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Subelliptic estimates for the oblique derivative problem

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We consider the equation

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} + \sum_{i=1}^{n} a_i(x)\, u_{x_i} + c(x)\, u = f, \qquad x \in \Omega,$$

where $\partial u/\partial \mathbf{v} = g$ on $\partial\Omega$, Ω is a bounded domain in $\mathbf{R}^n$ with boundary of class $C^3 \cap C^k$, L an elliptic operator in $\overline{\Omega}$, and $\mathbf{v}$ is a vector field on $\partial\Omega$ of class $C^3 \cap C^k$, which may touch $\partial\Omega$ in an $(n-2)$-dimensional submanifold Γ_0 of class C^3 with order or contact at most k.

Suppose that $\mathbf{v}$ does not touch Γ_0. Now $\mathbf{v}$ can be represented as $\mathbf{v} = b(x)\mathbf{n} + \boldsymbol{\tau}$, where $\mathbf{n}$ is the internal normal to $\partial\Omega$, $\boldsymbol{\tau}$ touches $\partial\Omega$, and $\boldsymbol{\tau} \neq 0$ on Γ_0, $b(x)$ vanishes on Γ_0.

We consider an arbitrary point A on Γ_0. We extend the field ν to a neighbourhood of A. Then three cases can happen:

1) $b(x)$ changes sign from "+" to "–" when moving in the positive direction along the characteristic of $\mathbf{v}$ through A;

2) $b(x)$ changes sign from "–" to "+";

3) $b(x)$ does not change sign.

Which case arises does not depend on A for every connected portion of Γ_0, that is, one can ask about three types of submanifolds Γ_0. On submanifolds of type 1 the additional condition $u|_{\Gamma_0} = u_0$ is prescribed. For subvarieties Γ_0 of types 2 and 3 the estimates do not contain terms involving u_0.

We assume that $a_{ij} \in C^3(\overline{\Omega})$, $a_i \in C^2(\overline{\Omega})$, $c \in C^1(\overline{\Omega})$, $0 < \alpha < 1$, and $0 < \alpha + \delta < 1$, where $\delta = k/(k+1)$.

Theorem. *Suppose that* $u \in C^2(\Omega) \cap C^{1+\alpha+\delta}(\overline{\Omega})$. *If*

$$g \in C^{1+\alpha+\delta}(\partial\Omega), \quad u_0 \in C^{2+\alpha}(\Gamma_0), \quad f \in C^{\alpha}(\overline{\Omega}),$$

then $u \in C^{2+\alpha}(\overline{\Omega})$, *and the estimate holds*:

$$|u|_{C^{2+\alpha}(\Omega)} \leqslant C\,(|f|_{C^{\alpha}(\Omega)} + |g|_{C^{1+\alpha+\delta}(\partial\Omega)} + |u_0|_{C^{2+\alpha}(\Gamma_0)} + |u|_{C^0(\Omega)}).$$

Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. *If* $g \in C^{\alpha+\delta}(\partial\Omega)$, $f \equiv 0$, *and* $u_0 \in C^{1+\alpha}(\Gamma_0)$, *then* $u \in C^{1+\alpha}(\overline{\Omega})$, *and the following estimate holds*:

$$|u|_{C^{1+\alpha}(\Omega)} \leqslant C\,(|g|_{C^{\alpha+\delta}(\partial\Omega)} + |u_0|_{C^{1+\alpha}(\Gamma_0)} + |u|_{C^0(\Omega)}).$$

The conditions on g, $\mathbf{v}$, L, and $\partial\Omega$, can be relaxed outside a neighbourhood of Γ_0.

Analogous estimates were proved in [1] and [3]. In [2] the oblique derivative problem in Sobolev spaces is treated; using examples from this paper one can show that better estimates cannot be obtained.

For the proof it suffices to consider $\Omega_1 \subset \Omega$ and to suppose that $u = 0$ in a neighbourhood of that part of $\partial\Omega_1$ that does not lie on $\partial\Omega$. Using a partition of unity one can obtain estimates for the whole domain Ω. One need only establish the estimates for such Ω_1 for which $\partial\Omega_1 \cap \Gamma_0 \neq \varnothing$.

The domain Ω_1 can be reduced to a form such that $\partial\Omega_1 \cap \partial\Omega \subset \{x_n = 0\}$, $x_n > 0$ in Ω_1, and $\partial/\partial\nu = \partial/\partial x_1 + a_n(x')\partial/\partial x_n$, where $x' = (x_1, \ldots, x_{n-1})$. Let $N = \{x \in \Omega_1 \mid x' \in \Gamma_0\}$.

If $x_0 \in N$, then there is a $k_0 \leqslant k$ such that $\partial^l/\partial x_1^l(a_n(x_0')) = 0$ for $l < k_0$ and $\partial^{k_0}/\partial x_1^{k_0}(a_n(x_0')) \neq 0$. We may assume that $\partial^{k_0}/\partial x_1^{k_0}(a_n(x')) \neq 0$ for $x \in \Omega_1$.

Then $a_n(x') = q(x')\cdot p_{k_0}(x_1)$ in Ω_1, where $|q| \geqslant c > 0$ in Ω_1 and $p_{k_0}(x_1)$ is a polynomial in x_1 of degree k_0 with coefficients depending on $(x_2; \ldots, x_{n-1})$ and leading coefficient 1. We extend $\mathbf{v}$ inside Ω_1. Then, using Lemma C.1 of [4], we find that $x_n(s) \geqslant c_0\,|s|^{k+1}$ on characteristics passing through points $x \in \Omega_1$ at $s = 0$, and emerging on $\partial\Omega_1$ for $x_n > 0$, or on N in the case of a subvariety Γ_0 of type 1. Here c_0 is independent of the initial point $x \in \Omega_1$, and s is a parameter on a characteristic.

Then the proof proceeds by using the Green's function for the operator

$$L_0 = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \text{ where } x \in \Omega_1.$$

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