A. N. Sergeev, The tensor algebra of the identity representation as a module over the Lie superalgebras $\mathfrak{Gl}(n, m)$ and $Q(n)$, *Mathematics of the USSR-Sbornik*, 1985, Volume 51, Issue 2, 419–427

DOI: 10.1070/SM1985v051n02ABEH002867

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Том 123 (165) (1984), Вып. 3

Math. USSR Sbornik
Vol. 51(1985), No. 2

THE TENSOR ALGEBRA OF THE IDENTITY REPRESENTATION AS A MODULE OVER THE LIE SUPERALGEBRAS $\mathfrak{Gl}(n,m)$ AND $Q(n)$

UDC 512

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ABSTRACT. Let T be the tensor algebra of the identity representation of the Lie superalgebras in the series $\mathfrak{Gl}$ and Q. The method of Weyl is used to construct a correspondence between the irreducible representations (respectively, the irreducible projective representations) of the symmetric group and the irreducible $\mathfrak{Gl}$- (respectively, Q-) submodules of T. The properties of the representations are studied on the basis of this correspondence. A formula is given for the characters on the irreducible Q-submodules of T.

Bibliography: 8 titles.

It is known (see [1]) that all the irreducible finite-dimensional representations of the complex Lie algebras in the series A_n can be obtained by decomposing the tensor powers of the identity representation. In the present article we use the method of Weyl to decompose the tensor powers of the identity representation of the Lie superalgebras $\mathfrak{Gl}(n,m)$ and $Q(n)$. We determine restrictions on the highest weight under which the corresponding irreducible module is a submodule of the tensor algebra. Moreover, a formula is given for the characters on the irreducible finite-dimensional $Q(n)$-modules appearing in the tensor algebra of the identity representation. Our results also explain the use of Young diagrams for describing subrepresentations of the supergroup $Gl(n,m)$ in the tensor algebra, as done in [7] and [8]. The ground field is always assumed to be the complex field. All operations, concepts, and constructions (tensor product, taking the commutant, commutativity, irreducibility, the enveloping algebra, and so on) are understood in the superalgebra sense (see [3]) unless otherwise specified.

§1. Notation and auxiliary constructions

1.1. Let $\{\pm 1\}$ be a two-element set, B the group of all mappings of $\mathbf{Z}_2^k$ into $\{\pm 1\}$, and $\mathfrak{S}_k$ the symmetric group of order k; $\mathfrak{S}_k$ acts in a natural way on B by permuting the factors in $\mathbf{Z}_2^k$. If $\mathfrak{A}$ is a free associative commutative superalgebra, with the free family $\{x_i\}_{i\in I}$ of generators, then a function $c\colon \mathbf{Z}_2^k \times \mathfrak{S}_k \to \{\pm 1\}$ is defined, and

$$c(p(x),\sigma)x_1\cdots x_k = x_{\sigma(1)}\cdots x_{\sigma(k)},$$

where $p(x) = (p(x_1),\ldots,p(x_k))$ is the parity vector of the elements x_i, and $\sigma\in\mathfrak{S}_k$. A trivial check shows that c is a cocycle of $\mathfrak{S}_k$ with values in B, i.e.,

$$c(p(x),\sigma\tau) = c(\sigma^{-1}p(x),\tau)c(p(x),\sigma).$$

1980 *Mathematics Subject Classification*. Primary 17A70, 17B10; Secondary 15A72, 20C30.

If V is a superspace of dimension (n, m), and W is its kth tensor power, then let

$$\pi(\sigma)(v_1 \otimes \cdots \otimes v_k) = c(\sigma^{-1}p(v), \sigma)v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}.$$

The fact that c is a cocycle gives us that π is a representation of $\mathfrak{S}_k$ in W. Let $L(V)$ be the space of endomorphisms of the vector space V. The preceding construction (with V replaced by $L(V)$) leads to a representation π_1 of $\mathfrak{S}_k$ in the space $L(V)^{\otimes k}$. On the other hand, the representation π determines a representation π_2 of $\mathfrak{S}_k$ in $L(W)$ according to the rule $\pi_2(\sigma)E = \pi(\sigma)E\pi(\sigma)^{-1}$, where $E \in L(W)$.

LEMMA 1. *The natural isomorphism $L(V)^{\otimes k} \to L(W)$ is an isomorphism of $\mathfrak{S}_k$-modules.*

Let $\dim V_{\overline{0}} = \dim V_{\overline{1}}$, and let P be an odd operator in $L(V)$: $P^2 = -1$. Denote by $\mathcal{H}_k$ the group with generators $a_1, \ldots, a_k, \varepsilon$ and relations $a_i^2 = \varepsilon$, $a_i a_j = \varepsilon a_j a_i$, $i \neq j$, and $\varepsilon^2 = 1$. Let $\varphi(a_i) = 1 \otimes \cdots \otimes P \otimes \cdots \otimes 1$ (the P in the ith position), and let $\varphi(\varepsilon) = -1$. Let $G_k = \mathfrak{S}_k \circ \mathcal{H}_k$ be the semidirect product with respect to the natural action of $\mathfrak{S}_k$ on $\mathcal{H}_k$ (permutation of the a_i's); then the representation π together with φ determine a representation π' of the group G_k in W. We endow G_k with a parity, taking $p(a_i) = \overline{1}$, $p(\varepsilon) = \overline{0}$ and $p(\sigma) = \overline{0}$ for $\sigma \in \mathfrak{S}_k$. Then the group algebra $\mathbf{C}[G_k]$ becomes a superalgebra, and π' is a representation preserving parity. If φ_1 is the representation of $\mathbf{Z}_2$ in $L(V)$ defined by the rule $\varphi_1(\overline{1})E = (-1)^{p(E)}PEP^{-1}$, $E \in L(V)$, then $\varphi_1^{\otimes k}$ determines a representation of $\mathcal{H}_k/\langle \varepsilon, 1 \rangle$ in $L(V)^{\otimes k}$. Together with the representation π_1 this determines a representation π_1' of G_k in $L(V)^{\otimes k}$. Moreover, π' determines a representation π_2' of G_k in $L(W)$ according to the rule

$$\pi_2'(g)E = (-1)^{p(g)p(E)}\pi'(g)E\pi'(g^{-1}),$$

where $g \in G_k$ and $E \in L(W)$.

LEMMA 2. *The natural isomorphism $L(V)^{\otimes k} \to L(W)$ is an isomorphism of G_k-modules.*

1.2. *Some modules over the group $\mathfrak{S}_k$.*

DEFINITION (see [2]). $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a partition of k if the λ_i are nonnegative integers with $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$, and $\lambda_1 + \cdots + \lambda_r = k$.

Let (k_1, k_2) be a partition of k, λ a partition of k_1, and μ a partition of k_2, and define

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\{1,\ldots,\lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \cdots,$$

$$\mathfrak{S}_\mu = \mathfrak{S}_{\{k_1+1,\ldots,k_1+\mu_1\}} \times \mathfrak{S}_{\{k_1+\mu_1+1,\ldots,k_1+\mu_1+\mu_2\}} \times \cdots.$$

Let $M^{\lambda,\mu}$ be the module over $\mathfrak{S}_k$ induced by the tensor product of the trivial representation of $\mathfrak{S}_\lambda$ and the alternating representation of $\mathfrak{S}_\mu$. If ν is a partition of k, then let S^ν be the irreducible $\mathfrak{S}_k$-module corresponding to the partition ν, and M^ν the module induced by the trivial representation of the subgroup $\mathfrak{S}_\nu$. In the notation of [2],

$$M^\nu = [\nu_1] \cdots [\nu_t], \qquad M^{\lambda,\mu} = [\lambda_1] \cdots [\lambda_r][1^{(\mu_1)}] \cdots [1^{(\mu_s)}].$$

By a *ν-tableau* we understand a filling of the Young diagram corresponding to ν by positive integers not exceeding k.

DEFINITION. A ν-tableau T has *type* (λ, μ) if i occurs λ_i times in T for $i \leq r$, and $r + j$ occurs μ_j times for $1 \leq j \leq s$.

Let $\mathcal{T}_0(\nu, (\lambda, \mu))$ be the set of ν-tableaux of type (λ, μ) such that 1) the numbers are nondecreasing downwards in the columns and from left to right in the rows, 2) the numbers $1, 2, \ldots, r$ are strictly increasing downwards in the columns, and 3) the numbers

$r+1, \ldots, r+s$ are strictly increasing from left to right in the rows. Let $\mathcal{T}_1(\nu, (\lambda, \mu))$ be the set of ν-tableaux of type (λ, μ) such that 1) the numbers are nondecreasing downwards in the columns and from left to right in the rows, and 2) the numbers $r+1, \ldots, r+s$ are strictly increasing in the rows. If ξ is a partition of k, then let $\mathcal{T}_2(\nu, \xi)$ be the set of semistandard ν-tableaux of type ξ (see [2]).

LEMMA 3. *The multiplicity of S^ν in $M^{\lambda,\mu}$ is equal to $|\mathcal{T}_0(\nu, (\lambda, \mu))|$.*

PROOF. In the notation of [2],

$$M^{\lambda,\mu} = [\lambda_1]\cdots[\lambda_r][1^{(\mu_1)}]\cdots[1^{(\mu_s)}] = \left(\sum m(\lambda,\xi)[\xi]\right)[1^{(\mu_1)}]\cdots[1^{(\mu_s)}]$$
$$= \left(\sum m(\lambda,\xi)[\xi][1^{(\mu_1)}]\right)[1^{(\mu_2)}]\cdots[1^{(\mu_s)}],$$

where $m(\lambda, \xi) = |\mathcal{T}_2(\lambda, \xi)|$. According to Lemma 21.5 in [2], $[\xi][1^{(\mu_1)}] = \sum d_\eta[\eta]$, where d_η is the number of ways of adding μ_1 cells to the diagram (of) ξ such that 1) no more than 1 cell is added to each row of ξ, and 2) the result is the diagram η (cells are added either from below or from the right). If the number $r+1$ is put in the cells added, then we see that the multiplicity of S^η in M^{λ,μ_1} is $|\mathcal{T}_0(\eta, (\lambda, \mu_1))|$. The lemma is proved by induction on the number of nonzero parts of μ.

LEMMA 4. $\dim \operatorname{Hom}_{\mathfrak{S}_k}(M^\nu, M^{\lambda,\mu}) = |\mathcal{T}_1(\nu, (\lambda, \mu))|$.

PROOF. We realize $M^{\lambda,\mu}$ as a subspace of $V^{\otimes k}$, where V is a superspace of dimension (r, s). It is generated by vectors of the form $e_{i_1} \otimes \cdots \otimes e_{i_k}$, where e_1 occurs λ_1 times, $\ldots, e_r$ occurs λ_r times, e_{r+1} occurs μ_1 times, $\ldots, e_{r+s}$ occurs μ_s times. The desired number is equal to the dimension of the space of $\mathfrak{S}_\nu$-invariant vectors in $M^{\lambda,\mu}$. With each $T \in \mathcal{T}_1(\nu, (\lambda, \mu))$ we associate the vector e_T obtained by averaging the vector $e_{i_1} \otimes \cdots \otimes e_{i_{\nu_1}} \otimes \cdots \otimes e_{i_k}$, where $i_1, \ldots, i_{\nu_1}$ are the numbers in the first row of T, etc., over the subgroup $\mathfrak{S}_\nu$. It is assumed that $e_1 < \cdots < e_{r+s}$; then the tensors in $V^{\otimes k}$ are ordered lexicographically, and the vector indicated above is the smallest among all the vectors occurring in e_T; moreover, it occurs in this decomposition with nonzero coefficient, in view of the conditions imposed on T. This implies that the e_T for different $T \in \mathcal{T}_1(\nu, (\lambda, \mu))$ are linearly independent. If v is any invariant vector, then $v = \sum \gamma_T e_T$, where T is a tableau not necessarily belonging to $\mathcal{T}_1(\nu, (\lambda, \mu))$. Since $e_{\sigma T} = \pm e_T$ for $\sigma \in \mathfrak{S}_k$, it can be assumed that the numbers in the rows of T are nondecreasing; moreover, if some row of T contains two identical numbers greater than r, then it is easy to see that $e_T = 0$. The theorem is proved.

1.3. *Symmetric functions.* We give some facts from the theory of symmetric functions; see [5] for details and proofs. Let $x_1, \ldots, x_N$ be a family algebraically independent over $\mathbf{C}$, let $\lambda \mapsto k$ mean that λ is a partition of k, and let $|\lambda|$ be the number of nonzero parts of λ. We define $s_l(x) = \sum x_i^l$, $s_\lambda(x) = s_{\lambda_1}(x)\cdots s_{\lambda_r}(x)$, and let $m_\lambda(x)$ be the symmetrization of the monomial $x_1^{\lambda_1}\cdots x_r^{\lambda_r}$. For $\mu \mapsto k$ and $\lambda \mapsto k$, write $\mu \le \lambda$ if and only if $\lambda_1 + \cdots + \lambda_i \ge \mu_1 + \cdots + \mu_i$ for all $i \ge 1$. Further, let

$$\Phi_l(t) = \prod_{j=1}^{l}(1-t^j), \qquad b_\lambda(t) = \prod_{i\ge 1} \Phi_{n_i}(\lambda),$$

where $n_i(\lambda)$ is the number of parts of λ equal to i, and let

$$Q_\lambda(x,t) = \frac{(1-t)^N}{\Phi_{n-r}(t)} \sum_{\sigma\in\mathfrak{S}_N} \sigma\left(x_1^{\lambda_1}\cdots x_N^{\lambda_N} \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j}\right), \tag{1}$$

where N is sufficiently large; $u_\lambda(x) = Q_\lambda(x, 0)$ is the Schur function.

PROPOSITION 1. *$Q_\lambda(x,t) = \sum_{\mu\le\lambda} \alpha_{\lambda\mu}(t) u_\mu(x)$, where $\alpha_{\lambda\mu}(t) \in \mathbf{Q}[t]$ and $\alpha_{\lambda\lambda}(t) = b_\lambda(t)$. In particular,*

$$\sum_{\sigma\in\mathfrak{S}_N} \sigma\left(\prod_{i<j}(x_i - tx_j)\right)\operatorname{sgn}\sigma = \frac{\Phi_N(t)}{(1-t)^N}\prod_{i<j}(x_i - x_j). \tag{2}$$

DEFINITION. $P_\lambda(x,t) = Q_\lambda(x,t)/b_\lambda(t)$ is the *Hall-Littlewood function.*
Let $\mathcal{L}_k(t) = \{\sum_{\lambda\vdash k} \alpha_\lambda(t) m_\lambda(x) \mid \alpha_\lambda(t) \in \mathbf{Q}[t]\}$.

PROPOSITION 2. *$\mathcal{L}_k(t)$, as a module over $\mathbf{Q}[t]$, has basis*

1) $\{Q_\lambda(x,t) \mid \lambda \vdash k\}$, 2) $\{P_\lambda(x,t) \mid \lambda \vdash k\}$, 3) $\{S_\lambda(x) \mid \lambda \vdash k\}$.

If $\lambda, \rho \vdash k$, then, by Proposition 2, there exist $X_\rho^\lambda(t) \in \mathbf{Q}[t]$ such that

$$s_\rho(x) = \sum_{\lambda\vdash k} X_\rho^\lambda(t) P_\lambda(x,t).$$

PROPOSITION 3. *If $\rho = (1^{\rho_1} 2^{\rho_2} \ldots)$, $e_\rho(t) = \prod_{i\ge1}(1-t^i)^{\rho_i}$ and $Z_\rho = \prod_{i\ge1}(i^{\rho_i})\rho_i$, then*

$$\sum_{\rho\vdash k} \frac{e_\rho(t)}{Z_\rho} X_\rho^\lambda(t) X_\rho^\mu(t) = \delta_{\lambda\mu} b_\lambda(t);$$

moreover,

$$Q_\lambda(x,t) = \sum \frac{e_\rho(t)}{Z_\rho} X_\rho^\lambda(t) S_\rho(x).$$

Let $Q_\lambda(x) = Q_\lambda(x,-1)$ for $\lambda = (\lambda_1, \ldots, \lambda_r)$, where $\lambda_1 > \cdots > \lambda_r > 0$. These functions were first introduced by Schur for computing the projective characters on $\mathfrak{S}_k$ (see [6]). The functions $Q_\lambda(x)$ satisfy the following recursion relations (see [4]):

$$Q_{(\lambda_1\cdots\lambda_r)} = Q_{(\lambda_1\lambda_2)}Q_{(\lambda_3\cdots\lambda_r)} - Q_{(\lambda_1\lambda_3)}Q_{(\lambda_2\lambda_4\cdots\lambda_r)} + \cdots + Q_{(\lambda_1\lambda_r)}Q_{(\lambda_2\cdots\lambda_{r-1})} \quad (r \text{ even}),$$

$$Q_{(\lambda_1\cdots\lambda_r)} = Q_{\lambda_1}Q_{(\lambda_2\cdots\lambda_r)} - Q_{\lambda_2}Q_{(\lambda_1\lambda_3\cdots\lambda_r)} + \cdots + Q_{\lambda_r}Q_{(\lambda_1\cdots\lambda_{r-1})} \quad (r \text{ odd})$$

and

$$Q_{(\lambda_1\lambda_2)} = Q_{\lambda_1}Q_{\lambda_2} - 2Q_{\lambda_1+1}Q_{\lambda_2-1} + \cdots + (-1)^{\lambda_2}Q_{\lambda_1+\lambda_2}.$$

1.4. *Irreducible characters on G_k.* We consider only representations preserving parity. A representation is said to be *absolutely irreducible* if its commutant reduces to the scalars. A group is said to be $\mathbf{Z}_2$*-graded* if a subgroup of it with index 2 is singled out, and the elements of the subgroup are assumed to be even by definition. The following simple proposition describes the correspondence between the conjugacy classes of elements in a $\mathbf{Z}_2$-graded group and its irreducible representations.

PROPOSITION 4. *Let G be a finite $\mathbf{Z}_2$-graded group, and G_0 the subgroup of even elements. Then the number of nonequivalent irreducible representations of G is equal to the number of conjugacy classes of elements of G lying in G_0. The number of nonequivalent irreducible representations of G that are not absolutely irreducible is equal to the number of conjugacy classes of elements of G lying in $G\backslash G_0$.*

In the case of the group G_k we are interested only in representations carrying ε into -1; therefore, it is necessary only to describe the classes C such that $C \cap \varepsilon C = \varnothing$.

LEMMA 5. *If $\sigma \in \mathfrak{S}_k$ and $a \in \mathcal{Y}_k$, then σa is not conjugate to $\varepsilon\sigma a$ if and only if the following conditions hold:*

1) $p(a) = \bar{0}$, all the cycles of σ have odd length, and $\sum_{j\in\tau} p(a_j) = \bar{0}$, where τ is any cycle of σ.

2) $p(a) = \bar{1}$, σ does not have multiple cycles, $\sum_{j\in\tau} p(a_j) = \bar{1}$, and the total number of cycles is odd.

Thus, to each partition ρ of k into odd addends there correspond two conjugacy classes of elements of G_k, and if one of them is C, the the other is εC. On the other hand, there is a bijection between partitions of k into odd addends and partitions of k into pairwise distinct addends. Such partitions are said to be *strict.* For each $\rho \vdash k$: $\rho = (1^{\rho_1} 3^{\rho_3} \cdots)$ we fix the conjugacy class C_ρ of elements that contains an element of $\mathfrak{S}_k$ of type ρ. Denote by X_ρ the value of the character X on the class C_ρ. Let

$$P_X = \sum \frac{X_\rho}{Z_\rho} s_\rho(x).$$

We call it the *polynomial of the representation with character* X. A simple argument shows that if X_1 is the character of a representation of G_{k_1}, X_2 is the character of a representation of G_{k_2}, and

$$P_{X_1} P_{X_2} = \sum \frac{X_\rho}{Z_\rho} s_\rho(x),$$

then X_ρ is the character of some representation of $G_{k_1+k_2}$. Consider the module induced by the module M^λ from the subalgebra $\mathbf{C}[\mathfrak{S}_k]$ to the algebra $\mathbf{C}[G_k]/\langle\varepsilon + 1\rangle$. The polynomial corresponding to it is equal to $Q_{\lambda_1} \cdots Q_{\lambda_r}$. Moreover, the commutant of this module contains a subalgebra isomorphic to the Clifford algebra of order r; therefore, $2^{(\delta(r)-r)/2} Q_{\lambda_1} \cdots Q_{\lambda_r}$ ($\delta(r) = 0$ if r is even, and 1 otherwise) is the polynomial of some representation of G_k. From the recursion relations for $Q_\lambda(x)$ it follows that $2^{(\delta(r)-r)/2} Q_\lambda(x)$ is also the polynomial of some (possible generalized) representation. Let Θ^λ be the character corresponding to it; then Proposition 3 shows that $\Theta^\lambda_\rho = 2^y X^\lambda_\rho(-1)$, where $y = n(\rho) + (\delta(r) - r)/2$, $n(\rho) = n_1(\rho) + n_2(\rho) + \cdots$, and

$$\sum \Theta^\lambda_\rho \Theta^\mu_\rho 2^{-n(\rho)}/Z_\rho = \delta_{\lambda\mu} 2^{\delta(|\lambda|)}.$$

But $|C_\rho| = |\varepsilon C_\rho| = |G_k|/Z_\rho \cdot 2^{n(\rho)+1}$, and so

$$\frac{1}{|G_k|} \sum \Theta^\lambda_\rho \Theta^\mu_\rho 2 |C_\rho| = \frac{1}{|G_k|} \sum_{g\in G_k} \Theta^\lambda(g)\Theta^\mu(g) = \langle\Theta^\lambda, \Theta^\mu\rangle = 2^{\delta(|\lambda|)}\delta_{\lambda\mu};$$

here $\langle\ ,\ \rangle$ is the (usual) inner product of characters. Using Proposition 4 and Lemma 5, we get

LEMMA 6. *Let λ run through the set of strict partitions of k. Then $\{\Theta^\lambda\}$ is a complete set of characters of pairwise nonequivalent irreducible representations of G_k. The representation corresponding to Θ^λ is absolutely irreducible if and only if $\delta(|\lambda|) = 0$.*

1.5. *The coproduct in the enveloping algebra.* Let $[1, k]$ denote the interval of integers from 1 to k. Let $f: [1, l] \to [1, k]$ be an arbitrary mapping, and define a $\sigma_f \in \mathfrak{S}_l$ by the following rule: put the values of f in increasing order $f(i_1) \le f(i_2) \le \cdots \le f(i_l)$, with $f(i_\alpha)$ first if $f(i_\alpha) = f(i_\beta)$ for $i_\alpha < i_\beta$, and then $\sigma_f(\alpha) = i_\alpha$. Next, let $R_j = f^{-1}(j)$ for $j \in [1, k]$. If $x \in \mathfrak{A}^k$, where $\mathfrak{A}$ is an associative superalgebra, then $R_i x = \prod_{j\in R_i} x_j$, where the product is taken in increasing order of indices, and where $R_i x$ is set equal to 1 if $R_i = \varnothing$.

LEMMA 7. *If* $c_k: U(\mathfrak{G}) \to U(\mathfrak{G})^{\otimes k}$ *is a k-fold coproduct and* $x_1, \dots, x_l \in \mathfrak{G}$, *then*

$$c_k\left(\prod_{i=1}^{l} x_i\right) = \sum_{f:[1,l]\to[1,k]} c(\sigma_f^{-1}p(x), \sigma_f^{-1}) R_1 x \otimes \cdots \otimes R_k x. \tag{3}$$

The simple but cumbersome proof of this lemma is omitted. Let $R = \{R_1, \dots, R_j\}$ be a partition of the interval $[1, l]$ into disjoint parts, let $|R|$ be the number of nonempty parts of R, and let R_f be the partition $\{f^{-1}(1), \dots, f^{-1}(k)\}$; then by transforming (3) we obtain

$$c_k\left(\prod_{i=1}^{l} x_i\right) = \sum_{j=1}^{l} \sum_{|R|=j} \frac{c(p(x), \sigma_f)}{(k-j)!} \omega(R_1 x \otimes \cdots \otimes R_j x \otimes 1 \otimes \cdots \otimes 1). \tag{4}$$

Here f is chosen so that $R = R_f$, and ω denotes supersymmetrization in $U(\mathfrak{G})^{\otimes k}$.

Let $\mathfrak{A}$ be an associative superalgebra, $\mathfrak{A}_\iota$ the Lie superalgebra connected with it, $U(\mathfrak{A}_\iota)$ its universal enveloping algebra, ψ: $U(\mathfrak{A}_\iota) \to \mathfrak{A}$ the homomorphism induced by the identity mapping of $\mathfrak{A}_\iota$ into $\mathfrak{A}$, and $\Psi = \psi^{\otimes k} \circ c_k$.

PROPOSITION 5. $\Psi(U(\mathfrak{A}_\iota))$ *coincides with the subalgebra of supersymmetric tensors in* $\mathfrak{A}^{\otimes k}$.

PROOF. It suffices to show that $\omega(x_1 \otimes \cdots \otimes x_k) \in \Psi(U(\mathfrak{A}_\iota))$, where $x_i \in \mathfrak{A}$. We carry out induction on the number $|\{i \mid x_i \neq 1\}| = l$. Since there exists a unique partition R^* of $[1, l]$ such that $|R^*| = l$, (4) gives us that

$$(k-l)!\omega(x_1 \otimes \cdots \otimes x_l \otimes 1 \otimes \cdots \otimes 1)$$
$$= \Psi\left(\prod_{i=1}^{l} x_i\right) \pm \sum_{j=1}^{l-1} \sum_{|R|=j} \frac{c(p(x), \sigma_f)}{(k-j)!} - \omega(\psi^{\otimes k}(R_1 x \otimes \cdots \otimes R_j x \otimes 1 \otimes \cdots \otimes 1)).$$

But

$$\omega(\psi^{\otimes k}(R_1 x \otimes \cdots \otimes R_j x \otimes 1 \otimes \cdots \otimes 1))$$
$$= \omega(\psi(R_1 x) \otimes \cdots \otimes \psi(R_j x) \otimes 1 \otimes \cdots \otimes 1) \in \Psi(U(\mathfrak{A}_\iota)),$$

because $j \in l$, and the induction hypothesis is applicable; hence also

$$\omega(x_1 \otimes \cdots \otimes x_l \otimes 1 \otimes \cdots \otimes 1) \in \Psi(U(\mathfrak{A}_\iota)).$$

§2. Decomposition of the tensor space

2.1. Let $G = \mathfrak{Gl}(n, m)$, let V be the identity representation of $\mathfrak{G}$, and let γ_k be the corresponding representation of $\mathfrak{G}$ and $U(\mathfrak{G})$ in W.

LEMMA 8. *If* $x \in \mathfrak{G}$ *and* $\sigma \in \mathfrak{S}_k$, *then* $\gamma_k(x)\pi(\sigma) = \pi(\sigma)\gamma_k(x)$.

PROOF. We can confine ourselves to the case where σ is a transposition of the form $(i, i+1)$ or $(1, k)$. In these cases the lemma can be verified by a direct computation.

THEOREM 1. $\pi(\mathfrak{S}_k)^! = \gamma_k(U(\mathfrak{G}))$ ($^!$ *denotes the commutant*).

PROOF. Lemma 8 shows that $\gamma_k(U(\mathfrak{G})) \subset \pi(\mathfrak{S}_k)^!$. Let $E \in \pi(\mathfrak{S}_k)^!$. Using Lemma 1, we identify $L(W)$ and $L(V)^{\otimes k}$, and then it can be assumed that E is a supersymmetric tensor. Proposition 5 shows that E lies in the image of $U(\mathfrak{G})$. The theorem is proved.

Thus, $W = \bigoplus_{\lambda \vdash k} S_\lambda \otimes V^\lambda$, where V^λ is an irreducible $\mathfrak{G}$-submodule of W. Fixing such a decomposition, we let $\{\lambda\}$ ($\{l\}$) denote the trace of the matrix

$$\operatorname{diag}(x_1, \dots, x_n, y_1, \dots, y_m)$$

in the module V^λ (in the lth symmetric power of V). Since the module $V^{\lambda_1} \otimes \cdots \otimes V^{\lambda_r}$ has a composition series whose factors are all isomorphic to the modules V^μ, and the multiplicities with which they appear are the same as the multiplicities of S^μ in M^λ, it follows that

$$\{\lambda_1\} \cdots \{\lambda_r\} = \sum m(\lambda, \mu)\{\mu\}. \tag{5}$$

Let $n(\nu, (\lambda, \mu)) = |\mathcal{T}_0(\nu, (\lambda, \mu))|$.

PROPOSITION 6. *$V^\nu \neq 0$ if and only if $\nu_{n+1} \leq m$.*

PROOF. Let $X = [1, n+m]^k$, and let the group $\mathfrak{S}_k$ act on X by permuting the factors. Let $Y = X/\mathfrak{S}_k$ be the orbit space; then $W = \bigoplus_{y,\lambda,\mu} M^{\lambda,\mu}$, where $y \in Y$, $\sum \lambda_i + \sum \mu_j = k$, $\lambda_{n+1} = 0$, and $\mu_{m+1} = 0$. Therefore,

$$\dim V^\nu = \dim V^\nu_{\bar{0}} + \dim V^\nu_{\bar{1}} = \sum_{y,\lambda,\mu} |\mathcal{T}_0(\nu, (\lambda, \mu))|. \tag{6}$$

If $\nu_{n+1} \leq m$, then there exist λ and μ such that $|\mathcal{T}_0(\nu, (\lambda, \mu))| \neq 0$; therefore, $V^\nu \neq 0$. Conversely, if $\dim V^\nu > 0$, then, by (6), there exist λ and μ such that $\mathcal{T}_0(\nu, (\lambda, \mu)) = \varnothing$. Since $\lambda_{n+1} = 0$, and since the numbers $1, \ldots, n$ are strictly increasing downwards in the columns for $T \in \mathcal{T}_0(\nu, (\lambda, \mu))$, the row of T with index $n+1$ contains numbers strictly greater than n. But $\mu_{m+1} = 0$; hence there are not more than m such numbers, and they are strictly increasing in the rows, so $\nu_{n+1} \leq m$. The proof is completed.

The next lemma gives an explicit expression for the function $\{\nu\}$.

LEMMA 9. $\{\nu\} = \sum n(\nu, (\lambda, \mu)) m_\lambda(x) m_\mu(y)$.

PROOF. Let N be the matrix formed from the elements $n(\nu, (\lambda, \mu))$, and M the matrix formed from the elements $m(\nu, \eta)$; then

$$\begin{aligned}(MN)_{\nu,(\lambda,\mu)} &= \sum m(\nu, \eta) n(\eta, (\lambda, \mu)) = \left\langle \sum n(\nu, \eta) \mathrm{X}_{S^\eta}, \sum n(\eta, (\lambda, \mu)) \mathrm{X}_{S^\eta} \right\rangle \\ &= \langle \mathrm{X}_{M^\nu}, \mathrm{X}_{M^{\lambda,\mu}} \rangle = \dim \operatorname{Hom}_{\mathfrak{S}_k}(M^\nu, M^{\lambda,\mu}) = |\mathcal{T}_1(\nu, (\lambda, \mu))|.\end{aligned}$$

The last equality is valid by Lemma 4. On the other hand, the coefficient of $x_1^{\lambda_1} \cdots x_n^{\lambda} y_1^{\mu_1} \cdots y_m^{\mu_m}$ in the product $\{\nu_1\} \cdots \{\nu_t\}$ is also equal to $|\mathcal{T}_1(\nu, (\lambda, \mu))|$. Therefore,

$$\{\nu_1\} \cdots \{\nu_t\} = \sum (MN)_{\nu(\lambda,\mu)} m_\lambda(x) m_\mu(y).$$

Consequently,

$$\begin{aligned}\{\nu\} &= \sum (M^{-1})_{\nu,\eta} \{\eta_1\} \cdots \{\eta_s\} = \sum (M^{-1})_{\nu,\eta} (MN)_{\eta,(\lambda,\mu)} m_\lambda(x) m_\mu(y) \\ &= \sum n(\nu, (\lambda, \mu)) m_\lambda(x) m_\mu(y),\end{aligned}$$

and the lemma is proved.

A consequence of Proposition 6 and Lemma 9 is

THEOREM 2. *The module V^ν is nonzero if and only if $\nu_{n+1} \leq m$, and its highest weight is $(\nu_1, \ldots, \nu_n, \langle \nu'_1 - n \rangle, \ldots, \langle \nu'_m - n \rangle)$, where ν'_i is the length of the ith column of the diagram of ν, and $\langle l \rangle = l$ if $l \geq 0$ and 0 otherwise. Moreover, the numbers $n(\nu, (\lambda, \mu))$ give the multiplicity of the weight (λ, μ) in the module V^ν.*

COROLLARY 1. *An irreducible $\mathfrak{G}$-module with highest weight Λ is a submodule of the tensor algebra of V if and only if the Λ_i are nonnegative integers (the labels of highest weight are taken with respect to a Cartan subalgebra consisting of diagonal matrices), and $\Lambda_n \geq |\{i \mid i > n,\ \Lambda_i \neq 0\}|$.*

COROLLARY 2. *There are multilinear $\mathfrak{G}$-invariant forms of p-vectors in V^* and q-vectors in V if $p = q$. In this case they are linear combinations of forms L_σ ($\sigma \in \mathfrak{S}_p$) such that*

$$L_\sigma(v_1, \ldots, v_p, u_1, \ldots, u_p) = v_1 \otimes \cdots \otimes v_p(\pi(\sigma)u_1 \otimes \cdots \otimes u_p).$$

The proof follows at once from the identification

$$(V^*)^{\otimes p} \otimes V^{\otimes q} \approx \operatorname{End}(V^{\otimes p}, V^{\otimes q}).$$

2.2. Let $\mathfrak{G} = Q(n)$, let V be the identity representation of it, and let P be an odd operator in the commutant of $\mathfrak{G}$ in V.

THEOREM 3. $\pi'(G_k)^! = \gamma_k(U(\mathfrak{G}))$.

PROOF. If $x \in \mathfrak{G}$ and $\sigma \in \mathfrak{S}_k$, then $\pi'(\sigma) = \pi(\sigma)$, and $\pi'(\sigma)\gamma_k(x) = \gamma_k(x)\pi'(\sigma)$ by Lemma 8. On the generators of $\mathcal{H}_k$ we have that

$$\pi'(a_i)\gamma_k(x) = (-1)^{p(x)}\gamma_k(x)\pi'(a_i),$$

and so $\gamma_k(U(\mathfrak{G})) \subset \pi'(G_k)^!$. Conversely, if $E \in \pi'(G_k)^!$, then (by Lemma 2) E, as an element of $L(V)^{\otimes k}$, is an invariant of the group G_k for the representation π'_1. In particular, E is in $\mathfrak{G}^{\otimes k}$ and is a supersymmetric tensor, but then Proposition 5 shows that $E \in \gamma_k(U(\mathfrak{G}))$, and the theorem is proved.

Thus, W is a completely reducible $\mathfrak{G}$-module. Let $\mathfrak{A}_i$ ($i = 1, 2$) be an associative superalgebra, and U_i a module over $\mathfrak{A}_i$, and suppose that the commutant of $\mathfrak{A}_i$ in U_i contains an odd operator P_i with $P_i^2 = -1$. Then the commutant of $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ in $U_1 \otimes U_2$ contains a subalgebra isomorphic to the Clifford algebra of order 2; therefore, $U_1 \otimes U_2 = U \oplus P(U)$, where U is some $\mathfrak{A}_1 \otimes \mathfrak{A}_2$-module and $P(U)$ is the module opposite to it. In this situation the module U will be denoted by $U_1 \otimes U_2 \cdot 2^{-1}$. If λ is a strict partition of k and T^λ is the irreducible G_k-module corresponding to it by Lemma 6, then

$$W = \bigoplus T^\lambda \otimes V^\lambda \cdot 2^{-\delta(|\lambda|)},$$

where V^λ is an irreducible $\mathfrak{G}$-submodule of W, and the direct sum is over the strict partitions of k. We fix one such decomposition. Denote by $\{\lambda\}$ the trace of the matrix $\operatorname{diag}(x_1, \ldots, x_n, x_1, \ldots, x_n) = E$ in the module V^λ. Consider the operator

$$\mathcal{L}_\lambda = \frac{1}{|G_k|} \sum_{g \in G_k} \Theta^\lambda(g) gE;$$

then

$$\begin{aligned} \operatorname{tr} \mathcal{L}_\lambda &= \sum_{\mu \vdash k} 2^{-\delta(|\mu|)} \operatorname{tr}(\mathcal{L}_\lambda|_{T^\lambda \otimes V^\lambda}) \\ &= \sum_{\mu \vdash k} 2^{-\delta(|\mu|)} \frac{1}{|G_k|} \sum_{g \in G_k} \Theta^\lambda(g)\Theta^\mu(g)\{\mu\} \\ &= 2^{-\delta(|\lambda|)} \langle \Theta^\lambda, \Theta^\lambda \rangle \{\lambda\} = \{\lambda\}. \end{aligned}$$

On the other hand, if $\sigma \in \mathfrak{S}_k$ and $\sigma = (1^{\rho_1} 3^{\rho_3} \cdots)$ is the cycle structure of σ, then $\operatorname{tr} \sigma E = 2^{n(\rho)} s_\rho(x)$, and, consequently,

$$\operatorname{tr} \mathcal{L}_\lambda = \sum_{\rho \vdash k} \frac{2^{-n(\rho)}}{Z_\rho} \Theta^\lambda_\rho 2^{n(\rho)} s_\rho(x) = \sum_{\rho \vdash k} \frac{\Theta^\lambda_\rho}{Z_\rho} s_\rho(x) = 2^y Q_\lambda(x),$$

where the last equality is valid according to Proposition 3, and $y = (\delta(|\lambda|) - |\lambda|)/2$. Proposition 1 and formula (1) show that $\{\lambda\} \not\equiv 0$ if and only if $\lambda_{n+1} = 0$. Let $|\lambda| = r \leq n$; then, by (2), $Q_\lambda(x,t)$ can be transformed to the form

$$Q_\lambda(x,t) = \frac{(1-t)^r}{\Delta(x)} \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}\sigma \cdot \sigma(x_1^{\lambda_1+n-1}\cdots x_n^{\lambda_n}) \times \prod_{i_1>1}\left(1-t\frac{x_{i_1}}{x_1}\right)\cdots\prod_{i_r>r}\left(1-t\frac{x_{i_r}}{x_r}\right), \tag{7}$$

where $\Delta(x) = \prod_{i<j}(x_i - x_j)$.

Let $J = J_{\overline{0}} \oplus J_{\overline{1}}$, a Cartan subalgebra consisting of diagonal matrices in $\mathfrak{G}_{\overline{0}}$ and $\mathfrak{G}_{\overline{1}}$, and let X_α and F_α be even and odd elements of weight α with respect to $J_{\overline{0}}$. Define $\overline{\alpha} \in J_{\overline{1}}^*$ by the rule $[h, F_\alpha] = \overline{\alpha}(h)X_\alpha$, where $h \in J_{\overline{1}}$. Let $\jmath$ be an odd nondegenerate form on J^* corresponding to an odd bilinear invariant form on $\mathfrak{G}$. Formula (7) and Theorem 3 give us the next theorem.

THEOREM 4. *The module V^λ is nonzero if and only if $\lambda_{n+1} = 0$. It is absolutely irreducible if and only if $\delta(|\lambda|) = 0$, its highest weight is equal to λ, and*

$$\operatorname{ch} V^\lambda = 2^z \sum \operatorname{sgn} w \cdot w\left(e^{\lambda+\rho}\prod_{\jmath(\overline{\alpha},\lambda)\neq 0}(1+e^{-\alpha})\right) 1\Big/\sum \operatorname{sgn} w \cdot e^{w\rho},$$

where $z = (|\lambda| + \delta(|\lambda|))/2$, α is a positive root, ρ is the half-sum of the positive roots, and the summation is over the elements of the Weyl group of the Lie algebra $\mathfrak{G}_{\overline{0}}$.

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Translated by H. H. McFADEN

**Editor's note.* The Russian original combines the authors, volume and page span of [8a] with the title of [8b].