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UNIMPROVABILITY OF ESTIMATES OF HÖLDER CONSTANTS FOR SOLUTIONS OF LINEAR ELLIPTIC EQUATIONS WITH MEASURABLE COEFFICIENTS

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ABSTRACT. The author considers the validity of an estimate in the norm of the Hölder spaces C^β for the solutions of linear elliptic equations $a_{ij}u_{x_ix_j} = 0$, where $\nu|l|^2 \le a_{ij}l_il_j \le \nu^{-1}|l|^2$ for all $l = (l_1, \dots, l_n) \in E_n$ ($n \ge 2$, $\nu = \text{const} > 0$). This estimate does not depend on the smoothness of the coefficients $a_{ij} = a_{ij}(x)$. It is known (MR **83c**:35059) that such an estimate holds for sufficiently small exponents $\beta \in (0,1)$ depending on n and ν. In this paper it is proved that this dependence is essential: for every $\beta_0 \in (0,1)$ one can exhibit a constant $\nu \in (0,1)$ and construct a sequence in E_3 of elliptic equations, of the indicated form with smooth coefficients, whose solutions converge uniformly in the unit ball to a function that does not belong to C^{β_0}.

Bibliography: 5 titles.

§1. Introduction. Formulation of the main result

Let E_n be Euclidean space of dimension $n \ge 2$. Consider in the unit ball $B_1 \subset E_n$ the general linear second-order elliptic equation

$$a_{ij}u_{x_ix_j} = 0, \tag{1.1}$$

where the coefficients $a_{ij} = a_{ij}(x)$ satisfy the conditions

$$a_{ij} = a_{ji}, \qquad \nu|l|^2 \le a_{ij}l_il_j \le \nu^{-1}|l|^2 \quad \text{for all } l = (l_1, \dots, l_n) \in E_n, \tag{1.2}$$

$\nu = \text{const} \in (0,1]$. Henceforth, it is assumed that repeated indices are summed over.

Let u be a bounded, sufficiently smooth solution of (1.1) in the ball B_1. From [1] and [2] we know that for a sufficiently small exponent $\beta = \beta(n,\nu) \in (0,1)$ in any concentric ball B_ρ of radius $\rho < 1$ we have the estimate

$$\langle u\rangle_{B_\rho}^{(\beta)} = \sup_{x \ne y \in B_\rho} \frac{|u(x) - u(y)|}{|x-y|^\beta} \le C \sup_{B_1} |u| \tag{1.3}$$

involving Hölder's constant, where the constant C depends only on n, ν, and ρ. Estimates of this sort, with constants β and C not depending on the smoothness of the coefficients a_{ij}, play a vital role in solvability questions for linear and nonlinear partial differential equations, and in the study of the qualitative properties of their solutions.

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The question arises whether (1.3) can be extended to the derivatives of the function u, i.e. whether the estimate

$$\sum_i \langle u_{x_i}\rangle_{B_\rho}^{(\beta_1)} \le C_1 \sup_{B_1} |u| \tag{1.4}$$

holds for $\rho < 1$ with some constants C_1, $\beta_1 > 0$ depending only on n, ν, and ρ. An affirmative answer to this question was given by Nirenberg [3] for $n = 2$ and arbitrary $\nu \in (0, 1]$ and by Cordes [4] for $n \ge 3$ and ν close to 1. For $n \ge 3$ and small ν the answer remained unknown.

REMARK 1.1. Estimate (1.3) easily follows from (1.4) for arbitrary $\beta \in (0, 1]$ with constant $C = C(\beta_1, C_1, \rho)$ whereas, in the constructions of [1] and [2], $\beta = \beta(n, \nu) \to 0$ as $\nu \to 0$.

The main result of this work is the following theorem.

THEOREM 1.1. *Let $0 < \beta_0 \le 1$, and let B_1 be the unit ball in E_3. Then there exists a constant $\nu_0 \in (0, 1)$, depending only on β_0, and for each $\varepsilon \in (0, 1)$ there exist functions a^ε_{ij}, $u^\varepsilon \in C^\infty(\overline{B}_1)$, $i, j = 1, 2, 3$, satisfying*

$$a^\varepsilon_{ij} u^\varepsilon_{x_i x_j} = 0 \quad \text{in } B_1; \tag{1.5}$$

$$a^\varepsilon_{ij} = a^\varepsilon_{ji}, \qquad \nu_0 |l|^2 \le a^\varepsilon_{ij} l_i l_j \le \nu_0^{-1} |l|^2 \quad \text{for all } l = (l_1, l_2, l_3) \in E_3; \tag{1.6}$$

$$\sup_{B_1} |u^\varepsilon| = 1, \qquad \lim_{\varepsilon \to 0+} \langle u^\varepsilon \rangle_{B_{1/2}}^{(\beta_0)} = +\infty. \tag{1.7}$$

Thus, for $n = 3$, $\rho = 1/2$, $\beta = \beta_0$, $0 < \nu \le \nu_0 = \nu_0(\beta_0)$, and any constant $C > 0$, estimate (1.3) cannot hold for all a_{ij}, $u \in C^\infty$ satisfying (1.1) and (1.2). Consequently, for (1.3) to hold in the case $0 < \nu \le \nu_0(\beta_0)$ it is necessary that $0 < \beta < \beta_0$, and because β_0 is arbitrary we have $\beta \to 0$ as $\nu \to 0$. From Theorem 1.1 and Remark 1.1 follows

COROLLARY 1.1. *Let $\nu_0 \in (0, 1)$ be the constant from Theorem 1.1 assumed at $\beta_0 = 1$. Then for any constant $\beta_1 \in (0, 1)$ the functions u^ε satisfy*

$$\lim_{\varepsilon \to 0+} \sum_i \langle u^\varepsilon_{x_i} \rangle_{B_{1/2}}^{(\beta_1)} = +\infty. \tag{a}$$

This means that, for $n = 3$, $0 < \nu \le \nu_0$, $0 < \beta_1 < 1$, and $\rho = 1/2$, estimate (1.4) does not hold for any constant C_1. Therefore, for $n \ge 3$, the conditions of Cordes type [4] on ν are essential for this estimate to hold.

The whole of this paper, in essence, is devoted to the proof of Theorem 1.1, and consists of eight sections. In §2, the proof is reduced to the search for functions $a_{ij}(x)$ and $u(x) = |x|^\alpha v(x)$, where $\alpha = \beta_0/2$, which satisfy (1.1) and (1.2) on the sphere $S = \partial B_1$ and obey certain additional conditions ensuring the smooth continuation of these functions to the entire ball B_1, preserving relations (1.1) and (1.2). Equality (1.1), together with (1.2), is in a sense a "saddle-shape" condition on the function u. In §3 another "saddle-shape" condition is given, which allows the functions a_{ij} to be ignored. The problem is thereby reduced to that of finding a function v satisfying conditions 2.1)–2.3) and 3.1). To elucidate the main points in the construction of this function, a simpler two-dimensional problem is considered in §4, in which the desired function is constructed according to the same plan as the function v in the main problem. In §5 condition 3.1) is transformed into condition 5.1), which is comparatively easy to check. In the next two sections the functions v_1 and v_2 are constructed, each of which satisfies 5.1) on its "own" portion of the sphere S. Finally, in §8, our desired function v is constructed by "patching" v_1 and v_2.

Throughout this paper the following notation is used. $E = E_3 = \{x = (x_1, x_2, x_3): -\infty < x_i < \infty\}$ is three-dimensional Euclidean space, $\dot{E} = E\backslash\{0\}$, $B_\rho(x) = \{y \in E: |y - x| < \rho\}$ is the ball of radius $\rho > 0$ with center at the point $x \in E$, $B_\rho = B_\rho(0)$, and $S = \{x \in E: |x| = 1\}$ is the unit sphere in E with center at 0. Depending on the context, $\partial\Gamma$ is the boundary of the set Γ on the sphere S (for $\Gamma \subset S$) or in E, and $\overline{\Gamma} = \Gamma \cup \partial\Gamma$.

As a rule, all the functions we consider are smooth (class C^∞) throughout their domains except, perhaps, at the origin in E. We denote by u_{x_i}, $u_{x_ix_j}$, u_φ, etc., the partial derivatives of u with respect to the corresponding variables; $u_{(l)} = u_{x_i}l_i$ and $u_{(l)(l)} = u_{x_ix_j}l_il_j$ are the first and second derivatives of $u(x)$ with respect to x in the direction of the vector $l = (l_1, l_2, l_3) \in E$. Only in §4 are the analogous objects in the plane E_2 considered.

In order not to overload the formulations of the assertions, some of the assumptions will be expressed immediately prior to the assertions. C and ε, generally, denote various positive constants; indices will be provided only to those constants which are used again later on.

The results presented here have been discussed on a number of occasions with N. V. Krylov; in the spring of 1985 they were reported at the All-Union Seminar on Nonlinear Problems of Mathematical Physics, and at the seminar led by O. A. Ladyzhenskaya and N. N. Ural′tseva in Leningrad. The author wishes to thank all those who took part in the discussion of the results.

§2. Information concerning the problem on the sphere

Let the constant $\beta_0 \in (0, 1]$ be given. Put $\alpha = \beta_0/2$. Let the function $v(x)$ defined on $\dot{E} = E\backslash\{0\}$ satisfy the following conditions:

2.1) $v \in C^\infty(\dot{E})$, $\sup|v| = 1$.

2.2) $v(\lambda x) = v(x)$ for all $\lambda > 0$, $x \neq 0$.

2.3) $v(\xi) \equiv -v(x)$, where $\xi = \xi(x) = \xi(x_1, x_2, x_3) = (x_2, x_1, -x_3)$.

2.4) There exist functions $a_{ij} \in C^\infty(S)$, $i, j = 1, 2, 3$, which satisfy, together with the function $u(x) = |x|^\alpha v(x)$, the relations (1.1) and (1.2) on S for some constant $\nu \in (0, 1]$ depending only on α.

From the equality $u(\lambda x) = |\lambda x|^\alpha v(\lambda x) = \lambda^\alpha |x|^\alpha v(x) = \lambda^\alpha u(x)$ it follows that

$$u_{x_ix_j}(\lambda x) = \lambda^{\alpha-2} u_{x_ix_j}(x) \quad \text{for all } \lambda > 0,\ x \neq 0. \tag{2.1}$$

Consequently, extending a_{ij} from S to $\dot{E}$ by the formula $a_{ij}(\lambda x) = a_{ij}(x)$, we find that the functions a_{ij}, $u \in C^\infty(\dot{E})$ also satisfy (1.1) and (1.2) on $\dot{E}$.

We shall show, relying on 2.1)–2.4), that the functions a_{ij} and u can be "adjusted" near the origin so that the resulting functions a_{ij}^ε and u^ε have all the required properties (1.5)–(1.7) with constant $\nu_0 = \nu$ from 2.4). For that, we write the linear transformation $\xi = \xi(x)$ from 2.3) in coordinates: $\xi_i = \xi_{ij}x_j$, where $\xi_{12} = \xi_{21} = -\xi_{33} = 1$ and $\xi_{ij} = 0$ for the remaining $i, j = 1, 2, 3$. From this notation easily follows

LEMMA 2.1. *Let $\xi = \xi(x)$, and let the functions $f, \varphi \in C^2(\overline{B}_1)$ be connected by the identity $\varphi(\xi) \equiv -f(x)$. Then $a_{ij}(x)f_{x_ix_j}(x) = 0$ if and only if $\bar{a}_{km}(\xi)\varphi_{\xi_k\xi_m}(\xi) = 0$, where $\bar{a}_{km}(\xi) = a_{ij}(x)\xi_{ki}\xi_{mj}$.*

In case $f = u$ we have $\varphi(\xi) = -u(x) = -|x|^\alpha v(x) = |\xi|^\alpha v(\xi) = u(\xi)$. By Lemma 2.1, equality (1.1) remains true on changing the matrices $a(x) = \|a_{ij}(x)\|$ to $\bar{a}(x) = \|\bar{a}_{ij}(x)\|$, and also to $A(x) = (a(x) + \bar{a}(x))/2$. From the identity $\xi(\xi(x)) \equiv x$ is easily follows that on passing twice from a to $\bar{a}$ the matrices $a(x)$ are unchanged: $\bar{\bar{a}}_{ij}(x) \equiv a_{ij}(x)$. Consequently,

$$\overline{A}(x) = \tfrac{1}{2}(\bar{a}(x) + \bar{\bar{a}}(x)) = \tfrac{1}{2}(\bar{a}(x) + a(x)) = A(x).$$

Changing $a(x)$ to $A(x)$, if necessary, we may assume that $\bar{a}_{ij}(x) = a_{ij}(x)$ in $\overline{B}_1 \backslash \{0\}$.

Next, for $\varepsilon \in (0,1)$ we introduce the functions $\chi^\varepsilon \in C^\infty([0,1])$ which satisfy the conditions $0 \le \chi^\varepsilon \le 1$, $\chi^\varepsilon = 1$ on $[0, \varepsilon/2]$, and $\chi^\varepsilon = 0$ on $[\varepsilon, 1]$. Then the functions $a^\varepsilon_{ij}(x) = \chi^\varepsilon(|x|)\delta_{ij} + (1 - \chi^\varepsilon(|x|))a_{ij}(x)$, where δ_{ij} is the Kronecker symbol, belong to $C^\infty(\overline{B}_1)$. In addition, from the constraints (1.2) on a_{ij}, (1.6) follows with constant $\nu_0 = \nu$ from (1.2).

As we know, there exists a unique solution $u^\varepsilon \in C^\infty(\overline{B}_1)$ of the Dirichlet problem for equation (1.5) with boundary condition $u^\varepsilon = v$ on $S = \partial B_1$. From the equality $\bar{a} = a$ it follows that $\overline{a}^\varepsilon_{ij}(x) = a^\varepsilon_{ij}(x)$. Applying Lemma 2.1 to the case $\varphi = u^\varepsilon$, $a_{ij} = a^\varepsilon_{ij}$, we find that the function $f(x) = -u^\varepsilon(\xi(x))$ is also a solution of this problem. By uniqueness of the solution, $u^\varepsilon(x) \equiv -u^\varepsilon(\xi(x))$ and, in particular, $u^\varepsilon(0) = 0$.

It remains for us to prove (1.7). The first of these two relations follows from the classical maximum principle: $\sup_{B_1} |u^\varepsilon| = \sup_S |v| = 1$. For the proof of the second, we note that the functions $u = u^\varepsilon$ obey (1.3) uniformly with respect to $\varepsilon \in (0, 1/2)$. Taking the equality $u^\varepsilon(0) = 0$ into account, this gives

$$\sup_{B_\varepsilon} |u^\varepsilon| \le \varepsilon^\beta \sup_{x \in B_\varepsilon} \frac{|u^\varepsilon(x) - u^\varepsilon(0)|}{|x|^\beta} \le \varepsilon^\beta \langle u^\varepsilon \rangle^{(\beta)}_{B_{1/2}} \le C\varepsilon^\beta \sup_{B_1} |u^\varepsilon| = C\varepsilon^\beta,$$

where the constants $C, \beta > 0$ depend only on ν. Since $a_{ij}(u^\varepsilon - u)_{x_i x_j} = 0$ in $B_1 \backslash B_\varepsilon$ and $u^\varepsilon - u = 0$ on ∂B_1, then by the maximum principle

$$\begin{aligned}\sup_{B_1} |u^\varepsilon - u| &= \sup_{B_\varepsilon} |u^\varepsilon - u| \le \sup_{B_\varepsilon} |u^\varepsilon| + \sup_{B_\varepsilon} |u| \\ &\le C\varepsilon^\beta + \varepsilon^\alpha \to 0 \quad \text{as } \varepsilon \to 0+.\end{aligned}$$

Finally, since $\alpha = \beta_0/2 < \beta_0$, we have $\langle u \rangle^{(\beta_0)}_{B_{1/2}} = \infty$, and from the convergence of u^ε to u it follows that $\lim_{\varepsilon \to 0+} \langle u^\varepsilon \rangle^{(\beta_0)}_{B_{1/2}} = \infty$. Relations (1.5)–(1.7) are proved.

§3. A sufficient condition for the function u to be "saddle-shaped"

In §2 we reduced the proof of Theorem 1.1 to the search for a function $v(x)$ satisfying conditions 2.1)–2.4), given the constant $\alpha \in (0,1)$. In this section we show that condition 2.4) can be replaced by the following condition:

3.1) For all $x \in S$ the function $u(x) = |x|^\alpha v(x)$ satisfies

$$\sup_{|l|=1} u_{(l)(l)}(x) \ge 3\nu_1 > 0, \qquad \inf_{|l|=1} u_{(l)(l)}(x) \le -3\nu_1 < 0, \tag{3.1}$$

where $\nu_1 > 0$ is a constant.

The possibility of replacing 2.4) by 3.1) follows immediately from the next Lemma 3.1. Roughly speaking, this lemma asserts that the "saddle-shape" conditions (1.1) and (1.2) on the function u can be replaced by (3.1).

LEMMA 3.1. *Let the smooth function u be defined in a neighborhood of the sphere S. For all $x \in S$ let inequalities* (3.1) *hold for some constant $\nu_1 > 0$. Then one can select functions $a_{ij} \in C^\infty(S)$ satisfying, together with the function u, conditions* (1.1) *and* (1.2) *on S with constant $\nu = \min\{1/2, \nu_1, (\sup_S |\Delta u|)^{-1}\nu_1\} > 0$, where Δ is the Laplace operator.*

PROOF. By the choice of ν we have $|\nu \Delta u| \le \nu_1$ on S. From the first inequality in (3.1) it follows that for any point $x \in S$ there is a vector $l = l(x)$, $|l| = 1$, for which $\nu\Delta u(x) + u_{(l)(l)}(x) \ge 2\nu_1$. In view of the smoothness of u, there is a constant $\varepsilon > 0$ independent of $x \in S$ such that

$$\nu \Delta u + u_{(l)(l)} \ge \nu_1 \quad \text{in the ball } B_\varepsilon(x) \text{ for all } x \in S,\ l = l(x). \tag{3.2}$$

For this ε we choose a partition of unity on S, i.e. a finite collection of smooth functions χ^k of compact support in E, $k = 1, 2, \dots$, such that $0 \le \chi^k \le 1$, $\sum_k \chi^k = 1$ on S, and for any k the support of χ^k is contained in the ball $B_\varepsilon(x^k)$ with center at a point $x^k \in S$. We choose $l^k = (l_1^k, l_2^k, l_3^k) = l(x^k)$ according to (3.2) and set

$$a_{ij}^{(1)}(x) = \nu\delta_{ij} + \chi^k(x) l_i^k l_j^k \in C^\infty(S).$$

For these functions, for all $l = (l_1, l_2, l_3) \in E$ and $x \in S$, we have

$$\nu|l|^2 \le a_{ij}^{(1)} l_i l_j = \nu|l|^2 + \chi^k(x)(l_i^k l_i)^2 \le (\nu + 1)|l|^2 \le \nu^{-1}|l|^2,$$

i.e. (1.2) is satisfied. In addition, by the choice of l^k and χ^k the smooth function

$$f_1 \equiv a_{ij}^{(1)} u_{x_i x_j} = (\nu\Delta u + u_{(l^k)(l^k)})\chi^k \ge \nu_1 \ge \nu \quad \text{on } S.$$

Analogously, from the second inequality in (3.1) it follows that for some functions $a_{ij}^{(2)} \in C^\infty(S)$ satisfying (1.2), the smooth function $f_2 = -a_{ij}^{(2)} u_{x_i x_j} \ge \nu$ on S. Finally, putting $a_{ij} = (f_1 a_{ij}^{(2)} + f_2 a_{ij}^{(1)})/(f_1 + f_2) \in C^\infty(S)$, we complete the proof of the lemma.

§4. The two-dimensional problem

To clarify the course of the argument later on, we consider in this section a problem on the plane E_2. The solution of this problem contains in simplified form some features of the solution of the main problem: find a function v on $\dot{E}$ satisfying conditions 2.1)–2.3) and 3.1). For points $x = (x_1, x_2) \in \dot{E}_2 = E_2 \backslash \{0\}$, we introduce polar coordinates r, φ: $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$. Note that condition (2.2) for the function v on $\dot{E}_2$ means simply that v is independent of r: $v = v(\varphi)$.

Let us proceed to the formulation of the problem. Let the constants $\alpha \in (0,1)$ and $\varphi_0 \in (0, \pi)$ be given. We must find a function $v(\varphi)$ defined on $[-\varphi_0, \varphi_0]$ and satisfying the following conditions:

4.1) $v \in C^\infty([-\varphi_0, \varphi_0])$.

4.2) $0 < v(\varphi) < 1$ for $0 < |\varphi| < \varphi_0$, $v(0) = 1$, $v(\pm\varphi_0) = 0$.

4.3) The function $u(x) = r^\alpha v(\varphi)$ on the arc $\gamma = \{x \in E_2 : r = 1, |\varphi| < \varphi_0\}$ satisfies (3.1) for some constant $\nu_1 > 0$.

The following lemma permits inequalities (3.1) in condition 4.3) to be changed to the single inequality

$$\alpha v v'' + (1-\alpha)(v')^2 + \alpha^2 v^2 \ge \nu_2 = \text{const} > 0, \tag{4.1}$$

where v' and v'' are derivatives of the function $v = v(\varphi)$.

LEMMA 4.1. *Let $0 < \alpha < 1$, and let the function $v(\varphi)$ be defined and twice continuously differentiable in a neighborhood of the point φ_*. At the point $x_* = (\cos\varphi_*, \sin\varphi_*) \in E_2$, let the following inequality hold for the function $u(x) = u(x_1, x_2) = r^\alpha v(\varphi)$:*

$$\left(\sum_{i,j} u_{x_i x_j}^2 \right)^{1/2} \le C_0 = \text{const}. \tag{4.2}$$

Then at the point x_* relations (3.1) *and* (4.1) *are equivalent in the following sense.* a) (3.1) *implies* (4.1) *with constant* $\nu_2 = 9\nu_1^2/(1-\alpha) > 0$. b) (4.1) *implies* (3.1) *with constant* $\nu_1 = (1-\alpha)\nu_2/3C_0 > 0$.

PROOF. Note that the left-hand sides in (3.1) are the eigenvalues λ_1 and λ_2 ($\lambda_1 > 0 > \lambda_2$) of the matrix $\|u_{x_i x_j}(x_*)\|$. Therefore, from (3.1) it follows that

$$(u_{x_1 x_1} u_{x_2 x_2} - u_{x_1 x_2}^2)(x_*) = \det(u_{x_i x_j}(x_*)) = \lambda_1 \lambda_2 \le -(3\nu_1)^2. \tag{4.3}$$

Let us find an expression for the left-hand side of (4.3) in terms of the values of v, v', and v'' at the point φ_*. Without loss of generality, we shall assume that $\varphi_* = 0$ and $x_* = (1,0)$. Since v does not depend on $r = |x|$, at x_* we have

$$v_{x_1} = v_{x_1x_1} = 0, \quad v_{x_2} = -v_{x_1x_2} = v', \quad v_{x_2x_2} = v'', \quad u_{x_1x_1} = \alpha(\alpha-1)v,$$
$$u_{x_1x_2} = (\alpha-1)v', \quad u_{x_2x_2} = v'' + \alpha v;$$
$$u_{x_1x_1}u_{x_2x_2} - u^2_{x_1x_2} = (\alpha-1)[\alpha v v'' + (1-\alpha)(v')^2 + \alpha^2 v^2].$$

From the last equality and (4.3) assertion a) follows.

To prove b) we carry out the preceding argument in reverse. As a result we find that (4.1) entails $\lambda_1\lambda_2 \le (\alpha-1)\nu_2 < 0$. In addition, by (4.2),

$$|\lambda_{1,2}| \le (\lambda_1^2 + \lambda_2^2)^{1/2} = \left(\sum_{i,j} u^2_{x_ix_j}\right)^{1/2} \le C_0,$$

which leads to the inequalities

$$\lambda_1 = \frac{-\lambda_1\lambda_2}{-\lambda_2} \ge \frac{(1-\alpha)\nu_2}{C_0} > 0, \qquad \lambda_2 = \frac{\lambda_1\lambda_2}{\lambda_1} \le \frac{(\alpha-1)\nu_2}{C_0} < 0,$$

equivalent to (3.1) with constant $\nu_1 = (1-\alpha)\nu_2/3C_0$. The lemma is proved.

REMARK 4.1. Under condition 4.1), estimate (4.2) holds on the arc γ for some constant C_0. Therefore, from Lemma 4.1 it follows that, in 4.1)–4.3), condition 4.3) can be changed to

4.3′) The function $v(\varphi)$ for $|\varphi| < \varphi_0$ satisfies (4.1) with some constant $\nu_2 > 0$.

The answer to the existence question for the solution of the problem posed at the beginning of this section is given by the following.

LEMMA 4.2. a) *If $0 < \varphi_0 \le \pi/2$, then there exist no functions $v(\varphi)$ satisfying* 4.1)-4.3).

b) *If $\pi/2 < \varphi_0 < \pi$, then such functions $v(\varphi)$ exist.*

PROOF. By the previous remark, condition 4.3′) can be considered in place of 4.3). Let us make the substitution $v = w^\alpha$. Inequality (4.1) then assumes the form

$$\alpha^2 w^{2\alpha-1}(w'' + w) \ge \nu_2 > 0. \tag{4.1′}$$

Hence, from 4.1) and 4.2) it follows that the function w is continuous on $[-\varphi_0, \varphi_0]$, smooth in $(-\varphi_0, \varphi_0)$, and in addition

$$w > 0, \quad w'' + w > 0 \text{ on } (-\varphi_0, \varphi_0), \quad w(\pm\varphi_0) = 0. \tag{4.4}$$

It is well known that relations (4.4) are only possible in the case when the length of the interval $(-\varphi_0, \varphi_0)$ is greater than π, i.e. $\varphi_0 > \pi/2$. With this, assertion a) is proved.

Let us proceed to the proof of b). Let $\pi/2 < \varphi_0 < \pi$. We need to construct a function $v = w^\alpha$ satisfying 4.1), 4.2) and 4.3′). We arbitrarily choose $q \in (\pi/2\varphi_0, 1)$ and put $v_1(\varphi) = (\varphi_0 - |\varphi|)/\pi = w_1^\alpha(\varphi)$ for $0 < |\varphi| \le \varphi_0$ and $v_2(\varphi) = w_2^\alpha(\varphi)$ for $|\varphi| \le \pi/2q$, where $w_2(\varphi) = \cos q\varphi$. Further, we choose a sufficiently small $h \in (0, 1/4)$ for which

$$2h \le w_1(\varphi) \le 1 - 2h \quad \text{for } 0 < |\varphi| \le \pi/2q \tag{4.5}$$

and define φ_1 and φ_2 from the relations

$$w_2(\varphi_1) = h, \quad w_2(\varphi_2) = 1 - h, \qquad \varphi_0 > \pi/2q > \varphi_1 > \varphi_2 > 0. \tag{4.6}$$

The desired function $v(\varphi)$ will be obtained by a smooth "patching" of v_1 and v_2. Since $(v_1'(\varphi))^2 = \pi^{-2}$ and $v_1''(\varphi) = 0$ for $\varphi_0 > |\varphi| > 0$, it follows that for these φ (4.1) and

(4.1′) hold for $v = v_1$ and $w = w_1$ with constant $\nu_2 = \pi^{-2}(1-\alpha) > 0$. In addition, since $w_2(\varphi) \geq h$ and

$$w_2^{2\alpha-1}(w_2'' + w_2)(\varphi) = (1-q^2)w_2^{2\alpha}(\varphi) \geq (1-q^2)h^{2\alpha}$$

for $|\varphi| \leq \varphi_1$, it follows that for $|\varphi| \leq \varphi_1$ (4.1) and (4.1′) hold for $v = v_2$ and $w = w_2$ with constant $\nu_2 = \alpha^2(1-q^2)h^{2\alpha} > 0$. We put $w(\varphi) = w_1(\varphi)$ for $\varphi_0 \geq |\varphi| \geq \varphi_1$, and $w(\varphi) = w_2(\varphi)$ for $|\varphi| \leq \varphi_2$.

It remains for us to extend the definition of $w(\varphi)$ to $\varphi_1 > |\varphi| > \varphi_2$ in such a way that the function $v = w^\alpha$ satisfies conditions 4.1), 4.2), and 4.3′). From (4.5), (4.6), and (4.1′) it follows that the functions $w = w_{1,2}$ satisfy

$$h \leq w(\varphi) \leq 1-h, \quad (w'' + w)(\varphi) \geq \nu_3 > 0 \quad \text{for } \varphi_1 \geq |\varphi| \geq \varphi_2 \tag{4.7}$$

for some constant ν_3. It is easy to see that it now suffices for us to construct a smooth function $w(\varphi)$ on $\varphi_1 \geq |\varphi| \geq \varphi_2$, which satisfies (4.7) and is such that $w(\varphi) = w_1(\varphi)$ for $|\varphi|$ close to φ_1, and $w(\varphi) = w_2(\varphi)$ for $|\varphi|$ close to φ_2. Such a function can be constructed using a smoothing of the function $\bar{w}(\varphi) = \max\{w_1(\varphi), w_2(\varphi)\}$. The behavior we require of $w(\varphi)$ for $|\varphi|$ near φ_1 or φ_2 is ensured by (4.5) and (4.6). We omit the details of the construction of w here, since they will be carried out in the somewhat more general situation in §8. We merely note that the inequality $w'' + w \geq \nu_3 > 0$ is obtained due to the fact that, roughly speaking, $\bar{w}'' = +\infty$ at points where $\bar{w}$ is not smooth, while at the remaining points $\bar{w}'' + \bar{w}$ is equal to $w_1'' + w_1$ or $w_2'' + w_2$. Thus, except for some of the details of the construction of the function $w(\varphi)$ for $\varphi_1 > |\varphi| > \varphi_2$, the lemma is proved.

COROLLARY 4.1. *There exist no smooth functions $v = v(\varphi)$ satisfying inequality* (4.1) *for some constant $\nu_2 > 0$ on the whole unit circle in E_2 and assuming both positive and negative values.*

PROOF. Assume such a function v exists. Then at least one of the two sets

$$\{x \in E_2 \colon r = 1,\ v(\varphi) > 0\} \quad \text{or} \quad \{x \in E_2 \colon r = 1,\ v(\varphi) < 0\} \tag{4.8}$$

contains an arc of length $2\varphi_0 < \pi$ at whose endpoints $v = 0$. Without loss of generality it can be assumed that this arc has the form $\gamma = \{x \in E_2 \colon r = 1, |\varphi| < \varphi_0\}$ and $v > 0$ on it; the general case reduces to this one by rotating the coordinate axes in E_2 and, if necessary, changing v to $-v$. Thus, we have the relations $v(\varphi) > 0$ for $|\varphi| < \varphi_0$ and $v(\pm\varphi_0) = 0$, contained in condition 4.2). Observe now that in the proof of Lemma 4.2a) from 4.2) only these relations were used. From here it is easy to obtain the required contradiction. The corollary is proved.

To conclude this section, let us point out the analogy between the constructions carried out here and the constructions in the rest of this paper. In proving the preceding corollary, we have seen that it is impossible for inequality (4.1) to hold simultaneously on two disjoint subsets (4.8) of the unit circle in E_2 if they are both nonempty. On the two-dimensional sphere $S \subset E = E_3$ such a possibility does exist. In contrast to the unit circle in E_2, S can be divided into two disjoint subsets, each of which would contain great circle arcs of length greater than π. Indeed, an example of such are the sets $S^+ = \{x \in S \colon v(x) > 0\}$ and $S^- = (x \in S \colon v(x) < 0\}$ for the function $v(x)$ in the final construction of §8. In preparation for the construction of $v(x)$, the analog of (4.1), relation (5.1), is derived in §5. The latter relation, in contrast to (4.1), permits some "freedom" in the choice of direction in E along which the derivatives of v are to be taken. On a considerable portion of S, these directions exactly correspond to the directions of arcs of length greater than π lying in S^+ or S^-. The construction, per se, of the desired function v, as well as of the auxiliary functions v_1 and v_2 in §§6 and 7, are in large part similar to the construction of the functions v, v_1, and v_2 in the proof of Lemma 4.2b).

§5. A differential inequality for the function v

By virtue of the considerations of §§2 and 3, it remains for us to construct a function $v(x)$ satisfying conditions 2.1)–2.3) and 3.1) with a given constant $\alpha \in (0,1)$. We now show that condition 3.1) can be changed to the condition

5.1) For all $x \in S$ the inequality

$$\sup_{l\in T(x)} [avv_{(l)(l)} + (1-\alpha)v_l^{(2)} + \alpha^2 v^2](x) \geq \nu_2 > 0 \tag{5.1}$$

holds for some constant ν_2, where $T(x) = \{l = (l_1, l_2, l_3) \in E\colon |l| = 1,\ l_i x_i = 0\}$ is the collection of all unit vectors in E that are tangent to S at x.

LEMMA 5.1. *Conditions* 2.1), 2.2) *and* 5.1) *entail condition* 3.1) *with constant* $\nu_1 = (1-\alpha)\nu_2/3C_0 > 0$, *where* $C_0 = \sup_S(\sum_{i,j} u^2_{x_i x_j})^{1/2}$ *and* $u(x) = |x|^\alpha v(x)$.

PROOF. Fix $x_0 \in S$. Performing an orthogonal transformation in E, if necessary, we may assume that $x_0 = (1,0,0)$ and that the supremum in (5.1) is achieved at $l = (0,1,0)$. Consider the functions $v(x)$ and $u(x) = |x|^\alpha v(x)$ on the plane $E_2 = \{x = (x_1,x_2,x_3) \in E\colon x_3 = 0\}$ as functions of the two variables x_1 and x_2. Introducing polar coordinates as in §4 ($x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$), we conclude from condition 2.2) that v depends only on φ and that at the point x_0, for $l = (0,1,0)$,

$$v_{(l)} = v_\varphi, \qquad v_{(l)(l)} = v_{\varphi\varphi}. \tag{5.2}$$

Consequently, inequality (5.1) at x_0 assumes the form of (4.1), and by Lemma 4.1b) inequalities (3.1) hold, even if the supremum and infimum there are taken over the set $\{l = (l_1,l_2,l_3) \in E\colon |l| = 1,\ l_3 = 0\} \subset \{l \in E\colon |l| = 1\}$. The lemma is proved.

Analogously to §4, on substituting $v = w^\alpha$, inequality (5.1) on the set $S^+ = \{x \in S\colon v(x) > 0\}$ turns into

$$\alpha^2 w^{2\alpha-1} \sup_{l\in T(x)} [w_{(l)(l)} + w](x) \geq \nu_2 > 0. \tag{5.1$'$}$$

The next lemma follows immediately from the equivalence of (5.1) and (5.1′) on S^+.

LEMMA 5.2. *In a neighborhood of a subset* $\Gamma \subset S$, *let the function* $v = w^\alpha$ *be smooth and satisfy* 2.2), *and, in addition, on* Γ *let*

$$1 \geq w \geq h = \text{const} > 0. \tag{5.3}$$

Then on Γ *relations* (5.1) *and*

$$\sup_{l\in T(x)} [w_{(l)(l)} + w](x) \geq \nu_3 > 0 \tag{5.4}$$

are equivalent in the following sense.

a) (5.1) *entails* (5.4) *with constant* $\nu_3 = \alpha^{-2}\nu_2 \min\{1, h^{1-2\alpha}\} > 0$.

b) (5.4) *entails* (5.1) *with constant* $\nu_2 = \alpha^2\nu_3 \min\{1, h^{2\alpha-1}\} > 0$.

In the next two sections we construct the functions v_1 and v_2, which are defined and have our required form on different overlapping subsets of $\dot{E}$. "Patching" v_1 and v_2 in §8, we obtain the desired function v satisfying all the conditions 2.1)–2.3) and 5.1).

§6. Construction of the function v_1

On the set $E_{(1)} = \{x = (x_1,x_2,x_3) \in E\colon x_1^2 + x_2^2 \neq 0\}$, we introduce spherical coordinates r, s, t ($r > 0$, $-\pi < s \leq \pi$, $|t| < \pi/2$): $x_1 = r\cos s\cos t$, $x_2 = r\sin s\cos t$, $x_3 = r\sin t$.

LEMMA 6.1. *The function v_1 defined on $E_{(1)}$ by the formula*

$$v_1(x) = \frac{1}{\pi}\arcsin\frac{x_3}{|x|} - \frac{x_1^2 - x_2^2}{4(x_1^2 + x_2^2)} = \frac{1}{\pi}t - \frac{1}{4}\cos 2s,$$

belongs to $C^\infty(E_{(1)})$ and everywhere in $E_{(1)}$ satisfies 2.2), 2.3), *and the inequality $|v_1| < 3/4$. In addition, for all $x \in S \cap E_{(1)}$,* (5.1) *holds with constant $\nu_2 = \pi^{-2}(1-\alpha) > 0$.*

It suffices to carry out the proof only for the last assertion in the lemma, since the remaining assertions are obvious. Fix $y = (y_1, y_2, y_3) \in S \cap E_{(1)}$. On the half-plane

$$\sigma = \{x \in E : r > 0,\ s = s_0 = \text{const},\ |t| < \pi/2\},$$

passing through the point y, v_1 is a linear function of one variable t. In addition, t is the angular parameter of the semicircle $S \cap \sigma$.

Pick a vector $l \in T(y)$ tangent to the semicircle $S \cap \sigma$ at the point y, i.e.

$$l = (-y_1y_3/\rho, -y_2y_3/\rho, \rho), \quad \text{where } \rho = (y_1^2 + y_2^2)^{1/2} > 0.$$

Then, analogously to (5.2), at the point y we have $v_{1(l)} = v_{1t} = 1/\pi$ and $v_{1(l)(l)} = v_{1tt} = 0$, whence follows the desired inequality (5.1) for $x = y$. Since y is an arbitrary point in $S \cap E_{(1)}$, the lemma is proved.

Let us further introduce certain auxiliary objects connected with the function v_1. The curve $S^0 = \{x \in S : v_1(x) = 0\}$ on the sphere S divides it into two parts S^+ and S^-, containing the corresponding "poles" $P^+ = (0,0,1)$ and $P^- = (0,0,-1)$. The function v_1 is defined everywhere on $S\backslash\{P^+, P^-\}$, where $v_1 > 0$ on $S^+\backslash\{P^+\}$ and $v_1 < 0$ on $S^-\backslash\{P^-\}$. It is easy to see that for sufficiently small $h_0 > 0$

$$S_1 \equiv \{x = (x_1, x_2, x_3) \in S : |x_1| \le h_0,\ x_3 \ge -h_0\} \subset S^+. \tag{6.1}$$

We henceforth fix such a constant h_0.

§7. Construction of the function v_2

On $\{x = (x_1, x_2, x_3) \in E : x_2^2 + x_3^2 \ne 0\}$ we now pass to spherical coordinates r, φ, θ ($r > 0$, $-\pi < \varphi \le \pi$, $|\theta| < \pi/2$):

$$x_1 = r\sin\theta, \quad x_2 = r\cos\theta\sin\varphi, \quad x_3 = r\cos\theta\cos\varphi. \tag{7.1}$$

Fix constants $p > 1$ and $q \in (0,1)$ so that $S_2 \equiv \{x \in S : |\varphi| < \pi/2q,\ |\theta| < \pi/2p\} \subset S_1$, the set from (6.1). Denote $E_{(2)} = \{x \in E : r > 0,\ |\varphi| < \pi/2q,\ |\theta| < \pi/2p\}$, so that $S_2 = S \cap E_{(2)}$. We set

$$v_2 = w_2^\alpha \quad \text{on } E_{(2)}, \text{ where } w_2 = \cos p\theta\cos q\varphi \in C^\infty(E_{(2)}). \tag{7.2}$$

LEMMA 7.1. *Let the constant $h \in (0,1)$ be given. Then the function $w = w_2$ given by* (7.2) *in spherical coordinates* (7.1) *satisfies inequality* (5.4) *on the set $\Gamma = (x \in S_2 : w_2(x) \ge h\}$ with constant $\nu_3 = (1-q^2)h > 0$.*

PROOF. Fix $y = (y_1, y_2, y_3) \in \Gamma$ and choose $l = (0, y_3/\rho, -y_2/\rho) \in T(y)$, where $\rho = (y_2^2 + y_3^2)^{1/2} > 0$. We will show that for this choice of l at the point y the equality

$$w_{(l)(l)} = \frac{1}{\cos^2\theta}w_{\varphi\varphi} - \tan\theta w_\theta \tag{7.3}$$

holds for any smooth, positively homogeneous function w. From this equality for $w = w_2 = \cos p\theta\cos q\varphi$ the lemma follows, since y is an arbitrary point of Γ and at it we have

$$w_{(l)(l)} + w = (1 - q^2/\cos^2\theta + p\tan\theta\tan p\theta)w \ge (1-q^2)w \ge (1-q^2)h.$$

To prove (7.3) we consider w as a function of the two variables x_2 and x_3 on the plane $\pi = \{x = (x_1, x_2, x_3) \in E\colon x_1 = y_1\}$. The spherical coordinates (7.1) correspond to polar coordinates on π: $x_2 = \rho\sin\varphi$, $x_3 = \rho\cos\varphi$, where $\rho = (x_2^2 + x_3^2)^{1/2} = r\cos\theta$.

Using the expression for the Laplace operator in polar coordinates ρ, φ, at the point y we have

$$w_{(l)(l)} = w_{x_2x_2} + w_{x_3x_3} - w_{\rho\rho} = \frac{1}{\rho^2}w_{\varphi\varphi} + \frac{1}{\rho}w_\rho = \frac{1}{\cos^2\theta}w_{\varphi\varphi} + \frac{1}{\cos\theta}w_\rho. \tag{7.4}$$

Here w_ρ and $w_{\rho\rho}$ are derivatives of w in the radial direction in the plane π. This direction makes an angle θ with the radial direction in E, along which the function w does not change: $w_r = 0$. Hence, resolving the unit vector in the radial direction in the plane π into its normal and tangential components to S at y, we easily obtain $w_\rho = -\sin\theta w_\theta$, which together with (7.4) gives (7.3). The lemma is proved.

From this lemma and Lemma 5.2b) follows

COROLLARY 7.1. *The function* $v_2 = w_2^\alpha$ *of* (7.2) *on the set* $\Gamma = \{x \in S_2\colon w_2(x) \geq h\}$, *where* $h = \mathrm{const} \in (0,1)$, *satisfies inequality* (5.1) *for some constant* $\nu_2 > 0$.

§8. Construction of the functions v. Concluding remarks

In §§2, 3, and 5, we successively reduced the proof of Theorem 1.1 to the problem of constructing a function v satisfying conditions 2.1)–2.3) and 5.1) with given constant $\alpha \in (0,1)$. We shall construct v on the sphere S; then its values on $\dot{E} = E\backslash\{0\}$ will be determined uniquely in view of the positive homogeneity condition 2.2).

The function v_1 from §6 satisfies our required conditions on the set $S\backslash\{P^+, P^-\}$, where the points $P^\pm = (0,0,\pm 1)$; however, the smoothness of v_1 "deteriorates" on approaching these points. The function $v_2 = w_2^\alpha$ of §7 has the "right" form on a subset $\Gamma \subset S_2 \subset S_1 \subset S^+$ containing the point P^+. Let us assume that we have succeeded in smoothly "patching" the functions v_1 and v_2 on the set Γ while preserving conditions 2.1), 2.2), and 5.1) (more precisely, the "restrictions" of these conditions to Γ) and that $v = v_1$ near $\partial\Gamma$. We extend the definition of v to the set $\xi(\Gamma) \ni P^-$ according to condition 2.3) and put $v = v_1$ on $S\backslash\Gamma\backslash\xi(\Gamma)$. Since v_1 also satisfies 2.3), we get $v = v_1$ near $\partial\xi(\Gamma)$, which ensures the smoothness of the function v constructed in this manner on the entire sphere S. From condition 2.3) it easily follows that if (5.1) holds on Γ then it also holds on $\xi(\Gamma)$. Consequently, the function v will satisfy all the conditions 2.1)–2.3) and 5.1).

All the aforesaid remains in force if, in place of v_1, one takes the function $\tilde{v}_1 = kv_1$ for an arbitrary choice of constant $k \in (0,1)$. We shall use, instead of v_1, the function $\tilde{v}_1 = kv_1$ with sufficiently small constant $k > 0$ whose choice we make more precise below. This simplifies the proof of inequality (8.6), on which we rely heavily in the process of constructing v.

So, the problem has been reduced to the following: construct a set $\Gamma_1 \subset S_2$ and define a function v satisfying conditions 2.1), 2.2), and 5.1) on Γ_1, where $v = \tilde{v}_1 = kv_1$ near $\partial\Gamma_1$, $k = \mathrm{const} \in (0,1)$. We divide the construction of such a function v into a series of steps.

1°. By virtue of the inclusion $S_2 \subset S_1 \subset S^+$ the function v_1 is strictly positive on $S_2\backslash\{P^+\}$. In addition, it is smooth outside any neighborhood of P^+ in S_2. Consequently, the function $w_1 = v_1^{1/\alpha}$ satisfies the inequalities

$$h_1 \leq w_1 \leq h_2, \quad |\operatorname{grad} w_1| \leq C_1 \quad \text{on the set } \{x \in S_2\colon w_2(x) \leq 1/2\} \tag{8.1}$$

for some constants $0 < h_1 < h_2 < 1$, $C_1 > 0$.

Further, for the function $w_2 = \cos p\theta \cos q\varphi$, from (7.2) by homogeneity we have $w_{2r} = 0$ and

$$\begin{aligned}|\operatorname{grad} w_2| &= \left(w_{2\theta}^2 + \frac{1}{\cos^2\theta} w_{2\varphi}^2\right)^{1/2} \\ &= \left(p^2 \tan^2 p\theta + \frac{q^2}{\cos^2\theta} \tan^2 q\varphi\right)^{1/2} w_2 \quad \text{on } S_2.\end{aligned} \tag{8.2}$$

Note that $w_2 > 0$ and S_2 and $w_2 = 0$ on ∂S_2. In addition, for small values of w_2 on S_2, i.e. near ∂S_2, the coefficient of w_2 on the right-hand side of (8.2) grows without bound. Taking what has been said into account, we choose a sufficiently small constant $H \in (0, 1/2)$ for which

$$|\operatorname{grad} w_2| \geq (2C_1/h_1 + 1)w_2 \quad \text{on the set } \{x \in S_2 \colon w_2(x) \leq H\}, \tag{8.3}$$

and, in addition, the constant $\varepsilon_1 = 2H/(h_1 + 2h_2) \in (0, 1)$.

We put $h = \varepsilon_1 h_1/2$ and consider the sets

$$\Gamma_1 = \{x \in S_2 \colon w_2(x) \geq h\}, \quad \Gamma_2 = \{x \in S_2 \colon w_2(x) > H\}, \quad \Gamma = \Gamma_1 \backslash \Gamma_2.$$

Since $h < \varepsilon_1 h_1 < \varepsilon_2 h_2 < H$, we have $\Gamma_1 \supset \Gamma_2$ and $\partial\Gamma = \partial\Gamma_1 \cup \partial\Gamma_2$. By (8.1) and (8.3) the following estimates hold for the functions $\tilde{w}_1 = \varepsilon_1 w_1$ and w_2:

$$0 < h \leq \tilde{w}_1, \quad w_2 \leq H < 1/2 \quad \text{on } \Gamma; \tag{8.4}$$

$$\tilde{w}_1 - w_2 \geq \varepsilon_1 h_1 - h \quad \text{on } \partial\Gamma_1, \qquad w_2 - \tilde{w}_1 \geq H - \varepsilon_1 h_2 = h \quad \text{on } \partial\Gamma_2; \tag{8.5}$$

$$\begin{aligned}|\operatorname{grad}(w_2 - \tilde{w}_1)| &\geq |\operatorname{grad} w_2| - \varepsilon_1 |\operatorname{grad} w_1| \\ &\geq (2C_1/h_1 + 1)h - \varepsilon_1 C_1 = h \quad \text{on } \Gamma.\end{aligned} \tag{8.6}$$

2°. Now we set $k = \varepsilon_1^\alpha \in (0, 1)$ and $\tilde{v}_1 = \tilde{w}_1^\alpha = \varepsilon_1^\alpha w_1^\alpha = kv_1$ on Γ. Let the desired function v coincide with v_2 on Γ_2. It remains for us to extend the definition of $v = w^\alpha$ on Γ so that, besides 2.1), 2.2), and 5.1), the conditions

$$w = \tilde{w}_1 \quad \text{near } \partial\Gamma_1, \qquad w = w_2 \quad \text{near } \partial\Gamma_2, \tag{8.7}$$

hold, which guarantee a smooth "patching" of the function $v = w^\alpha$ with $\tilde{v}_1 = \tilde{w}_1^\alpha$ on $\partial\Gamma_1$ and with $v_2 = w_2^\alpha$ on $\partial\Gamma_2$. For that, by Lemma 5.4b), it suffices that w be smooth and, besides (8.7), that it satisfy (5.3) and (5.4) on Γ for some constant $\nu_3 > 0$. Taking (8.4) into account, we conclude from Lemmas 6.1, 5.2a), and 7.1 that $\tilde{w}_1$ and w_2 satisfy (5.3) and (5.4) on Γ for some positive constants ν_3' and ν_3'' respectively.

We construct w by smoothing the function

$$\bar{w} = \max\{\tilde{w}_1, w_2\} = (\tilde{w}_1 + w_2 + |\tilde{w}_1 - w_2|)/2.$$

To that end we choose an even nonnegative function $\varsigma(t) \in C^\infty(E_1)$ such that $\varsigma(t) = 0$ for $|t| \geq 1$, $\varsigma(t) > 0$ for $|t| < 1$, and $\int_{E_1} \varsigma(t)\, dt = 2$, and we put

$$\mu(t) = -t + \int_{-1}^{t} (t - s)\varsigma(s)\, ds \in C^\infty(E_1).$$

It is easy to see that $\mu(t) = |t|$ for $|t| \geq 1$, $1 > \mu(t) > |t|$ for $|t| < 1$, $|\mu'(t)| = |-1 + \int_{-1}^{t} \varsigma(s)\, ds| \leq 1$ for all t, and $\mu''(t) = \varsigma(t) > 0$ for $|t| < 1$. The function $\varepsilon\mu(t/\varepsilon)$, coinciding with $|t|$ for $|t| \geq \varepsilon$, will serve as an approximation for $|t|$.

3°. We prove that, relying on the properties of the function μ listed above, for a suitable choice of constant $\varepsilon \in (0, h)$ the smooth function

$$w = \tfrac{1}{2}(\tilde{w}_1 + w_2 + \varepsilon\mu(t/\varepsilon)) \quad \text{on } \Gamma, \text{ where } t = w_2 - \tilde{w}_1, \tag{8.8}$$

satisfies (8.7), (5.3), and (5.4). Relations (8.7) and (5.3) hold for an arbitrary $\varepsilon \in (0, h)$. Indeed, from (8.5) it follows that

$$w = \tfrac{1}{2}(\tilde{w}_1 + w_2 + |t|) = \max\{\tilde{w}_1, w_2\} \quad \text{near } \partial\Gamma = \partial\Gamma_1 \cup \partial\Gamma_2,$$

which means the validity of (8.7). Inequalities (5.3) easily follow from (8.4):

$$h \le \max\{\tilde{w}_1, w_2\} \le w \le \max\{\tilde{w}_1, w_2\} + \varepsilon/2 < (1+\varepsilon)/2 < 1 \quad \text{on } \Gamma.$$

To complete the whole construction, it remains to prove that for sufficiently small $\varepsilon \in (0, h)$ the function w satisfies (5.4) on Γ for some constant $\nu_3 > 0$. As was noted above, (5.4) holds for $\tilde{w}_1$ and w_2 on Γ for some constants $\nu_3', \nu_3'' > 0$. Let us choose $\nu_3 = \min\{\nu_3', \nu_3''\}/2 > 0$. On differentiating (8.8) twice in the direction of an arbitrary vector $l \in E$, we find that

$$2w_{(l)(l)} = (1-\mu')\tilde{w}_{1(l)(l)} + (1+\mu')w_{2(l)(l)} + (1/\varepsilon)\mu'' t_{(l)}^2, \tag{8.9}$$

where μ' and μ'' are the derivatives of μ at the point $t/\varepsilon = (w_2 - \tilde{w}_1)/\varepsilon$. Further, since $\tilde{w}_1$ and w_2 are smooth of Γ,

$$|\tilde{w}_{1(l)(l)}| \le C_2, |w_{2(l)(l)}| \le C_2 \quad \text{on } \Gamma \text{ for all } l \in E,\ |l| = 1 \tag{8.10}$$

for some constant $C_2 > 0$.

Put $\varepsilon_2 = \nu_3/2C_2 > 0$. Since $|\mu'| \le 1$, at each point in Γ one of three relations hold: a) $0 \le 1 + \mu' < \varepsilon_2$; b) $0 \le 1 - \mu' < \varepsilon_2$; or c) $|\mu'| \le 1 - \varepsilon_2 < 1$. Let us consider these three cases separately. In case a), taking account of (8.10) and the nonnegativity of the last term in (8.9), we have

$$w_{(l)(l)} \ge \tilde{w}_{1(l)(l)} + (1+\mu')(w_{2(l)(l)} - \tilde{w}_{1(l)(l)}) \ge \tilde{w}_{1(l)(l)} - 2\varepsilon_2 C_2$$

on Γ for all $|l| = 1$, whence by choice of ν_3 and ε_2 we get (5.4):

$$\sup_{|l|=1} w_{(l)(l)} + w \ge \sup_{|l|=1} \tilde{w}_{1(l)(l)} + \tilde{w}_1 - 2\varepsilon_2 C_2 \ge \nu_3' - 2\varepsilon_2 C_2 \ge \nu_3 \quad \text{on } \Gamma.$$

Case b) is analyzed analogously.

Consider the remaining case c), $|\mu'| \le 1 - \varepsilon_2 < 1$. From the properties of μ it follows that in this case $\mu'' \ge \varepsilon_3 = \text{const} > 0$. Therefore, by (8.9), (8.10), and (8.6) we find that

$$\begin{aligned}\sup_{|l|=1} w_{(l)(l)} + w &\ge -C_1 + \frac{\varepsilon_3}{2\varepsilon} \sup_{|l|=1} t_{(l)}^2 = -C_1 + \frac{\varepsilon_2}{2\varepsilon} |\operatorname{grad}(w_2 - \tilde{w}_1)|^2 \\ &\ge -C_1 + \varepsilon_3 h^2/2\varepsilon \ge \nu_3 \quad \text{on } \Gamma,\end{aligned}$$

if $\varepsilon > 0$ is sufficiently small, as was required.

REMARK 8.1. As was shown in §2, the function $u(x) = |x|^\alpha v(x)$ is the uniform limit of smooth functions $u^\varepsilon(x)$ which are solutions of (1.5) with coefficients $a_{ij}^\varepsilon(x)$ satisfying (1.6) with constant $\nu_0 > 0$ depending only on $\alpha \in (0,1)$. By (2.1) the derivatives $u_{x_i x_j}(x)$ are positively homogeneous functions of order $\alpha - 2$. Consequently, for all $\rho \in (0,1)$ and $i, j = 1, 2, 3$, we have

$$\|u_{x_i x_j}\|_{L_p(B_\rho)} = \left(\int_{B_\rho} |u_{x_i x_j}(x)|^p \, dx \right)^{1/p} = \infty, \quad \text{if } (2-\alpha)p \ge 3. \tag{8.11}$$

In addition, in the constructions of §2, $a_{ij}^\varepsilon(x) = a_{ij}(x)$ for $x \in B_1 \backslash B_{\varepsilon_0}$, $0 < \varepsilon \le \varepsilon_0 < 1$. Consequently, by well-known properties of the solutions of elliptic equations with smooth coefficients, from (1.5) and the uniform convergence of u^ε to u we deduce the uniform convergence of $u^\varepsilon_{x_ix_j}$ to $u_{x_ix_j}$ as $\varepsilon \to 0+$ on $B_1 \backslash B_{\varepsilon_0}$ for any $\varepsilon_0 \in (0,1)$. Therefore, from (8.11) it follows that

$$\lim_{\varepsilon \to 0+} \|u^\varepsilon_{x_ix_j}\|_{L_p(B_\rho)} = \infty, \quad \text{if } (2-\alpha)p \geq 3.$$

Since α can be made arbitrarily small, there exists no estimate for the derivatives $u_{x_ix_j}$ of the possible solutions u of (1.1) in the ball $B_1 \subset E = E_3$ in the norm of $L_p(B_\rho)$, $\rho \in (0,1)$, in terms of the norm of u in $C(B_1)$ if $p > 3/2$ and the coefficients a_{ij} obey only conditions (1.2).

REMARK 8.2. Estimates for Hölder constants of the form (1.3) for the solutions of elliptic equations under certain natural assumptions also hold near ∂D for the Dirichlet problem in the domain $D \subset E_n$ (see [2], Theorem 4.3), and for the problem with oblique derivative [5]. Our results also imply the unimprovability of the estimates of the Hölder constants in the case of the problem with oblique derivative. Indeed, the functions $u(x) = |x|^\alpha v(x)$ and $u^\varepsilon(x)$ of §2 are even with respect to each of the variables x_1 and x_2. We use evenness with respect to x_1. Choose a region $D \subset B_1$ with smooth boundary so that the portion of the boundary $\partial D \cap B_{1/2}$ lies entirely in the plane $\{x_1 = 0\}$. For $x \in \partial D$ let $l = l(x)$ be the unit inward normal vector to ∂D at x. We view the functions $u^\varepsilon(x)$ as solutions of (1.5) in D. We have $u^\varepsilon_{(l)} = 0$ on $\partial D \cap B_{1/2}$ by evenness of u^ε. In addition, since the coefficients a^ε_{ij} of (1.5) on $B_1 \backslash B_{1/4}$ are smooth and do not depend on $\varepsilon \in (0, 1/4)$, for such ε the functions u^ε on $\partial D \backslash B_{1/2}$ are uniformly bounded together with their derivatives of all orders.

Thus, the smoothness of $u^\varepsilon_{(l)}$ on ∂D does not deteriorate as $\varepsilon \to 0+$, whereas the function $u = \lim_{\varepsilon \to 0+} u^\varepsilon$ has a singularity at the point $0 \in \partial D$.

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