial a_{11}^2} + a_{13}^2 \frac{\partial}{\partial a_{13}^2} -\\
&-a_{22}^2 \frac{\partial}{\partial a_{22}^2} + a_{33}^2 \frac{\partial}{\partial a_{33}^2} - a_{12}^3 \frac{\partial}{\partial a_{12}^3} - 2a_{22}^3 \frac{\partial}{\partial a_{22}^3} - a_{23}^3 \frac{\partial}{\partial a_{23}^3},\\
D_3^{(2)} = &-a_{13}^1 \frac{\partial}{\partial a_{13}^1} - a_{23}^1 \frac{\partial}{\partial a_{23}^1} - 2a_{33}^1 \frac{\partial}{\partial a_{33}^1} - a_{13}^2 \frac{\partial}{\partial a_{13}^2} - a_{23}^2 \frac{\partial}{\partial a_{23}^2} -\\
&-2a_{33}^2 \frac{\partial}{\partial a_{33}^2} + a_{11}^3 \frac{\partial}{\partial a_{11}^3} + a_{12}^3 \frac{\partial}{\partial a_{12}^3} + a_{22}^3 \frac{\partial}{\partial a_{22}^3} - a_{33}^3 \frac{\partial}{\partial a_{33}^3},
\end{aligned}$$

$$D_4^{(2)} = a_{11}^2 \frac{\partial}{\partial a_{11}^1} + (a_{12}^2 - a_{11}^1)\frac{\partial}{\partial a_{12}^1} + a_{13}^2 \frac{\partial}{\partial a_{13}^1} + (a_{22}^2 - 2a_{12}^1)\frac{\partial}{\partial a_{22}^1} + (a_{23}^2 - a_{13}^1)\frac{\partial}{\partial a_{23}^1} +$$

$$+ a_{33}^2 \frac{\partial}{\partial a_{33}^1} - a_{11}^2 \frac{\partial}{\partial a_{12}^2} - 2a_{12}^2 \frac{\partial}{\partial a_{22}^2} - a_{13}^2 \frac{\partial}{\partial a_{23}^2} - a_{11}^3 \frac{\partial}{\partial a_{12}^3} - 2a_{12}^3 \frac{\partial}{\partial a_{22}^3} - a_{13}^3 \frac{\partial}{\partial a_{23}^3},$$

$$D_5^{(2)} = a_{11}^3 \frac{\partial}{\partial a_{11}^1} + a_{12}^3 \frac{\partial}{\partial a_{12}^1} + (a_{13}^3 - a_{11}^1)\frac{\partial}{\partial a_{13}^1} + a_{22}^3 \frac{\partial}{\partial a_{22}^1} + (a_{23}^3 - a_{12}^1)\frac{\partial}{\partial a_{23}^1} +$$

$$+ (a_{33}^3 - 2a_{13}^1)\frac{\partial}{\partial a_{33}^1} - a_{11}^2 \frac{\partial}{\partial a_{13}^2} - a_{12}^2 \frac{\partial}{\partial a_{23}^2} - 2a_{13}^2 \frac{\partial}{\partial a_{33}^2} - a_{11}^3 \frac{\partial}{\partial a_{13}^3} - a_{12}^3 \frac{\partial}{\partial a_{23}^3} - 2a_{13}^3 \frac{\partial}{\partial a_{33}^3},$$

$$D_6^{(2)} = -2a_{12}^1 \frac{\partial}{\partial a_{11}^1} - a_{22}^1 \frac{\partial}{\partial a_{12}^1} - a_{23}^1 \frac{\partial}{\partial a_{13}^1} + (a_{11}^1 - 2a_{12}^2)\frac{\partial}{\partial a_{11}^2} + (a_{12}^1 - a_{22}^2)\frac{\partial}{\partial a_{12}^2} +$$

$$+ (a_{13}^1 - a_{23}^2)\frac{\partial}{\partial a_{13}^2} + a_{22}^1 \frac{\partial}{\partial a_{22}^2} + a_{23}^1 \frac{\partial}{\partial a_{23}^2} + a_{33}^1 \frac{\partial}{\partial a_{33}^2} - 2a_{12}^3 \frac{\partial}{\partial a_{11}^3} - a_{22}^3 \frac{\partial}{\partial a_{12}^3} - a_{23}^3 \frac{\partial}{\partial a_{13}^3},$$

$$D_7^{(2)} = -a_{12}^1 \frac{\partial}{\partial a_{13}^1} - a_{22}^1 \frac{\partial}{\partial a_{23}^1} - 2a_{23}^1 \frac{\partial}{\partial a_{33}^1} + a_{11}^3 \frac{\partial}{\partial a_{11}^2} + a_{12}^3 \frac{\partial}{\partial a_{12}^2} + (a_{13}^3 - a_{12}^2)\frac{\partial}{\partial a_{13}^2} +$$

$$+ a_{22}^3 \frac{\partial}{\partial a_{22}^2} + (a_{23}^3 - a_{22}^2)\frac{\partial}{\partial a_{23}^2} + (a_{33}^3 - 2a_{23}^2)\frac{\partial}{\partial a_{33}^2} - a_{12}^3 \frac{\partial}{\partial a_{13}^3} - a_{22}^3 \frac{\partial}{\partial a_{23}^3} - 2a_{23}^3 \frac{\partial}{\partial a_{33}^3},$$

$$D_8^{(2)} = -2a_{13}^1 \frac{\partial}{\partial a_{11}^1} - a_{23}^1 \frac{\partial}{\partial a_{12}^1} - a_{33}^1 \frac{\partial}{\partial a_{13}^1} - 2a_{13}^2 \frac{\partial}{\partial a_{11}^2} - a_{23}^2 \frac{\partial}{\partial a_{12}^2} - a_{33}^2 \frac{\partial}{\partial a_{13}^2} +$$

$$+ (a_{11}^1 - 2a_{13}^3)\frac{\partial}{\partial a_{11}^3} + (a_{12}^1 - a_{23}^3)\frac{\partial}{\partial a_{12}^3} + (a_{13}^1 - a_{33}^3)\frac{\partial}{\partial a_{13}^3} + a_{22}^1 \frac{\partial}{\partial a_{22}^3} + a_{23}^1 \frac{\partial}{\partial a_{23}^3} + a_{33}^1 \frac{\partial}{\partial a_{33}^3},$$

$$D_9^{(2)} = -a_{13}^1 \frac{\partial}{\partial a_{12}^1} - 2a_{23}^1 \frac{\partial}{\partial a_{22}^1} - a_{33}^1 \frac{\partial}{\partial a_{23}^1} - a_{13}^2 \frac{\partial}{\partial a_{12}^2} - 2a_{23}^2 \frac{\partial}{\partial a_{22}^2} - a_{33}^2 \frac{\partial}{\partial a_{23}^2} +$$

$$+ a_{11}^2 \frac{\partial}{\partial a_{11}^3} + (a_{12}^2 - a_{13}^3)\frac{\partial}{\partial a_{12}^3} + a_{13}^2 \frac{\partial}{\partial a_{13}^3} + (a_{22}^2 - 2a_{23}^3)\frac{\partial}{\partial a_{22}^3} + (a_{23}^2 - a_{33}^3)\frac{\partial}{\partial a_{23}^3} + a_{33}^2 \frac{\partial}{\partial a_{33}^3}. \quad (5)$$

According to [2] is proved the following

Theorem 1. *The polynomial $\theta(a)$ in the coefficients of the system (1) is a centro-affine invariant [3] of the system (1) with weight g iff the equalities*

$$d_i(\theta) = -g\theta \quad (i = \overline{1,3}), \quad d_j(\theta) = 0 \quad (j = \overline{4,9}), \tag{6}$$

hold, where $d_i \quad (i = \overline{1,3})$ and $d_j \quad (j = \overline{4,9})$ are the operators (2)–(5).

Definition 1. *The set of polynomial invariants $\{\theta_s(a),\ s \in B\}$ of the system (1) with respect to the $GL(3,\mathbb{R})$-group is called a functional basis of invariants of the system (1) with respect to this group if any invariant $\theta(a)$ of the system (1) with respect to the $GL(3,\mathbb{R})$-group can be written as a univocal function of the invariants $\theta_s(a)$. (Here B is some set of finite or transfinite natural numbers.)*

Definition 2. *A functional basis of invariants of the system (1) with respect to the $GL(3,\mathbb{R})$- group is called minimal if any invariant could not be removed out, overwise it is not a functional basis anymore.*

With the aid of Theorem 1 we obtaine

Lemma 1. *The number of elements μ in a functional basis of centro-affine invariants for the system (1) is equal to 22 (i.e. $\mu = 22$).*

Proof. We observe, according to equalities (6), that any invariant $\theta(a)$ satisfies a non-homogeneous linear system of partial differential equations of the first order. In the theory of equations (see for example [6]) it is known that the number of functionally independent solutions (invariants) of the system (6) is equal to

$$\mu = N - rank M_1 + 1, \tag{7}$$

where N is the number of coefficients in the system (1), and M_1 is the matrix, constructed on coordinate vectors of the operators (2)–(5). As for coefficients of the system (1) we have $N = 30$, and the general rank of the matrix M_1 is equal to 9, according to equality (7) we obtain that the number of functionally independent solutions (invariants) of the system (6) is equal to 22. Lemma 1 is proved.

Remark, with the aid of respective combinations of the operators (3)-(5), the truth of equalities of the type (7) can be showed for any subsystem of the system (1). Taking into consideration this fact and equality (7) we obtain that for μ holds

Lemma 2. *The number of elements μ in the basis of centro-affine invariants for the three-dimensional differential system is given in the Table 1.*

Table 1

μ	*Differential system*	*Number of the system*
0	$\dot{x}^j = a^j \quad (j = \overline{1,3})$	(8)
3	$\dot{x}^j = a^j_\alpha x^\alpha \quad (j, \alpha = \overline{1,3})$	(9)
10	$\dot{x}^j = a^j_{\alpha\beta} x^\alpha x^\beta \quad (j, \alpha, \beta = \overline{1,3})$	(10)
4	$\dot{x}^j = a^j + a^j_\alpha x^\alpha \quad (j, \alpha = \overline{1,3})$	(11)
13	$\dot{x}^j = a^j + a^j_{\alpha\beta} x^\alpha x^\beta \quad (j, \alpha, \beta = \overline{1,3})$	(12)
19	$\dot{x}^j = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \quad (j, \alpha, \beta = \overline{1,3})$	(13)

Remark 1. For the system (8) we have $\mu = 0$, as writing the equation (6) with the operators $D_i^{(0)}(\theta) = -g\theta \ \ (i = \overline{1,3}),\ \ D_j(\theta) = 0 \ \ (j = \overline{4,9})$ from (3) we obtain that the constant is the unique solution of this system. Such invariants will not be considered further.

To construct invariants of the system (1) and (9)–(13) we will use their notation with the aid of the convolution and alternation [4]. Further the unit three-vector ε^{pqr} with coordinates $\varepsilon^{123} = -\varepsilon^{132} = \varepsilon^{312} = -\varepsilon^{321} = \varepsilon^{231} = -\varepsilon^{213} = 1$ and $\varepsilon^{pqr} = 0 \ \ (p, q, r = \overline{1,3})$ will be used in other cases.

2 Centro-affine invariants of functional bases for the systems (9)–(13) and (1)

Theorem 2. *The expressions*

$$\theta_1 = a^\alpha_\alpha, \quad \theta_2 = a^\alpha_\beta a^\beta_\alpha, \quad \theta_3 = a^\alpha_\gamma a^\beta_\alpha a^\gamma_\beta, \tag{14}$$

form a functional basis of centro-affine invariants of the system (9).

Proof. We observe that the invariants (14) satisfy $D_i^{(1)}(\theta_j) = 0 \ \ (i = \overline{1,9};\ \ j = \overline{1,3})$, where $D_i^{(1)}$ is from (4). One can verify that the Jacobi matrix for the polynomials from (14) has the general rank 3. Hence the indicated invariants are functionally independent and according to Table 1 form a functional basis of centro-affine invariants of the system (9). Theorem 2 is proved.

From (5) with the aid of information from Table 1 we obtain

Theorem 3. *The expressions*

$$\begin{gathered} i_1 = a^\alpha_{p\gamma} a^\beta_{q\alpha} a^\gamma_{r\beta} \varepsilon^{pqr}, \quad i_2 = a^\alpha_{ps} a^\beta_{qt} a^\gamma_{ru} a^\delta_{\alpha\delta} a^\mu_{\beta\mu} a^\nu_{\gamma\nu} \varepsilon^{pqr} \varepsilon^{stu}, \\ i_3 = a^\alpha_{\beta p} a^\beta_{\delta q} a^\gamma_{rs} a^\delta_{\nu t} a^\mu_{\gamma u} a^\nu_{\alpha\mu} \varepsilon^{pqr} \varepsilon^{stu}, \quad i_4 = a^\alpha_{\delta p} a^\beta_{\nu q} a^\gamma_{\beta r} a^\delta_{\mu s} a^\mu_{\gamma t} a^\nu_{\alpha u} \varepsilon^{pqr} \varepsilon^{stu}, \\ i_5 = a^\alpha_{ps} a^\beta_{qt} a^\gamma_{\alpha r} a^\delta_{\nu u} a^\mu_{\beta\mu} a^\nu_{\gamma\delta} \varepsilon^{pqr} \varepsilon^{stu}, \quad i_6 = a^\alpha_{ps} a^\beta_{qt} a^\gamma_{\delta r} a^\delta_{\alpha u} a^\mu_{\beta\nu} a^\nu_{\gamma\mu} \varepsilon^{pqr} \varepsilon^{stu}, \\ i_7 = a^\alpha_{ps} a^\beta_{qt} a^\gamma_{\beta r} a^\delta_{\mu u} a^\mu_{\gamma\nu} a^\nu_{\alpha\delta} \varepsilon^{pqr} \varepsilon^{stu}, \quad i_8 = a^\alpha_{ps} a^\beta_{qt} a^\gamma_{\alpha r} a^\delta_{\delta u} a^\mu_{\beta\gamma} a^\nu_{\mu\nu} \varepsilon^{pqr} \varepsilon^{stu}, \\ i_9 = a^\alpha_{ps} a^\beta_{qt} a^\gamma_{\beta r} a^\delta_{\gamma u} a^\mu_{\alpha\mu} a^\nu_{\delta\nu} \varepsilon^{pqr} \varepsilon^{stu}, \quad i_{10} = a^\alpha_{ps} a^\beta_{qt} a^\gamma_{\nu r} a^\delta_{\beta u} a^\mu_{\delta\mu} a^\nu_{\gamma\alpha} \varepsilon^{pqr} \varepsilon^{stu} \end{gathered} \tag{15}$$

form a functional basis of centro-affine invariants of the system (10).

Jacobi matrix for the invariants $i_1 - i_{10}$ is calculated when

$$\begin{gathered} a^1_{11} = a^1_{12} = a^1_{13} = a^1_{22} = 1, \quad a^1_{23} = 2, \quad a^1_{33} = 3, \quad a^2_{11} = -1, \\ a^2_{12} = 6, \quad a^2_{13} = -1, \quad a^2_{22} = 0, \quad a^2_{23} = 5, \quad a^2_{33} = 0, \\ a^3_{11} = a^3_{12} = a^3_{13} = 1, \quad a^3_{22} = 7, \quad a^3_{23} = 4, \quad a^3_{33} = 0, \end{gathered}$$

its rank is equal to 10, that shows the functional independence of $i_1 - i_{10}$.

Using the operators $D_i^{(0)} + D_i^{(1)}$ $(i = \overline{1,9})$ from (3)–(4) and $\mu = 4$ from Table 1, is proved

Theorem 4. *The expressions (14) and*

$$\theta_4 = a^\alpha_\mu a^\beta_\alpha a^\gamma_\nu a^\delta a^\mu a^\nu \varepsilon_{\beta\gamma\delta} \tag{16}$$

form a functional bases of centro-affine invariants of the system (11).

Using the operators $D_i^{(0)} + D_i^{(2)}$ $(i = \overline{1,9})$ from (3), (5) and $\mu = 13$ from Table 1, is proved

Theorem 5. *The expressions (15) with any three invariants from the following four*

$$i_{11} = a^\alpha_{\alpha\beta} a^\beta, \quad i_{12} = a^\alpha_{\alpha\beta} a^\beta_{\gamma\delta} a^\gamma a^\delta, \quad i_{13} = a^\alpha_{\beta\gamma} a^\beta_{\alpha\delta} a^\gamma a^\delta, \quad i_{14} = a^\alpha_{\beta\nu} a^\beta_{\alpha\gamma} a^\gamma_{\delta\mu} a^\delta a^\mu a^\nu \tag{17}$$

form a functional basis of centro-affine invariants of the system (12).

Jacobi matrix for elements of a basis of centro-affine invariants of the system (12) from (15) and (17) is calculated when

$$a^1 = 3, \quad a^2 = 5, \quad a^3 = 7, \quad a^1_{11} = a^1_{12} = a^1_{13} = a^1_{22} = 1, \quad a^1_{23} = 2,$$
$$a^1_{33} = 3, \quad a^2_{11} = -1, \quad a^2_{12} = 6, \quad a^2_{13} = -1, \quad a^2_{22} = 0, \quad a^2_{23} = 5,$$
$$a^2_{33} = 0, \quad a^3_{11} = a^3_{12} = a^3_{13} = 1, \quad a^3_{22} = 7, \quad a^3_{23} = 4, \quad a^3_{33} = 0,$$

its rank is equal to 13, that shows the functional independence of the indicated invariants.

Remark 2. The invariants (17) for the system (12) are obtained from tensorial expressions of the comitants K_1, K_6, K_7 and K_{17}, respectively, from the monograph [4, p. 141–142] after the substitution x^- for a^-.

Using the operators $D_i^{(1)} + D_i^{(2)}$ $(i = \overline{1,9})$ from (4)–(5) and $\mu = 19$ from Table 1 is proved

Theorem 6. *The expressions (14), (15) and*

$$i_{15} = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta r} a^\delta_{\gamma\delta} \varepsilon^{pqr}, \quad i_{16} = a^\alpha_p a^\beta_{\delta q} a^\gamma_{\gamma r} a^\delta_{\alpha\beta} \varepsilon^{pqr}, \quad i_{17} = a^\alpha_p a^\beta_{\delta q} a^\gamma_{\alpha r} a^\delta_{\beta\gamma} \varepsilon^{pqr},$$

$$i_{18} = a^\alpha_\gamma a^\beta_{\alpha p} a^\gamma_{\beta q} a^\delta_{\delta r} \varepsilon^{pqr}, \quad i_{19} = a^\alpha_p a^\beta_\alpha a^\gamma_{\beta q} a^\delta_{\mu r} a^\mu_{\gamma\delta} \varepsilon^{pqr}, \quad i_{20} = a^\alpha_p a^\beta_\gamma a^\gamma_{\mu q} a^\delta_{\beta r} a^\mu_{\alpha\delta} \varepsilon^{pqr}, \tag{18}$$

form a functional basis of the centro-affine invariants of the system (13).

Jacobi matrix for elements of a basis of centro-affine invariants of the system (13) from (14), (15) and (18) is calculated when

$$a^1_1 = -7, \quad a^1_2 = 5, \quad a^1_3 = -9, \quad a^2_1 = 4, \quad a^2_2 = 5, \quad a^2_3 = 7,$$

$$a_1^3 = -3, \quad a_2^3 = -2, \quad a_3^3 = 5, \quad a_{11}^1 = a_{12}^1 = a_{13}^1 = a_{22}^1 = 1,$$

$$a_{23}^1 = 2, \quad a_{33}^1 = 3, \quad a_{11}^2 = -1, \quad a_{12}^2 = 6, \quad a_{13}^2 = -1, \quad a_{22}^2 = 0,$$

$$a_{23}^2 = 5, \quad a_{33}^2 = 0, \quad a_{11}^3 = a_{12}^3 = a_{13}^3 = 1, \quad a_{22}^3 = 7, \quad a_{23}^3 = 4, \quad a_{33}^3 = 0,$$

it's rank is equal to 19, that shows the functional independence of indicated invariants.

Remark 3. The invariants

$$i_{21} = a_\beta^\alpha a_{\alpha\gamma}^\beta a^\gamma, \quad i_{22} = a_\gamma^\alpha a_{\alpha\beta}^\beta a^\gamma, \quad i_{23} = a_\gamma^\alpha a_\delta^\beta a_{\alpha\beta}^\gamma a^\delta,$$

$$i_{24} = a_\beta^\alpha a_{\alpha\gamma}^\beta a_{\delta\mu}^\gamma a^\delta a^\mu, \quad i_{25} = a_\gamma^\alpha a_{\alpha\beta}^\beta a_{\delta\mu}^\gamma a^\delta a^\mu \tag{19}$$

for the system (1) are obtained from tensorial expressions of the comitants K_3, K_4, K_8, K_{12} and K_{13}, respectively, from the monograph [4, p. 141–142] after the substitution x^- for a^-.

Using the operators $D_i^{(0)} + D_i^{(1)} + D_i^{(2)}$ $(i = \overline{1,9})$ from (3)–(5), Remark 3 and the statement of Lemma 1 about the number of centro-affine invariants in functional basis (22 elements) for the system (1), is proved

Theorem 7. *The expressions (14)–(16) and (18), with i_{11} from (17) and i_{21} from (19) form a functional basis of centro-affine invariants of the system (1).*

Jacobi matrix for elements of a basis of centro-affine invariants of the system (1) from (14)–(16) and (18) with i_{11} from (17), i_{21} from (19) is calculated when

$$a^1 = 3, \quad a^2 = 5, \quad a^3 = 7, \quad a_1^1 = -7, \quad a_2^1 = 5, \quad a_3^1 = -9,$$

$$a_1^2 = 4, \quad a_2^2 = 5, \quad a_3^2 = 7, \quad a_1^3 = -3, \quad a_2^3 = -2, \quad a_3^3 = 5,$$

$$a_{11}^1 = a_{12}^1 = a_{13}^1 = a_{22}^1 = 1, \quad a_{23}^1 = 2, \quad a_{33}^1 = 3,$$

$$a_{11}^2 = -1, \quad a_{12}^2 = 6, \quad a_{13}^2 = -1, \quad a_{22}^2 = 0, \quad a_{23}^2 = 5,$$

$$a_{33}^2 = 0, \quad a_{11}^3 = a_{12}^3 = a_{13}^3 = 1, \quad a_{22}^3 = 7, \quad a_{23}^3 = 4, \quad a_{33}^3 = 0,$$

its rank is equal to 22, that shows the functional independence of indicated invariants.

One can verify

Remark 4. With the aid of the invariants (14)-(19) it is possible to construct other functional bases of centro-affine invariants of the system (1) consisting of 22 elements.

References

[1] Popa M.N. *Algebraic methods for differential systems.* Editura the Flower Power, Universitatea din Piteşti, Seria Matematică Aplicată şi Industrială, N 15, 2004 (in Romanian).

[2] Popa M., Georgescu A., Rocşoreanu C. *A Lie algebra of a differential generalized FitzHugh-Nagumo system.* Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2003, N 1(41), p. 18–30.

[3] Gherstega N., Popa M. *Lie algebra of the operators and three-dimensional polynomial differential systems.* Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2005, N 2(48), p. 51–64.

[4] Sibirsky K.S. *Introduction to the Algebraic Theory of Invariants of Differential Equations.* Kishinev, Shtiintsa, 1982 (in Russian; published in English in 1988).

[5] Vulpe N.I. *Polynomial bases of comitants of differential systems and their applications in qualitative theory.* Kishinev, Shtiintsa, 1986 (in Russian).

[6] Ovsyannikov L.V. *Group analysis of differential equations.* Moscow, Nauka, 1978 (English transl. by Academic Press, 1982).

Department of Mathematics
State University of Tiraspol
str. Gh. Iablocichin 5
MD-2069 Chisinau
Moldova
E-mail: *gherstega@mail.ru*

Received February 10, 2006