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ON REDUCTIVE OPERATORS AND OPERATOR ALGEBRAS

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Abstract. We prove a theorem on the structure of weakly closed reductive operator algebras. The proof essentially relies on a known result of V. I. Lomonosov on transitive operator algebras containing a nonzero compact operator. We deduce a number of corollaries which apply to the reductivity problem.

Bibliography: 20 titles.

1. Let A be an operator on the Hilbert space H. A subspace $L \subset H$ is called a *reducing subspace* for A if it itself and its orthogonal complement are invariant with respect to A. If all invariant subspaces of A are reducing, then A is called a *reductive operator.*

It is clear that all Hermitian operators are reductive. In a series of papers, beginning with Wermer's paper [1], the conditions for the reductivity of a normal operator were studied. It still is not known whether reductive operators exist which are not normal. As Dyer, Pedersen and Porcelli [2] showed recently, this question is equivalent to the invariant subspace problem (do there exist operators without nontrivial invariant subspaces?). The fact that the reductivity of an operator implies its normality was proved for certain special classes of operators by Ando [4] (compact operators), Rosenthal [5] (polynomially compact operators), Moore [10] and Nordgren, Radjavi and Rosenthal [7] (perturbations of normal operators by compact ones).

A more general problem is represented by the study of reductive operator algebras, i.e., algebras whose invariant subspaces reduce all of their elements. The main question here is formulated analogously to the foregoing one: do there exist nonsymmetric weakly closed reductive algebras? This question, under various supplementary conditions on the algebra, was examined in the papers [6], [8], [9], [11], [12], and [16].

The class of reductive algebras contains the class of transitive ones (those having no nontrivial invariant subspaces). In particular, from the symmetry of all weakly closed reductive algebras would follow the positive resolution of Burnside's problem: every transitive algebra is weakly dense in the algebra of all operators. In recent times, substantial progress has taken place in the theory of transitive algebras: Lomonosov [3] has proved that Burnside's problem can be solved positively under the condition that the algebra in question contain a nonzero compact operator.

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In the present paper, Lomonosov's theorem is used in the proof of the following theorem: *every weakly closed reductive algebra is the direct sum of a W^*-algebra and a reductive algebra which does not contain nonzero compact operators.* The second basic result of the paper is the proof of the reductivity of the commutant of a reductive commutative algebra. Several corollaries are derived from these theorems, one of which strengthens the above-mentioned results of [4] and [5] as follows: if a reductive operator commutes with a nondegenerate compact operator, then it is normal.

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2. Below, we will denote by $B(H)$ the algebra of all bounded linear operators on the separable Hilbert space H, and by $C(H)$ the ideal of compact (completely continuous) operators. In $B(H)$, we consider the following topologies: the uniform (u), the weak (w), and the ultraweak (uw); the closure of a set in any of these topologies is denoted by a bar, supplemented by the appropriate symbol; the same applies to the notation for convergence in each of the topologies. The identity operator on H is denoted by 1_H, and the commutant of a set of operators $\mathfrak{A} \subset B(H)$ is denoted by $\mathfrak{A}'$, while the set of (closed) subspaces invariant with respect to $\mathfrak{A}$ is denoted by lat $\mathfrak{A}$. If L is a subspace of H, then the restriction to L of an operator $A \in B(H)$ (or a set $\mathfrak{A} \subset B(H)$) is denoted by $A|L$ (respectively, $\mathfrak{A}|L$). The algebra $B(L)$ is assumed to be imbedded in $B(H)$; this means that an operator $A \in B(L)$ is identified with the operator on H which coincides with A on L and vanishes on $L^{\perp}$.

For any operator $A \in B(H)$ and any natural number n, we denote by $A^{(n)}$ the operator on the space $H^{(n)} = H \oplus \cdots \oplus H$ (with n terms) which acts by the formula

$$A^{(n)}\left(\bigoplus_{k=1}^{n} x_k\right) = \bigoplus_{k=1}^{n} Ax_k.$$

If $\mathfrak{A} \subset B(H)$, then $\mathfrak{A}^{(n)}$ denotes the set of all operators $A^{(n)}$ where $A \in \mathfrak{A}$.

We recall that an operator $A \in B(H)$ is called *nondegenerate* if it does not vanish on any nontrivial reducing subspace.

The proof of the following lemma uses an idea from Ando [4].

LEMMA 2.1. *Let $\mathfrak{A}$ be a reductive algebra of operators on H containing a nonzero compact operator K. Then there exists a subspace $H_0 \in$ lat $\mathfrak{A}$ such that the algebra $\mathfrak{A}|H_0$ is transitive and $K|H_0 \neq 0$.*

PROOF. Let $\mathfrak{N} = \{L \in \text{lat } \mathfrak{A}\colon \ \|K|L\| = \|K\|\}$; we will prove that the set $\mathfrak{N}$ of subspaces, ordered by inclusion, contains minimal elements. Let $\{L_\alpha\}$ be a descending chain of elements of $\mathfrak{N}$; by Zorn's lemma, it is sufficient to prove that $\bigcap_\alpha L_\alpha \in \mathfrak{N}$. Since a compact operator achieves its norm, there exist vectors $x_\alpha \in L_\alpha$, $\|x_\alpha\| = 1$, $\|Kx_\alpha\| = \|K|L_\alpha\| = \|K\|$. By virtue of the weak compactness of the unit sphere, there exists a weakly convergent subnet $\{x_{\alpha_\beta}\}$, $x_{\alpha_\beta} \longrightarrow x$. Clearly, $x \in \bigcap_\alpha L_\alpha$ and $\|x\| \leqslant 1$. Further, since a compact operator converts a weakly convergent bounded chain into a strongly convergent one, it follows that $Kx_{\alpha_\beta} \longrightarrow Kx$, $\|Kx\| = \|K\|$, $\|x\| = 1$, and consequently $\bigcap_\alpha L_\alpha \in \mathfrak{N}$.

Let H_0 be any minimal element of $\mathfrak{N}$. If the algebra $\mathfrak{A}|H_0$ is not transitive, then there exist (by virtue of the reductivity of $\mathfrak{A}$) subspaces H_1, $H_2 \in \operatorname{lat} \mathfrak{A}$ such that $H_0 = H_1 \oplus H_2$. Since $\|K|H_0\| = \sup_{i=1,2}\|K|H_i\|$, one of the spaces H_i lies in $\mathfrak{N}$, which contradicts the minimality of H_0. Thus, the algebra $\mathfrak{A}|H_0$ is transitive. The lemma is proved.

We will call $\mathfrak{A}$-invariant subspaces, the restrictions of $\mathfrak{A}$ to which are transitive, *irreducible.*

COROLLARY 2.2. *Let $\mathfrak{A}$ be a reductive algebra, $L \in \operatorname{lat} \mathfrak{A}$ and $K \in \mathfrak{A} \cap C(H)$. If $K|M \neq 0$ for all $M \in \operatorname{lat} \mathfrak{A}$, $0 \neq M \subset L$, then there exists a decomposition of L into an orthogonal sum of subspaces irreducible with respect to $\mathfrak{A}$.*

LEMMA 2.3. *If the algebra $\mathfrak{A} \subset B(H)$ is reductive and K is a nonzero compact operator in $\mathfrak{A}$, then there exists an operator $A \in \mathfrak{A}$ such that $\sigma(AK) \neq \{0\}$.*

PROOF. Let H_0 be an irreducible $\mathfrak{A}$-invariant subspace such that $K_0 = K|H_0 \neq 0$. By Lomonosov's lemma [3], there exists in the algebra $\mathfrak{A}|H_0$ an operator A_0 such that the spectrum of A_0K_0 is nonzero. Consequently, $\sigma(AK) \neq \{0\}$ for every operator $A \in \mathfrak{A}$ such that $A|H_0 = A_0$.

LEMMA 2.4. *If the uniformly closed reductive algebra $\mathfrak{A}$ contains a nonzero compact operator, then it contains a finite-dimensional (not necessarily orthogonal) projection E such that*

$$\dim EH = \inf \{\dim AH : 0 \neq A \in \mathfrak{A}\}.$$

PROOF. We put $d = \inf\{\dim AH\colon 0 \neq A \in \mathfrak{A}\}$. By Lemma 2.3, $\mathfrak{A}$ contains finite-dimensional operators—projections (not necessarily orthogonal) onto the characteristic subspaces of a compact operator with nonzero spectrum; that is, $d < \infty$. Let $A \in \mathfrak{A}$ and $\dim AH = d$; then by virtue of Lemma 2.3 there exists an operator $B \in \mathfrak{A}$ such that $\sigma(BA) \neq 0$. If E is a projection onto some nonzero characteristic subspace of the operator BA, then $d \leqslant \dim EH \leqslant \dim BAH \leqslant \dim AH = d$. The lemma is proved.

LEMMA 2.5. *Let the uniformly closed algebra $\mathfrak{A} \subset B(H)$ be reductive and $\mathfrak{A} \cap C(H) \neq 0$. Then there exists a nonzero orthogonal projection $P \in \mathfrak{A}' \cap \mathfrak{A}''$ satisfying the following conditions:*

a) *The algebra $\mathfrak{A}|PH$ is unitarily equivalent to the algebra $\mathfrak{A}_0^{(n)}$ ($n < \infty$), where $\mathfrak{A}_0$ is an algebra of operators on some Hilbert space H_0, containing $C(H_0)$.*

b) $C(PH) \cap (\mathfrak{A}|PH) \subset \mathfrak{A}$.

PROOF. Let E be the projection whose existence was established in Lemma 2.4. We put $\widetilde{H} = \overline{\mathfrak{A}EH}$; one easily sees that $\widetilde{H} \in \operatorname{lat}\mathfrak{A} \cap \operatorname{lat} \mathfrak{A}'$, from which it follows that the orthogonal projection P onto the subspace $\widetilde{H}$ lies in $\mathfrak{A}' \cap \mathfrak{A}''$. Applying Corollary 2.2 to the subspace $\widetilde{H}$ and the operator E, we obtain a decomposition $\widetilde{H} = \bigoplus_0^n H_k$, where the H_k are irreducible $\mathfrak{A}$-invariant subspaces (the number of terms is finite, since $\dim EH < \infty$).

We will prove that the algebras $\mathfrak{A}|\widetilde{H}$ and $(\mathfrak{A}|H_0)^{(n+1)}$ are unitarily equivalent. If $n = 0$, there is nothing to prove; therefore we assume below that $n \geqslant 1$.

We intorduce the notation $\hat{H} = \bigoplus_1^n H_k$, $\widetilde{\mathfrak{A}} = \mathfrak{A}|\widetilde{H}$, $\hat{\mathfrak{A}} = \mathfrak{A}|\hat{H}$, $\mathfrak{A}_k = \mathfrak{A}|H_k$, $A_k = A|H_k$, $\widetilde{A} = A|\widetilde{H}$ and $\hat{A} = A|\hat{H}$ $(A \in \mathfrak{A}; 0 \leqslant k \leqslant n)$.

We shall prove that all the subspaces E_kH are one-dimensional. In fact, from the transitivity of the algebras $\mathfrak{A}_k$ follows the transitivity of the algebras $(E_k\mathfrak{A}_kE_k)|E_kH$; consequently the finite-dimensional algebra $E\mathfrak{A}E$ has a sufficient set of irreducible representations, i.e., it is semisimple. At the same time, the algebra $E\mathfrak{A}E$ contains no nontrivial projections (by the choice of E); by virtue of the Wedderburn theorem, $\dim E\mathfrak{A}E = 1$, and hence $\dim E_kH = 1$.

We choose vectors x_k, $x_k \neq 0$, $x_k \in E_kH$, $k = 0, 1, \ldots, n$. We put $\widetilde{x} = \Sigma_0^n x_k$, $\hat{x} = \Sigma_1^n x_k$, $\Gamma = \mathfrak{A}\widetilde{x}$, and $D_k = \mathfrak{A}x_k$. The subspace D_k is dense in H_k, since the algebra $\mathfrak{A}_k$ is transitive. The subspace Γ is closed in $\widetilde{H}$, since the correspondence $A\widetilde{x} \longrightarrow AE$ is a homeomorphism of Γ onto the closed left ideal $\mathfrak{A}E$ of the algebra $\mathfrak{A}$.

We note that vectors in Γ are uniquely determined by their projections on H_0 (in fact, if $A \in \mathfrak{A}$, $Ax_0 = 0$, and $A\widetilde{x} \neq 0$, then $AE \neq 0$ and $\dim AEH < \dim EH$, which contradicts the choice of E). Therefore, we can define an operator $T\colon H_0 \longrightarrow \hat{H}$ by putting $TAx_0 = A\hat{x}$. Clearly, for all $A \in \mathfrak{A}$ we have the inclusion $\hat{A}T \subset TA_0$. The graph of the operator T coincides with Γ, from which it follows that T is closed. Therefore, the subspace $\Gamma^\perp \cap \widetilde{H}$ is the graph of the operator $-T^*$. Since $\Gamma \in \operatorname{lat} \mathfrak{A}$ and the algebra $\mathfrak{A}$ is reductive, we have $\Gamma^\perp \cap \widetilde{H} \in \operatorname{lat} \mathfrak{A}$. Consequently, for all $A \in \mathfrak{A}$ we have the inclusion $A_0T^* \subset T^*\hat{A}$, which together with the preceding implies the commutability of the operator T^*T with $\mathfrak{A}_0$. The spectral subspaces of the operator T^*T are invariant for $\mathfrak{A}_0$, from which it follows, by virtue of the transitivity of $\mathfrak{A}_0$, that T^*T is a multiple of 1_{H_0}. In particular, the operator T is bounded, and $D_0 = H_0$.

We now define operators $T_k\colon H_0 \longrightarrow H_k$, $0 \leqslant k \leqslant n$, putting $T_kAx_0 = Ax_k$ $(A \in \mathfrak{A})$. The operators T_k are everywhere defined and bounded (since T is bounded), with $A_kT_k = T_kA_0$. Regarding the graphs of these operators as $\mathfrak{A}$-invariant subspaces and using the reductivity of $\mathfrak{A}$, we obtain that all of the operators $T_k^*T_k$ are multiples of 1_{H_0}. The isometric mapping of the space $H_0^{(n+1)}$ onto $\widetilde{H}$ which carries an arbitrary vector $\bigoplus_0^n y_k \in H_0^{(n+1)}$ into the vector $\Sigma_0^n T_k/\|T_k\| y_k \in \widetilde{H}$ establishes, as is easily verified, a unitary equivalence between the algebras $\mathfrak{A}_0^{(n+1)}$ and $\widetilde{\mathfrak{A}}$.

From what has been proved, it follows that the algebra $\overline{\widetilde{\mathfrak{A}}}^u$ is unitarily equivalent to the algebra $\overline{\mathfrak{A}_0^{(n+1)}}^u$. Since $\mathfrak{A}_0$ is a transitive algebra containing the nonzero finite rank operator E_0, it follows that $C(H_0) \subset \overline{\mathfrak{A}}_0^u$ (see [14]). Hence the algebra $\overline{\widetilde{\mathfrak{A}}}^u \cap C(\widetilde{H})$ is unitarily equivalent to the algebra $C(H_0)^{(n+1)}$, and, in particular, is topologically simple. We now put $\mathfrak{J} = \{A \in \mathfrak{A}\colon A|\widetilde{H}^\perp = 0\}$; $\mathfrak{J}$ is a closed ideal in the algebra $\mathfrak{A}$, and therefore $\widetilde{\mathfrak{J}}$ is a uniformly complete ideal in the algebra $\widetilde{\mathfrak{A}}$, and consequently a closed ideal in the algebra $\overline{\widetilde{\mathfrak{A}}}^u$ $(\subset B(\widetilde{H}))$. Since $\overline{\widetilde{\mathfrak{A}}}^u \cap C(\widetilde{H})$ is a simple algebra and the ideal $\widetilde{\mathfrak{J}} \cap C(\widetilde{H})$ is nonzero $(E \in \widetilde{\mathfrak{J}} \cap C(\widetilde{H}))$, it follows that $\overline{\widetilde{\mathfrak{A}}}^u \cap C(\widetilde{H}) = \widetilde{\mathfrak{J}} \cap C(\widetilde{H}) \subset \widetilde{\mathfrak{J}}$. Hence we conclude, first, that $\overline{\mathfrak{A}}_0^u \cap C(H_0) \subset \mathfrak{A}_0$, i.e., $C(H_0) \subset \mathfrak{A}_0$, and second, that $\widetilde{\mathfrak{A}} \cap C(\widetilde{H}) \subset \mathfrak{A}$. The lemma is proved.

Corollary 2.6. *If the reductive algebra* $\mathfrak{A} \subset B(H)$ *is ultraweakly closed and* $\mathfrak{A} \cap C(H) \neq \{0\}$, *then there exists a nonzero projection* $P \in \mathfrak{A} \cap \mathfrak{A}'$ *such that* $\mathfrak{A}|PH$ *is a factor of type* I *with a finite commutant.*

PROOF. Retaining the notation of Lemma 2.5, we note that the algebra $\overline{\widetilde{\mathfrak{A}}}^{uw}$ is unitarily equivalent to $(\overline{\mathfrak{A}}_0^{uw})^{(n+1)}$. But $\overline{\mathfrak{A}}_0^{uw} = B(H_0)$ since $C(H_0) \subset \mathfrak{A}_0$; consequently, $\overline{\widetilde{\mathfrak{A}}}^{uw}$ is a factor of type I with a finite commutant. Further, we have $\overline{\widetilde{\mathfrak{A}} \cap C(\widetilde{H})}^{uw} = \overline{\widetilde{\mathfrak{A}}}^{uw}$ (since the algebra $\widetilde{\mathfrak{A}} \cap C(\widetilde{H})$ is unitarily equivalent to $C(H_0)^{(n+1)}$); therefore, $\overline{\widetilde{\mathfrak{A}}}^{uw} \subset \overline{\mathfrak{A}}^{uw} = \mathfrak{A}$. Hence it follows that $\widetilde{\mathfrak{A}} = \overline{\widetilde{\mathfrak{A}}}^{uw}$, and, in particular, $1_{\widetilde{H}} \in \widetilde{\mathfrak{A}}$. Therefore, $P \in \mathfrak{A} \cap \mathfrak{A}'$. The corollary is proved.

THEOREM 2.7. *Every ultraweakly (weakly) closed reductive algebra is unitarily equivalent to the direct sum of a type* I *W^*-algebra with finite commutant and discrete center, and an ultraweakly (respectively, weakly) closed reductive algebra which contains no nonzero compact operators.*

PROOF. Let the algebra $\mathfrak{A}$ be reductive and ultraweakly closed. We denote by $\mathfrak{M}$ the class of orthogonal projections $P \in \mathfrak{A} \cap \mathfrak{A}'$ such that $\mathfrak{A}|PH$ is a factor of type I with finite commutant. Let $\{P_n\}_1^N$ $(N \leqslant \infty)$ be a maximal system of pairwise orthogonal projections in $\mathfrak{M}$, and $E = \Sigma_1^N P_n$. It is clear that $\mathfrak{A}|EH$ is a type I W^*-algebra with finite commutant and discrete center. By virtue of the maximality of the system $\{P_n\}_1^N$, $(1 - E)$ does not majorize nonzero projections in $\mathfrak{M}$; by Corollary 2.6, this implies that the algebra $\mathfrak{A}|(1 - E)H$ contains no nonzero compact operators. Since $E \in \mathfrak{A} \cap \mathfrak{A}'$, the fact that $\mathfrak{A}|(1 - E)H$ is closed is trivial.

COROLLARY 2.8. *If a reductive weakly closed algebra contains a nondegenerate compact operator (in particular, one with zero null space or dense range), then it is symmetric.*

LEMMA 2.9. *A uniformly closed reductive algebra of compact operators is symmetric.*

PROOF. Let $\mathfrak{A}$ be a reductive closed subalgebra of the C^*-algebra $C(H)$. It follows from Theorem 2.7 that $\overline{\mathfrak{A}}^{uw}$ is a symmetric algebra, and consequently $\overline{\mathfrak{A}}^{uw} \cap C(H)$ is a C^*-algebra. But since every u-continuous functional on $C(H)$ can be extended to a uw-continuous functional on $B(H)$, it follows that $\mathfrak{A} = \overline{\mathfrak{A}}^{u} = \overline{\mathfrak{A}}^{uw} \cap C(H)$. The lemma is proved.

COROLLARY 2.10. *If the uniformly closed algebra $\mathfrak{A}$ is reductive, then the algebra $\mathfrak{A} \cap C(H)$ is symmetric.*

PROOF. If $\mathfrak{A} \cap C(H) \neq 0$, then, using Lemma 2.5, it is possible to find a set of mutually orthogonal projections $P_n \in \mathfrak{A}' \cap \mathfrak{A}''$ with the following properties:

a) $C(P_nH) \cap \mathfrak{A}|P_nH \subset \mathfrak{A}$;

b) $\mathfrak{A}|P_nH \subset \overline{(\mathfrak{A}|P_nH) \cap C(P_nH)}^{uw}$;

c) the algebra $\mathfrak{A}|(1 - \Sigma_nP_n)H$ contains no nonzero compact operators.

Hence it follows that

$$\overline{\mathfrak{A} \cap C(H)}^{uw} = \Big(1 - \sum_n P_n\Big)\overline{\mathfrak{A}}^{uw}.$$

But $(1 - \Sigma_nP_n)\overline{\mathfrak{A}}^{uw}$ is a reductive algebra; consequently, the algebra $\mathfrak{A} \cap C(H)$ is reductive. By Lemma 2.9, the algebra $\mathfrak{A} \cap C(H)$ is symmetric.

COROLLARY 2.11. *If $\mathfrak{A}$ is a uniformly closed subalgebra of a W^*-algebra R, weakly dense in R, then the algebra $\mathfrak{A} \cap C(H)$ is symmetric.*

PROOF. It is enough to note that the density of $\mathfrak{A}$ in R implies the reductivity of $\mathfrak{A}$, and to apply Corollary 2.10.

3. LEMMA 3.1. *Let $\mathfrak{A}$ be a reductive algebra, and P and Q mutually orthogonal projections in $\mathfrak{A}'$. Then $P\mathfrak{A}'Q \subset \mathfrak{A}'^*$.*

PROOF. It is required to prove that if $A \in \mathfrak{A}$ and $B \in \mathfrak{A}'$, then $PBQA^* = A^*PBQ$. We consider the subspace $L = (1 + QB)PH$. Since the operator $T = 1 + QBP$ is invertible and $L = TPH$, L is closed. Since $(1 + QB)P \in \mathfrak{A}'$, it follows that $L \in \operatorname{lat} \mathfrak{A} = \operatorname{lat} \mathfrak{A}^*$. Therefore for any $x \in H$ there exists a $y \in H$ such that $A^*Px + A^*QBPx = Py + QBPy$. Taking into account that $A^*P = PA^*$, we hence obtain $PA^*x = Py$ and $A^*QBPx = QBPy = QBPA^*x$. Consequently, because x was arbitrary, $A^*QBP = QBPA^*$. The lemma is proved.

THEOREM 3.2. *If a commutative algebra is reductive, then its commutant is a reductive algebra.*

PROOF. Let $\mathfrak{A}$ be a commutative reductive algebra, $L \in \operatorname{lat} \mathfrak{A}' \subset \operatorname{lat} \mathfrak{A}$, P the orthogonal projection onto L, and $Q = 1 - P$. Then $P\mathfrak{A}'Q = 0$, and, by virtue of Lemma 3.1,

$$(Q\mathfrak{A}'P)^* = (Q^2\mathfrak{A}'P^2)^* = P(Q\mathfrak{A}'P)^*Q \subset P\mathfrak{A}'Q = 0.$$

Consequently, $P\mathfrak{A}'Q = Q\mathfrak{A}'P = 0$. The theorem is proved.

COROLLARY 3.3. *The commutant of a reductive operator is a reductive algebra.*

COROLLARY 3.4. *A reductive operator which commutes with a nondegenerate compact operator is normal.*

PROOF. If the operator A satisfies the conditions of this assertion, then, by virtue of Corollary 3.3, $(A)'$ is a reductive algebra containing a nondegenerate compact operator. By Corollary 2.8, the algebra $(A)'$ is symmetric, i.e., the operator A is normal.

COROLLARY 3.5. *A reductive operator which commutes with a compact operator also commutes with its adjoint.*

PROOF. It is enough to apply Corollaries 3.3 and 2.10.

Corollary 3.5 is an analog of Fuglede's theorem [13] on normal operators. We note that it is easy to deduce from it the proof of a theorem announced in [7] and [10]:

COROLLARY 3.6. *A reductive operator which can be represented as the sum of a normal operator and a compact operator which commute is normal.*

PROOF. Let $A = B + C$, where A is a reductive operator, B a normal operator, C a compact operator, and $BC = CB$. Then $C \in (A)'$, and hence $C^* \in (A)'$ (by Corollary 3.5). Further, $B \in (A)'$, and thus, by Fuglede's theorem, $B^* \in (A)'$. Consequently, $A^* = B^* + C^* \in (A)'$, i.e., A is a normal operator.

COROLLARY 3.7. *A commutative weakly closed reductive subalgebra of a finite type I W^*-algebra is symmetric.*

PROOF. If R is a finite W^*-algebra of type I and $\mathfrak{A}$ is a reductive commutative subalgebra of R, then by Theorem 3.2, $\mathfrak{A}'$ is a reductive algebra containing R'. It is easy to deduce from the results of Hoover [12] that a reductive weakly closed algebra containing a type I W^*-algebra with finite commutant is symmetric. Consequently, $\mathfrak{A}'$ is a symmetric algebra, and hence $\mathfrak{A}$ consists of normal operators. By a theorem of Sarason [15], if $\mathfrak{A}$ is weakly closed, then it is symmetric. The corollary is proved.

In [10], the following assertion is made: a reductive operator which is similar to a normal operator is normal. With the help of Theorem 3.2, this result can be strengthened by replacing "similar" by "quasi-similar." We introduce some additional notation.

Let H_1 and H_2 be Hilbert spaces, and $B(H_1, H_2)$ the space of all bounded linear operators mapping H_1 to H_2. For any $A_1 \in B(H_1)$ and $A_2 \in B(H_2)$, we put

$$\operatorname{int}(A_1, A_2) = \{T \in B(H_1, H_2) : TA_1 = A_2T\}.$$

The operator A_1 is called a *quasi-affine transformation* of the operator A_2 (we write $A_1 \underset{\sim}{\rightarrow} A_2$) if $\operatorname{int}(A_1, A_2)$ contains a quasi-invertible operator (i.e., an operator with zero null space and dense range). If $A_1 \underset{\sim}{\rightarrow} A_2$ and $A_2 \underset{\sim}{\rightarrow} A_1$, then the operators A_1 and A_2 are called *quasi-similar.*

We put, for any set $S \subset B(H_1, H_2)$,

$$\operatorname{ref} S = \bigcap_{x \in H_1} \{T \in B(H_1, H_2) : Tx \in \overline{Sx}\}.$$

If $\operatorname{ref} S = S$, then the set S is called *reflexive.*

We need a lemma from [17]; since the proof is omitted in [17], we will prove it here.

LEMMA 3.8. *If $B_1 \underset{\sim}{\rightarrow} A_1$, $A_2 \underset{\sim}{\rightarrow} B_2$, and the set $\operatorname{int}(B_1, B_2)$ is reflexive, then $\operatorname{int}(A_1, A_2)$ is also reflexive.*

PROOF. From the immediately verifiable inclusion

$$\operatorname{int}(A_2, B_2) \cdot \operatorname{int}(A_1, A_2) \cdot \operatorname{int}(B_1, A_1) \subset \operatorname{int}(B_1, B_2),$$

we conclude that for any $T_1 \in \operatorname{int}(B_1, A_1)$ and $T_2 \in \operatorname{int}(A_2, B_2)$ we have the inclusion

$$T_2 \operatorname{int}(A_1, A_2) T_1 \subset \operatorname{int}(B_1, B_2).$$

Consequently,

$$\operatorname{ref}(T_2 \operatorname{int}(A_1, A_2) T_1) \subset \operatorname{ref}(\operatorname{int}(B_1, B_2)) = \operatorname{int}(B_1, B_2).$$

Since

$$\overline{T_2 \operatorname{ref}(\operatorname{int}(A_1, A_2)) T_1 x} = \overline{T_2 \operatorname{int}(A_1, A_2) T_1 x}$$

for all $x \in H_{B_1}$ (H_{B_1} is the space on which B_1 acts), we have

$$T_2 \operatorname{ref}(\operatorname{int}(A_1, A_2)) T_1 \subset \operatorname{ref}(T_2 \operatorname{int}(A_1, A_2) T_1)$$

and consequently

$$T_2 \operatorname{ref}(\operatorname{int}(A_1, A_2)) T_1 \subset \operatorname{int}(B_1, B_2).$$

This means that $B_2T_2XT_1 = T_2XT_1B_1$ for every $X \in \operatorname{ref}(\operatorname{int}(A_1, A_2))$, whence $T_2A_2XT_1 = T_2XA_1T_1$. Choosing the operators T_1 and T_2 to be quasi-invertible, we find that $A_2X = XA_1$, i.e., $X \in \operatorname{int}(A_1, A_2)$. The lemma is proved.

COROLLARY 3.9. *Let $B_1 \rightarrow A \rightarrow B_2$. If the operator A is reductive and B_1 and B_2 are normal, then A is normal.*

PROOF. It follows easily from Fuglede's theorem that $\operatorname{int}(B_1, B_2)$ is a reflexive set of operators. By virtue of Lemma 3.8, $(A)' = \operatorname{int}(A, A)$ is a reflexive algebra. By Corollary 3.3, the algebra $(A)'$ is reductive. Like every reflexive reductive algebra, $(A)'$ is symmetric, i.e., A is a normal operator.

COROLLARY 3.10. *A reductive operator which is quasi-similar to a normal operator is normal.*

Corollaries 3.9 and 3.10 permit one to obtain other normality conditions for reductive operators. Thus, for example, from Corollary 3.10 and Theorem 2.5.3 of [20] there follows immediately the unitariness of every reductive contraction A for which $A^n x \not\rightarrow 0$ and $A^{*n} x \not\rightarrow 0$ for $x \neq 0$. We now apply Corollary 3.9 in order to generalize Suzuki's theorem [19] on the existence of invariant subspaces of operators of the form AB, where $A, B \geqslant 0$.

COROLLARY 3.11. *A reductive operator which can be represented as the product of two positive operators is normal.*

PROOF. Let A be a reductive operator and $A = B_1B_2$, where $B_1, B_2 \geqslant 0$. We note that $\operatorname{Ker} B_2 \in \operatorname{lat} A$; therefore, by virtue of the reductivity of A, $\overline{B_2H} \in \operatorname{lat} A$, whence $B_1B_2\overline{B_2H} \subset \overline{B_2H}$, i.e., $B_1\overline{B_2H} \subset \overline{B_2H}$. Noting that the operator $A^* = B_2B_1$ is also reductive, we obtain that $B_2\overline{B_1H} \subset \overline{B_1H}$. Thus, the subspace $H_0 = \overline{B_1H} \cap \overline{B_2H}$ is invariant with respect to B_1 and B_2, and hence the restriction of A to H_0 is again a reductive operator which can be represented as the product of two positve operators. Since $A|H_0^\perp = 0$, it is sufficient to consider the case when $H_0 = H$. In this case, as is easily verified, $B_1^{1/2}B_2B_1^{1/2} \rightarrow A \rightarrow B_2^{1/2}B_1B_2^{1/2}$, and the assertion to be proved follows from Corollary 3.10.

We remark that a stronger assertion holds: a reductive operator representable as the product of two positive operators is itself positive. This is inferred from Corollary 3.11 with the aid of the following general lemma: if the product of a selfadjoint operator A with a positive operator B is a normal operator, then A and B are commutable. The proof easily reduces to the case when the null space of the operator B is trivial; further, since $B^{1/2}(AB) = (B^{1/2}AB^{1/2})B^{1/2}$ and the operators AB and $B^{1/2}AB^{1/2}$ are normal, by Fuglede's theorem, $B^{1/2}(AB)^* = (B^{1/2}AB^{1/2})^*B^{1/2}$, i.e., $B^{1/2}BA = B^{1/2}AB$, whence $BA = AB$.

A second remark: the p[illegible] of Corollary 3.9 uses the reductivity of the commutant of the operator and not the reductivity of the operator itself; based on this, it is easy to establish for operators of the form AB, where $A, B \geqslant 0$, the existence of nontrivial hyperinvariant (i.e., invariant with respect to the commutant) subspaces.

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