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ON THE FREE BOUNDARY PROBLEM OF MAGNETOHYDRODYNAMICS

Abstract. The paper proves the solvability of a free boundary problem of magnetohydrodynamics for a viscous incompressible fluid in a simply connected domain. The solution is obtained in Sobolev–Slobodetskii spaces $W_2^{2+l,1+l/2}$, $1/2 < l < 1$.

Dedicated to Professor G. A. Seregin on the occasion of his jubilee

1. Introduction

In the present paper, we are concerned with the simplest free boundary problem of magnetohydrodynamics. It consists of finding a bounded variable domain $\Omega_{1t} \subset \mathbb{R}^3$ filled with a viscous incompressible electrically conducting capillary fluid, together with the vector field of velocity $\mathbf{v}(x,t) = (v_1, v_2, v_3)$, the scalar pressure $p(x,t)$ and the magnetic field $\mathbf{H}(x,t)$ satisfying the system of equations of magnetohydrodynamics. The boundary Γ_t of Ω_{1t} is the free surface of the fluid, which is subject to capillary forces. It is assumed that the fluid is surrounded by a vacuum region Ω_{2t} and that the domain $\Omega = \Omega_{1t} \cup \Gamma_t \cup \Omega_{2t}$ is independent of time and bounded by a fixed perfectly conducting surface S such that $S \cap \Gamma_t = \varnothing$. Both Ω_{1t} and Ω are simply connected. The magnetic field should be found not only in Ω_{1t} but also in Ω_{2t}.

The problem can be presented in the following form (see [1–3]):

Key words and phrases: Free boundary problems, magnetohydrodinamics, Sobolev spaces.

$$\begin{cases} \mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla\cdot T(\mathbf{v},p) - \nabla\cdot T_M(\mathbf{H}) = 0, \\ \nabla\cdot\mathbf{v}(x,t) = 0, \quad x\in\Omega_{1t}, \ \ t>0, \\ \mu_1\mathbf{H}_t + \alpha^{-1}\,\mathrm{rotrot}\,\mathbf{H} - \mu_1\,\mathrm{rot}(\mathbf{v}\times\mathbf{H}) = 0, \\ \nabla\cdot\mathbf{H}(x,t) = 0, \quad x\in\Omega_{1t}, \\ \mathrm{rot}\,\mathbf{H} = 0, \quad \nabla\cdot\mathbf{H}(x,t) = 0, \quad x\in\Omega_{2t}, \\ \big(T(\mathbf{v},p) + [T_M(\mathbf{H})]\big)\mathbf{n} = \sigma\mathbf{n}H, \quad V_n = \mathbf{v}\cdot\mathbf{n}, \quad x\in\Gamma_t, \\ [\mu\mathbf{H}\cdot\mathbf{n}] = 0, \quad [\mathbf{H}_\tau] = 0, \quad x\in\Gamma_t, \\ \mathbf{H}\cdot\mathbf{n} = 0, \quad x\in S, \\ \mathbf{v}(x,0) = \mathbf{v}_0(x), \quad x\in\Omega_{10}, \quad \mathbf{H}(x,0) = \mathbf{H}_0(x), \quad x\in\Omega_{10}\cup\Omega_{20}, \end{cases} \tag{1.1}$$

where $T(\mathbf{v},p)$ is the viscous stress tensor:

$$T(\mathbf{v},p) = -pI + \nu S(\mathbf{v}), \quad S(\mathbf{v}) = \nabla\mathbf{v} + (\nabla\mathbf{v})^T$$

is the doubled rate-of-strain tensor, $T_M(\mathbf{H}) = \mu(\mathbf{H}\otimes\mathbf{H} - \frac{1}{2}|\mathbf{H}|^2 I)$, $x\in\Omega$, is the magnetic stress tensor, μ is a piece-wise constant function equal to μ_i in Ω_{it}, $\mathbf{n}$ is the exterior normal to Γ_t and to S, V_n is the velocity of evolution of Γ_t in the direction $\mathbf{n}$, $\mathbf{H}_\tau = \mathbf{H} - \mathbf{n}(\mathbf{n}\cdot\mathbf{H})$ is the tangential component of $\mathbf{H}$, H is the doubled mean curvature of Γ_t negative for convex domains. The parameters ν, μ_i, α, σ (the kinematic viscosity, magnetic permeabilities, conductivity, coefficient of the surface tension) are positive constants. By $[u]$ we denote the jump on Γ_t of the function given in Ω_{it}: $[u] = u^{(1)} - u^{(2)}$, and $u^{(i)} = u(x,t)|_{x\in\Omega_{it}}$.

In the case where the domains Ω_{it} are independent of t, $\mathbf{v}$ satisfies the no-slip condition on $\Gamma = \partial\Omega_1$ and Ω_1 and Ω are simply connected, the problem of magnetohydrodynamics has been studied in the paper [4]. The case of multi-connected domains is considered in [5–7], in particular, it is found that in this case some additional orthogonality conditions for $\mathbf{H}$ may appear.

Since Ω_{1t} and Ω are simply connected, the equations $\mathrm{rot}\,\mathbf{H}^{(2)} = 0$ and $\nabla\cdot\mathbf{H}^{(2)} = 0$ imply $\mathbf{H}^{(2)}(x,t) = \nabla\varphi(x,t)$, where φ is a solution of the Neumann problem

$$\nabla^2\varphi(x,t) = 0, \quad x\in\Omega_{2t}, \quad \mu_2\frac{\partial\varphi}{\partial n}\Big|_{\Gamma_t} = \mu_1\mathbf{H}^{(1)}\cdot\mathbf{n}, \quad \frac{\partial\varphi}{\partial n}\Big|_S = 0, \tag{1.2}$$

hence $\mathbf{H}^{(2)}$ is completely defined by $\mathbf{H}^{(1)}$.

Moreover, we have $\mathbf{H}_\tau^{(1)}|_{\Gamma_t} = \nabla_\tau \varphi$, which can be regarded as a nonlocal boundary condition

$$\mathbf{H}_\tau^{(1)} = \mathcal{B}(\mathbf{H}^{(1)} \cdot \mathbf{n})$$

for $\mathbf{H}^{(1)}$.

In particular, $\mathbf{H}_0^{(2)} = \nabla \varphi_0(x)$ with φ_0 satisfying the relations

$$\nabla^2 \varphi_0(x) = 0, \quad x \in \Omega_{20}, \quad \mu_2 \frac{\partial \varphi_0}{\partial n}\Big|_{\Gamma_0} = \mu_1 \mathbf{H}_0^{(1)} \cdot \mathbf{n}, \quad \frac{\partial \varphi_0}{\partial n}\Big|_S = 0.$$

We assume that Γ_0 is located in the neighborhood of a fixed smooth closed surface $\mathcal{G}$ of arbitrary shape and can be regarded as the normal perturbation of $\mathcal{G}$:

$$\Gamma_0 = \{x = y + \mathbf{N}(y)\rho_0(y), \quad y \in \mathcal{G}\},$$

where $\mathbf{N}(y)$ is the exterior normal to $\mathcal{G}$ and $\rho_0(y)$ is a small function given on $\mathcal{G}$. In addition, we assume that the free boundary Γ_t, $t > 0$, is given by a similar equation

$$x = y + \mathbf{N}(y)\rho(y,t), \quad y \in \mathcal{G}, \tag{1.3}$$

with unknown $\rho(y,t)$.

As usual, the free boundary problem (1.1) is written as a nonlinear problem in a fixed domain. This is achieved by the transformation

$$x = e_\rho(y) : x = y + \mathbf{N}^*(y)\rho^*(y,t) : \mathcal{F}_i \to \Omega_{it}, \tag{1.4}$$

where $\mathbf{N}^*(y)$ and $\rho^*(y,t)$ are extensions of $\mathbf{N}$ and ρ from $\mathcal{G}$ into Ω such that $\mathbf{N}^*(y)$ is a sufficiently regular non-vanishing vector field in Ω and ρ^* has a small $C^1(\Omega)$-norm and vanishes on S. Moreover, ρ^* can be chosen in such a way that $\frac{\partial \rho^*(x,t)}{\partial N}\Big|_{\mathcal{G}} = 0$ and

$$\begin{aligned} &\|\rho^*(\cdot,t)\|_{W_2^{r+1/2}(\Omega)} \le c\|\rho\|_{W_2^r(\mathcal{G})}, \quad r \in (0, l+5/2], \\ &\|\rho_t^*(\cdot,t)\|_{W_2^{r+1/2}(\Omega)} \le c\|\rho_t\|_{W_2^r(\mathcal{G})}, \quad r \in (0, l+3/2], \quad l \in (1/2, 1). \end{aligned} \tag{1.5}$$

The existence of $\rho^*(x,t)$ with these properties follows from well known imbedding and extension theorems for the elements of Sobolev–Slobodetskii spaces.

By $\mathcal{F}_1$ we mean the domain bounded by $\mathcal{G}$ and $\mathcal{F}_2 = \Omega \setminus \overline{\mathcal{F}_1}$. We denote by $\mathcal{L} = \mathcal{L}(y, \rho^*)$ the Jacobi matrix of the transformation $x = e_\rho(y)$ and we set $L = det\mathcal{L}$, $\widehat{\mathcal{L}} = L\mathcal{L}^{-1}$; finally, $l_{ij}(y, \rho^*)$, $l^{ij}(y, \rho^*)$ $\widehat{L}_{ij}(y, \rho^*)$ are the elements of $\mathcal{L}$, $\mathcal{L}^{-1}$, $\widehat{\mathcal{L}}$, respectively.

Now we write the transformed system (1.1). The mapping (1.4) converts $\nabla_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ in $\widetilde{\nabla} = \mathcal{L}^{-T}(y, \rho)\nabla_y$, where $\mathcal{L}^{-T} = (\mathcal{L}^{-1})^T$ and the sign T means transposition. Moreover, if $\mathbf{u}(y, t) = \mathbf{v}(e_\rho, t)$, then

$$\mathbf{u}_t(y, t) = \mathbf{v}_t(x, t)|_{x=e_\rho} + (x_t \cdot \nabla_x)\mathbf{v}(x, t)|_{x=e_\rho},$$

which implies

$$\mathbf{v}_t(x, t) \circ e_\rho = \mathbf{u}_t(y, t) - \rho_t^*(y, t)(\mathcal{L}^{-1}\mathbf{N}^*(y) \cdot \nabla_y)\mathbf{u}(y, t),$$

where $\mathbf{v}_t(x, t) \circ e_\rho = \mathbf{v}_t(e_\rho(y), t)$. In addition, we have $\nabla_x \cdot \mathbf{v}(x, t)|_{x=e_\rho(y)} = \mathcal{L}^{-T}\nabla_y \cdot \mathbf{u}(y, t)$, hence the Navier–Stokes equations can be written in the form

$$\begin{cases} \mathbf{u}_t - \rho_t^*(\mathcal{L}^{-1}\mathbf{N}^*(y) \cdot \nabla)\mathbf{u} + (\mathcal{L}^{-1}\mathbf{u} \cdot \nabla)\mathbf{u} - \widetilde{\nabla} \cdot \widetilde{T}(\mathbf{u}, q) - \widetilde{\nabla} \cdot T_M(\mathbf{H} \circ e_\rho) = 0, \\ \widehat{\mathcal{L}}^T\nabla \cdot \mathbf{u} = \nabla \cdot \widehat{\mathcal{L}}\mathbf{u} = 0, \quad y \in \mathcal{F}_1, \ \ t > 0, \end{cases}$$

where $q(y, t) = p(e_\rho, t)$.

Now we turn to the equations for $\mathbf{H}$. We use the formula

$$\mathrm{rot}_x\, \mathbf{H} = \frac{1}{L}\mathcal{L}\,\mathrm{rot}_y\, \mathcal{L}^T\widetilde{\mathbf{H}},$$

where $\widetilde{\mathbf{H}} = \mathbf{H} \circ e_\rho$. Hence the third equation in (1.1) becomes

$$\mu_1(\widetilde{\mathbf{H}}_t - \rho_t^*(\mathcal{L}^{-1}\mathbf{N}^* \cdot \nabla)\widetilde{\mathbf{H}}) + \alpha^{-1}\frac{1}{L}\mathcal{L}\,\mathrm{rot}_y\, \frac{1}{L}\mathcal{L}^T\mathcal{L}\,\mathrm{rot}_y\, \mathcal{L}^T\widetilde{\mathbf{H}} - \mu_1\frac{1}{L}\mathcal{L}\,\mathrm{rot}_y\, \mathcal{L}^T(\mathbf{u} \times \widetilde{\mathbf{H}}) = 0. \qquad (1.6)$$

We introduce a new vector field $\mathbf{h} = \widehat{\mathcal{L}}\widetilde{\mathbf{H}}$, multiply (1.6) by $\widehat{\mathcal{L}}$ from the left, and use the algebraic formula

$$(\mathcal{L}\mathbf{f} \times \mathcal{L}\mathbf{g}) = \widehat{\mathcal{L}}^T(\mathbf{f} \times \mathbf{g}).$$

It is clear that $\nabla_y \cdot \mathbf{h} = \nabla_y \cdot \widehat{\mathcal{L}}\widetilde{\mathbf{H}} = L\mathcal{L}^{-T}\nabla_y \cdot \widetilde{\mathbf{H}} = L(\nabla_x \cdot \mathbf{H} \circ e_\rho) = 0$, so we arrive at

$$\begin{cases} \mu_1\Big(\mathbf{h}_t - \dfrac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\mathbf{h} - \rho_t^*\widehat{\mathcal{L}}(\mathcal{L}^{-1}\mathbf{N}^*(y)\cdot\nabla)\dfrac{1}{L}\mathcal{L}\mathbf{h}\Big) \\ \ + \alpha^{-1}\operatorname{rot}\dfrac{1}{L}\mathcal{L}^T\mathcal{L}\operatorname{rot}\dfrac{1}{L}\mathcal{L}^T\mathcal{L}\mathbf{h} - \mu_1\operatorname{rot}(\mathcal{L}^{-1}\mathbf{u}\times\mathbf{h}) = 0, \\ \nabla\cdot\mathbf{h} = 0, \quad y \in \mathcal{F}_1, \\ \operatorname{rot}\dfrac{1}{L}\mathcal{L}^T\mathcal{L}\mathbf{h} = 0, \quad \nabla\cdot\mathbf{h} = 0, \quad y \in \mathcal{F}_2. \end{cases}$$

Since $\widetilde{\mathbf{H}}_t(y,t) = (\mathbf{H}_t(x,t) + (x_t\cdot\nabla_x)\mathbf{H})\circ e_\rho$, the expression

$$\frac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\mathbf{h} + \rho_t^*\widehat{\mathcal{L}}(\mathcal{L}^{-1}\mathbf{N}^*(y)\cdot\nabla)\frac{1}{L}\mathcal{L}\mathbf{h} = \mathbf{h}_t - \widehat{\mathcal{L}}(\mathbf{H}_t(x,t)|_{x=e_\rho}) \tag{1.7}$$

is divergence free.

Now we turn to the boundary conditions. We note that the vectors $\mathbf{n}(e_\rho)$ and $\mathbf{N}(y)$ are connected by

$$\mathbf{n}(e_\rho(y)) = \frac{\widehat{\mathcal{L}}^T(y,\rho)\mathbf{N}(y)}{|\widehat{\mathcal{L}}^T(y,\rho)\mathbf{N}(y)|},$$

so that the kinematic boundary condition $V_n = \mathbf{v}\cdot\mathbf{n}$ can be written in terms of ρ as

$$\rho_t = \frac{\mathbf{u}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{N}} = \frac{\mathbf{u}\cdot\widehat{\mathcal{L}}^T\mathbf{N}}{\Lambda(y,\rho)},$$

where

$$\Lambda = \mathbf{N}\cdot\widehat{\mathcal{L}}\mathbf{N} = 1 - \rho\mathcal{H} + \rho^2\mathcal{K}$$

and $\mathcal{H}$, $\mathcal{K}$ are the doubled mean curvature and the Gaussian curvature of $\mathcal{G}$, respectively. Since $\widetilde{\mathbf{H}}\cdot\mathbf{n}(e_\rho) = |\widehat{\mathcal{L}}^T\mathbf{N}|^{-1}(\mathbf{h}\cdot\mathbf{N})$ and

$$\widetilde{\mathbf{H}} - \mathbf{n}(\widetilde{\mathbf{H}}\cdot\mathbf{n}) = \widetilde{\mathbf{H}} - \frac{\widehat{\mathcal{L}}^T\mathbf{N}}{|\widehat{\mathcal{L}}^T\mathbf{N}|^2}(\widehat{\mathcal{L}}\widetilde{\mathbf{H}}\cdot\mathbf{N}) = \widehat{\mathcal{L}}^{-1}\Big(\mathbf{h} - \frac{\widehat{\mathcal{L}}\widehat{\mathcal{L}}^T\mathbf{N}}{|\widehat{\mathcal{L}}^T\mathbf{N}|^2}(\mathbf{h}\cdot\mathbf{N})\Big),$$

we have the following boundary conditions for $\mathbf{h}$:

$$\begin{cases} [\mu\mathbf{h}\cdot\mathbf{N}] = 0, \quad [\mathbf{h}_\tau] = \Big(\dfrac{\widehat{\mathcal{L}}\widehat{\mathcal{L}}^T\mathbf{N}}{|\widehat{\mathcal{L}}^T\mathbf{N}|^2} - \mathbf{N}\Big)[\mathbf{h}\cdot\mathbf{N}], \quad y \in \mathcal{G}, \\ \mathbf{h}\cdot\mathbf{n} = 0, \quad y \in S \end{cases}$$

Putting all the equations together, we obtain

$$\begin{cases}
\mathbf{u}_t-\rho_t^*(\mathcal{L}^{-1}\mathbf{N}^*(y)\cdot\nabla)\mathbf{u}+(\mathcal{L}^{-1}\mathbf{u}\cdot\nabla)\mathbf{u}-\widetilde{\nabla}\cdot\widetilde{T}(\mathbf{u},q)-\widetilde{\nabla}\cdot T_M(\frac{\mathcal{L}}{L}\mathbf{h})=0,\\
\nabla\cdot\widehat{\mathcal{L}}\mathbf{u}=0, \quad y\in\mathcal{F}_1, \ \ t>0,\\
\mu_1\Big(\mathbf{h}_t-\frac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\mathbf{h}-\rho_t^*\widehat{\mathcal{L}}(\mathcal{L}^{-1}\mathbf{N}^*(y)\cdot\nabla)\frac{1}{L}\mathcal{L}\mathbf{h}\Big)\\
\ +\alpha^{-1}\operatorname{rot}\frac{1}{L}\mathcal{L}^T\mathcal{L}\operatorname{rot}\frac{1}{L}\mathcal{L}^T\mathcal{L}\mathbf{h}-\mu_1\operatorname{rot}(\mathcal{L}^{-1}\mathbf{u}\times\mathbf{h})=0,\\
\nabla\cdot\mathbf{h}=0, \quad y\in\mathcal{F}_1,\\
\operatorname{rot}\frac{1}{L}\mathcal{L}^T\mathcal{L}\mathbf{h}=0, \quad \nabla\cdot\mathbf{h}=0, \quad y\in\mathcal{F}_2,\\
\widetilde{T}(\mathbf{u},q)\mathbf{n}(e_\rho)+[T_M(\frac{1}{L}\mathcal{L}\mathbf{h})]\mathbf{n}(e_\rho)=\sigma H(e_\rho)\mathbf{n}(e_\rho),\\
\rho_t=\dfrac{\mathbf{u}\cdot\widehat{\mathcal{L}}^T\mathbf{N}}{\Lambda(y,\rho)}, \quad y\in\mathcal{G},\\
[\mu\mathbf{h}\cdot\mathbf{N}]=0, \quad [\mathbf{h}_\tau]=\Big(\dfrac{\widehat{\mathcal{L}}\widehat{\mathcal{L}}^T\mathbf{N}}{|\widehat{\mathcal{L}}^T\mathbf{N}|^2}-\mathbf{N}\Big)[\mathbf{h}\cdot\mathbf{N}], \quad y\in\mathcal{G},\\
\mathbf{h}\cdot\mathbf{n}=0, \quad y\in S,\\
\mathbf{u}(y,0)=\mathbf{u}_0(y), \quad y\in\mathcal{F}_1, \quad \mathbf{h}(y,0)=\mathbf{h}_0(y), \quad y\in\mathcal{F}_1\cup\mathcal{F}_2,\\
\rho(y,0)=\rho_0(y), \quad y\in\mathcal{G},
\end{cases} \tag{1.8}$$

where $\widetilde{T}(\mathbf{u},q)$ is the transformed stress tensor: $\widetilde{T}=-qI+\nu\widetilde{S}(\mathbf{u})$, $\widetilde{S}(\mathbf{u})=(\widetilde{\nabla}\mathbf{u})+(\widetilde{\nabla}\mathbf{u})^T$, $\mathbf{u}_0(y)=\mathbf{v}_0(e_{\rho_0}(y))$, $\mathbf{h}_0(y)=\widehat{\mathcal{L}}(y,\rho_0)\mathbf{H}_0(e_{\rho_0}(y))$, ρ is an additional unknown function. We note that

$$\left(\frac{\widehat{\mathcal{L}}\widehat{\mathcal{L}}^T\mathbf{N}}{|\widehat{\mathcal{L}}^T\mathbf{N}|^2}-\mathbf{N}\right)\cdot\mathbf{N}=0.$$

We make further transformations of (1.8). We write the boundary condition

$$\widetilde{T}(\mathbf{u},q)\mathbf{n}(e_\rho)+[T_M(\frac{1}{L}\mathcal{L}h)]\mathbf{n}(e_\rho)=\sigma H(e_\rho)\mathbf{n}(e_\rho)$$

for the tangential and normal parts separately, which gives

$$\widetilde{S}(\mathbf{u})\mathbf{n}-\mathbf{n}(\mathbf{n}\cdot\widetilde{S}(\mathbf{u})\mathbf{n})=0,$$

$$-q+\nu\mathbf{n}\cdot\widetilde{S}(\mathbf{u})\mathbf{n}+[\mathbf{n}\cdot T_M(\frac{1}{L}\mathcal{L}\mathbf{h})\mathbf{n}]=\sigma H(e_\rho), \quad y\in\mathcal{G}.$$

For $H(x)$ we use the formula

$$H(x) = -\nabla_\tau \cdot \mathbf{n}(x), \quad x \in \Gamma_t,$$

where ∇_τ is the tangential part of the gradient.

Due to (1.7), the vector field $\mathbf{n}$ is defined (by extended ρ and $\mathbf{N}$) in a certain neighborhood of $\mathcal{G}$, and $H = -\nabla_x \cdot \mathbf{n}$, because $\mathbf{n}\frac{\partial}{\partial n}\mathbf{n} = 0$. This implies

$$H(e_\rho) = -\mathcal{L}^{-T}(y,\rho)\nabla_y \cdot \frac{\widehat{\mathcal{L}}^T(y,\rho)\mathbf{N}}{|\widehat{\mathcal{L}}^T(y,\rho)\mathbf{N}|} = \mathcal{H}(y) + \int\limits_0^1 \frac{d}{ds} H_s(y)\,ds$$

$$= -\mathfrak{L}\rho - \int\limits_0^1 (1-s)\frac{d^2}{ds^2}\mathcal{L}^{-T}(y,s\rho)\nabla \cdot \frac{\widehat{\mathcal{L}}^T(y,s\rho)\mathbf{N}}{|\widehat{\mathcal{L}}^T(y,s\rho)\mathbf{N}|} ds + \mathcal{H}(y),$$

where H_s is the curvature of the surface

$$\Gamma_{s,t} = \{x = y + s\mathbf{N}(y)\rho(y,t), \quad y \in \mathcal{G}\},$$

hence $\mathcal{H} = H_0$, $H = H_1$. The expression

$$-\mathfrak{L}\rho = \Delta_{\mathcal{G}}\rho + (\mathcal{H}^2 - 2\mathcal{K})\rho,$$

where $\Delta_{\mathcal{G}}$ is the Laplace-Beltrami operator on $\mathcal{G}$, is the first variation of $H(e_\rho) - \mathcal{H}(y)$ with respect to ρ:

$$-\mathfrak{L}\rho = \frac{d}{ds} H_s|_{s=0}.$$

Finally, we separate linear and nonlinear parts in the equations (1.8) and obtain

$$\begin{cases}
\mathbf{u}_t(y,t) - \nu\nabla^2\mathbf{u} + \nabla q = \mathbf{l}_1(\mathbf{u},q,\mathbf{h},\rho), \\
\nabla \cdot \mathbf{u} = l_2(\mathbf{u},\rho), \quad y \in \mathcal{F}_1,\ t > 0, \\
\Pi_{\mathcal{G}} S(\mathbf{u})\mathbf{N} = \mathbf{l}_3(\mathbf{u},\rho), \\
-q + \nu\mathbf{N}\cdot S(\mathbf{u})\mathbf{N}(y) + \sigma\mathfrak{L}(\rho) = l_4(\mathbf{u},\mathbf{h},\rho) + l_5(\rho) + \sigma\mathcal{H}(y), \\
\rho_t + \mathbf{V}(x)\cdot\nabla_\tau\rho - \mathbf{u}\cdot\mathbf{N}(y) = l_6(\mathbf{u},\rho), \quad y \in \mathcal{G}, \\
\mu_1\mathbf{h}_t + \alpha^{-1}\,\mathrm{rotrot}\,\mathbf{h} = \mathbf{l}_7(\mathbf{h},\mathbf{u},\rho), \\
\nabla\cdot\mathbf{h} = 0, \quad y \in \mathcal{F}_1, \\
\mathrm{rot}\,\mathbf{h} = \mathrm{rot}\,\mathbf{l}_8(\mathbf{h},\rho), \quad \nabla\cdot\mathbf{h} = 0, \quad y \in \mathcal{F}_2, \\
[\mu\mathbf{h}\cdot\mathbf{N}] = 0, \quad [\mathbf{h}_\tau] = \mathbf{l}_9(\mathbf{h},\rho), \quad y \in \mathcal{G}, \\
\mathbf{h}\cdot\mathbf{n} = 0, \quad y \in S, \\
\mathbf{u}(y,0) = \mathbf{u}_0(y), \quad y \in \mathcal{F}_1, \quad \mathbf{h}(y,0) = \mathbf{h}_0(y), \quad y \in \mathcal{F}_1 \cup \mathcal{F}_2, \\
\rho(y,0) = \rho_0(y), \quad y \in \mathcal{G},
\end{cases} \tag{1.9}$$

where

$$\begin{cases} \mathbf{l}_1(\mathbf{u},q,\rho)=\nu(\widetilde{\nabla}^2-\nabla^2)\mathbf{u}+(\nabla-\widetilde{\nabla})q+\rho_t^*(\mathcal{L}^{-1}\mathbf{N}^*(y)\cdot\nabla)\mathbf{u} \\ -(\mathcal{L}^{-1}\mathbf{u}\cdot\nabla)\mathbf{u}+\widetilde{\nabla}\cdot T_M(\frac{\mathcal{L}}{L}\mathbf{h}), \\ l_2(\mathbf{u},\rho)=(I-\widehat{\mathcal{L}}^T)\nabla\cdot\mathbf{u}=\nabla\cdot\mathbf{L}(\mathbf{u},\rho), \\ \mathbf{L}(\mathbf{u},\rho)=(I-\widehat{\mathcal{L}})\mathbf{u}, \quad y\in\mathcal{F}_1, \\ \mathbf{l}_3(\mathbf{u},\rho)=\Pi_{\mathcal{G}}(\Pi_{\mathcal{G}}S(\mathbf{u})\mathbf{N})(y)-\Pi\widetilde{S}(\mathbf{u})\mathbf{n}(e_\rho(y)), \\ l_4(\mathbf{u},\mathbf{h},\rho)=\nu(\mathbf{N}\cdot S(\mathbf{u})\mathbf{N}-\mathbf{n}\cdot\widetilde{S}(\mathbf{u})\mathbf{n})-[T_M(\frac{\mathcal{L}}{L}\mathbf{h})]\mathbf{n}, \\ l_5(\rho)=-\int_0^1(1-s)\frac{d^2}{ds^2}\mathcal{L}^{-T}(y,s\rho)\nabla\cdot\frac{\mathcal{L}^T(y,s\rho)\mathbf{N}}{|\mathcal{L}^T(y,s\rho)\mathbf{N}|}\,ds, \\ l_6(\mathbf{u},\mathbf{h},\rho)=(\frac{\widehat{\mathcal{L}}^T\mathbf{N}}{\Lambda(y,\rho)|}+\nabla_\tau\rho-\mathbf{N})\cdot\mathbf{u}+(\mathbf{V}-\mathbf{u})\cdot\nabla_\tau\rho, \quad y\in\mathcal{G}, \\ \mathbf{l}_7(\mathbf{h},\rho)=\alpha^{-1}\operatorname{rot}(\operatorname{rot}\mathbf{h}-\frac{1}{L}\mathcal{L}^T\mathcal{L}rot\frac{1}{L}\mathcal{L}^T\mathcal{L}\mathbf{h})+\frac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\mathbf{h} \\ \qquad +\rho_t^*\widehat{\mathcal{L}}(\mathcal{L}^{-1}\mathbf{N}\cdot\nabla)\frac{1}{L}\mathcal{L}\mathbf{h}+\mu_1 rot(\mathcal{L}^{-1}\mathbf{u}\times\mathbf{h}), \quad y\in\mathcal{F}_1, \\ \mathbf{l}_8(\mathbf{h},\rho)=(I-\frac{1}{L}\mathcal{L}^T\mathcal{L})\mathbf{h}, \quad y\in\mathcal{F}_2, \\ l_9(\mathbf{h},\rho)=(\frac{\widehat{\mathcal{L}}\widehat{\mathcal{L}}^T\mathbf{N}}{|\widehat{\mathcal{L}}^T\mathbf{N}|^2}-\mathbf{N})[\mathbf{h}\cdot\mathbf{N}]=[\mathbf{A}(\mathbf{h},\rho)], \quad y\in\mathcal{G}, \end{cases} \tag{1.10}$$

$$\Pi\mathbf{f}=\mathbf{f}-\mathbf{n}(\mathbf{n}\cdot\mathbf{f}), \quad \Pi_{\mathcal{G}}\mathbf{g}=\mathbf{g}-\mathbf{N}(\mathbf{g}\cdot\mathbf{N}),$$

$$\mathbf{A}^{(i)}(\mathbf{h},\rho^*)=\Big(\frac{\widehat{\mathcal{L}}(y,\rho^*)\widehat{\mathcal{L}}^T(y,\rho^*)\mathbf{N}^*}{|\widehat{\mathcal{L}}^T(y,\rho^*)\mathbf{N}^*|^2}-\frac{\mathbf{N}^*}{|\mathbf{N}^*|^2}\Big)\mathbf{h}^{(i)}\cdot\mathbf{N}^*. \tag{1.11}$$

The vector field $\mathbf{V}(x)$ depends on $\mathbf{u}_0$; its role is to improve the estimate of l_6 for small t (see [8, 9]).

Our aim is to prove local in time solvability of the problem (1.9), (1.10) without making the smallness assumptions on $\mathbf{v}_0$ and $\mathbf{h}_0$. In Sec.2 we study a linearized problem (1.9), with all the expressions (1.10) replaced by given functions. It is easily seen that it is decomposed into "hydrodynamic" and "magnetic" parts. Since the first of them is studied in [8], we consider the second problem, for which we obtain coercive estimates of the solution.

In the proof we use the ideas of the paper [4], devoted to the problem of magnetohydrodynamics in fixed simply-connected domains. Finally, in Sec. 3 and 4 we obtain the main result of the paper.

2. Linear problems

The proof of the solvability of the problem (1.9), (1.10) is based on the analysis of non-homogeneous linear problems

$$\begin{cases} \mathbf{v}_t - \nu\nabla^2\mathbf{v} + \nabla p = \mathbf{f}(y,t), \\ \nabla\cdot\mathbf{v} = f(y,t), \quad y\in\mathcal{F}_1, \; t>0, \\ T(\mathbf{v},p)\mathbf{N}(y) + \sigma\mathbf{N}(y)\mathfrak{L}\rho = \mathbf{d}(y,t), \\ \rho_t + \mathbf{V}\cdot\nabla_\tau\rho - \mathbf{v}\cdot\mathbf{N}(y) = g(y,t), \quad y\in\mathcal{G}, \\ \mathbf{v}(y,0) = \mathbf{v}_0(y), \quad y\in\mathcal{F}_1, \quad \rho(y,0)=\rho_0(y), \quad y\in\mathcal{G}, \end{cases} \tag{2.1}$$

$$\begin{cases} \mu_1\mathbf{H}_t + \alpha^{-1}\,\mathrm{rot}\,\mathrm{rot}\,\mathbf{H} = \mathbf{G}(y,t), \\ \nabla\cdot\mathbf{H} = 0, \quad y\in\mathcal{F}_1, \\ \mathrm{rot}\,\mathbf{H} = \mathbf{j}(y,t), \quad \nabla\cdot\mathbf{H}=0, \quad y\in\mathcal{F}_2, \\ [\mu\mathbf{H}\cdot\mathbf{N}] = 0, \quad [\mathbf{H}_\tau] = \mathbf{a}(y,t), \quad y\in\mathcal{G}, \quad \mathbf{H}\cdot\mathbf{n}(y) = 0, \quad y\in S, \\ \mathbf{H}(y,0) = \mathbf{H}_0(y), \quad y\in\mathcal{F}_1\cup\mathcal{F}_2, \end{cases} \tag{2.2}$$

and of an auxiliary problem

$$\begin{cases} \mathrm{rot}\,\mathbf{h}(y) = \mathbf{j}(y), \quad \nabla\cdot\mathbf{h}=0, \quad y\in\mathcal{F}_1\cup\mathcal{F}_2, \\ [\mu\mathbf{h}\cdot\mathbf{N}] = 0, \quad [\mathbf{h}_\tau] = \mathbf{a}(y), \quad y\in\mathcal{G}, \quad \mathbf{h}\cdot\mathbf{N}(y)=0, \quad y\in S. \end{cases} \tag{2.3}$$

We solve these problems in the Sobolev–Slobodetskii spaces. We recall the definition of the corresponding norms. Let Ω be a domain in $\mathbb{R}^n$. The (isotropic) Sobolev space $W_2^l(\Omega)$ with $l>0$ is the space of functions $u(x)$, $x\in\Omega$, with the norm

$$\|u\|^2_{W_2^l(\Omega)} = \sum_{0\le|j|\le l}\|D^ju\|^2_{L_2(\Omega)} \equiv \sum_{0\le|j|\le l}\int_\Omega |D^ju(x)|^2dx,$$

if $l=[l]$, i.e. l is an integral number, and

$$\|u\|^2_{W_2^l(\Omega)} = \|u\|^2_{W_2^{[l]}(\Omega)} + \sum_{|j|=[l]}\int_\Omega\int_\Omega |D^ju(x)-D^ju(y)|^2\frac{dx\,dy}{|x-y|^{n+2\lambda}},$$

if $l = [l] + \lambda$, $\lambda \in (0,1)$. As usual, $D^j u$ denotes a (generalized) partial derivative $\frac{\partial^{|j|}u}{\partial x_1^{j_1} \dots, \partial x_n^{j_n}}$ where $j = (j_1, j_2, \dots, j_n)$ and $|j| = j_1 + \dots + j_n$. The anisotropic space $W_2^{l,l/2}(Q_T)$, $Q_T = \Omega \times (0,T)$, can be defined as

$$L_2((0,T), W_2^l(\Omega)) \cap W_2^{l/2}((0,T), L_2(\Omega))$$

and supplied with the norm

$$\|u\|^2_{W_2^{l,l/2}(Q_T)} = \int_0^T \|u(\cdot,t)\|^2_{W_2^l(\Omega)}\,dt + \int_\Omega \|u(x,\cdot)\|^2_{W_2^{l/2}(0,T)}\,dx. \tag{2.4}$$

There exist many other equivalent norms in $W_2^{l,l/2}(Q_T)$; some of them will be used below. Sobolev spaces of functions given on smooth surfaces, in particular, on $\mathcal{G}$ and on $G_T = \mathcal{G} \times (0,T)$, are introduced in a standard way, with the help of local maps and partition of unity. We also find it convenient to introduce the spaces $W_2^{l,0}(Q_T) = L_2((0,T), W_2^l(\Omega))$ and $W_2^{0,l/2}(Q_T) = W_2^{l/2}((0,T), L_2(\Omega))$; the squares of norms in these spaces coincide, respectively, with the first and the second term in (2.4).

In order to obtain uniform estimates of the solutions of (2.1) and (2.2) for small T, we introduce in $W_2^{l,l/2}(Q_T)$ equivalent norms defined by

$$\|u\|_{\widehat{W}_2^{l,l/2}(Q_T)} = \|u\|_{W_2^{l,l/2}(Q_T)},$$

if $l/2$ is an integer or $l/2 = [l/2] + \lambda$, $\lambda \in (1/2, 1)$,

$$\|u\|^2_{\widehat{W}_2^{l,l/2}(Q_T)} = \|u\|^2_{W_2^{l,l/2}(Q_T)} + \frac{1}{T^{2\lambda}} \|\frac{\partial^{[l/2]}}{\partial t^{[l/2]}} u\|^2_{L_2(Q_T)},$$

if $\lambda \in (0, 1/2)$, and

$$\|u\|^2_{H^{l,l/2}(Q_T)} = \|u\|^2_{\widehat{W}_2^{l,l/2}(Q_T)} + \sum_{0 \le j < (l-1)/2} \sup_{t<T} \|D_t^j u(\cdot,t)\|^2_{W_2^{l-1-2j}(Q_T)}.$$

Similar norms can be introduced on the manifold $G_T = \mathcal{G} \times (0,T)$. The advantages furnished by working with H-norms are discussed in [9, Propositions 1.1 and 1.2].

The following result is obtained in [8].

Theorem 1. *Assume that* $l \in (1/2,1)$, $\mathbf{V} \in W_2^{l+3/2}(\mathcal{G})$ *and that the data of the problem* (2.1) *possess the following regularity properties:* $\mathbf{f} \in W_2^{l,l/2}(Q_T^1)$, $f \in W_2^{l+1,0}(Q_T^1)$, $f(x,t) = \nabla \cdot \mathbf{F}(x,t)$, $\mathbf{F}_t \in W_2^{0,l/2}(Q_T^1)$, $\mathbf{d}\cdot\mathbf{N} \in W_2^{l+1/2,0}(G_T)\cap W_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))$, $\mathbf{d}-\mathbf{N}(\mathbf{d}\cdot\mathbf{N}) \in W_2^{l+1/2,l/2+1/4}(G_T)$, $g \in W_2^{l+3/2,l/2+3/4}(G_T)$, $\mathbf{v}_0 \in W_2^{l+1}(\mathcal{F}_1)$, $\rho_0 \in W_2^{l+2}(\mathcal{G})$ *where* $T < \infty$, $Q_T^1 = \mathcal{F}_1 \times (0,T)$, $G_T = \mathcal{G} \times (0,T)$. *Moreover, let the compatibility conditions*

$$\nabla \cdot \mathbf{v}_0(x) = f(x,0), \quad x \in \mathcal{F}, \quad \nu \Pi_{\mathcal{G}} S(\mathbf{v}_0)\mathbf{N} = \Pi_{\mathcal{G}}\mathbf{d}(x,0), \quad x \in \mathcal{G}$$

be satisfied. Then the problem (2.1) *has a unique solution* $\mathbf{v}, p, \rho$ *such that* $\mathbf{v} \in W_2^{l+2,l/2+1}(Q_T^1)$, $\nabla p \in W_2^{l,l/2}(Q_T^1)$, $p \in W_2^{l+1/2,0}(G_T) \cap W_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))$, $\rho \in W_2^{l+5/2,0}(G_T) \cap W_2^{l/2}(0,T;W_2^{3/2}(\mathcal{G}))$, $\rho_t \in W_2^{l+3/2,l/2+3/4}(G_T)$, *and the solution satisfies the inequality*

$$\begin{aligned}
&\|\mathbf{v}\|_{H^{l+2,l/2+1}(Q_T^1)} + \|\nabla p\|_{\widehat{W}_2^{l,l/2}(Q_T^1)} + \|p\|_{W_2^{l+1/2,0}(G_T)} \\
&+ \|p\|_{\widehat{W}_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))} \\
&+ \|\rho\|_{W_2^{l+5/2,0}(G_T)} + \|\rho\|_{\widehat{W}_2^{l/2}(0,T;W_2^{5/2}(\mathcal{G}_T))} + \|\rho_t\|_{H^{l+3/2,l/2+3/4}(G_T))} \\
&\le c\Big(\|\mathbf{f}\|_{\widehat{W}_2^{l,l/2}(Q_T^1)} + \|f\|_{W_2^{l+1,0}(Q_T^1)} + \|\mathbf{F}_t\|_{\widehat{W}_2^{0,1+l/2}(Q_T^1)} \\
&+ \|\Pi_{\mathcal{G}}\mathbf{d}\|_{H^{l+1/2,l/2+1/4}(G_T)} + \|\mathbf{d}\cdot\mathbf{N}\|_{W_2^{l+1/2,0}(G_T)} \\
&+ \|\mathbf{d}\cdot\mathbf{N}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))} \\
&+ \|g\|_{H^{l+3/2,l/2+3/4}(G_T)} + \|\mathbf{v}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\Big)
\end{aligned} \tag{2.6}$$

with the constant independent of T, *when* T *is bounded.*

Now we turn to the problem (2.3).

Theorem 2. *If* $\mathbf{j}^{(i)} \in W_2^{l+1}(\mathcal{F}_i)$, $\mathbf{a} \in W_2^{l+3/2}(\mathcal{G})$, *and the compatibility conditions*

$$\begin{aligned}
&\mathbf{N}\cdot\mathbf{a} = 0, \quad [\mathbf{j}\cdot\mathbf{N}] = \mathbf{N}\cdot \operatorname{rot}\mathbf{a}, \quad y \in \mathcal{G}, \\
&\nabla\cdot\mathbf{j}^{(i)} = 0, \quad y \in \mathcal{F}_i, \quad i = 1,2,
\end{aligned} \tag{2.7}$$

are satisfied, then the problem (2.3) *has a unique solution* $\mathbf{h} \in W_2^{2+l}(\mathcal{F}_1)\cap W_2^{2+l}(\mathcal{F}_2)$, *and*

$$\sum_{i=1}^{2} \|\mathbf{h}^{(i)}\|_{W_2^{l+2}(\mathcal{F}_i)} \le c\Big(\sum_{i=1}^{2} \|\mathbf{j}^{(i)}\|_{W_2^{l+1}(\mathcal{F}_i)} + \|\mathbf{a}\|_{W_2^{l+3/2}(\mathcal{G})}\Big). \tag{2.8}$$

Proof. We construct the solution in the form

$$\mathbf{h}(x) = \mathbf{a}^*(x) + \nabla\psi(x) + \xi(x), \tag{2.9}$$

where $\mathbf{a}^*$ is the extension of $\mathbf{a}$ into $\mathcal{F}_1$ such that

$$\|\mathbf{a}^*\|_{W_2^{l+2}(\mathcal{F}_1)} \leq c\|\mathbf{a}\|_{W_2^{l+3/2}(\mathcal{G})} \tag{2.10}$$

(we set $\mathbf{a}^*(x) = 0$ for $x \in \mathcal{F}_2$). The function $\psi(x)$ we define as a solution of the problem

$$\begin{cases} \nabla^2\psi = -\nabla\cdot\mathbf{a}^*(x), \quad x \in \mathcal{F}_1 \cup \mathcal{F}_2, \\ [\psi] = 0, \quad [\mu\dfrac{\partial\psi}{\partial N}] = 0, \quad x \in \mathcal{G}, \quad \dfrac{\partial\psi}{\partial N}\Big|_S = 0. \end{cases} \tag{2.11}$$

Finally,

$$\begin{aligned} &\operatorname{rot}\xi = \mathbf{j}(x) - \operatorname{rot}\mathbf{a}^*(x), \quad \nabla\cdot\xi = 0, \quad x \in \mathcal{F}_1 \cup \mathcal{F}_2, \\ &[\mu\xi\cdot\mathbf{N}] = 0, \quad [\xi_\tau] = 0, \quad x \in \mathcal{G}, \quad \xi\cdot\mathbf{n}|_S = 0. \end{aligned} \tag{2.12}$$

The solution of (2.12) has the form

$$\begin{aligned} \xi(x) &= \xi_1 + \nabla\omega, \\ \xi_1(x) &= \frac{1}{4\pi}\operatorname{rot}\int\limits_\Omega \frac{\mathbf{j}(y) - \operatorname{rot}\mathbf{a}^*(y)}{|x-y|}\,dy, \\ \nabla^2\omega(x) &= 0, \quad x \in \mathcal{F}_1 \cup \mathcal{F}_2, \\ [\omega] &= 0, \quad [\mu\frac{\partial\omega}{\partial N}] = -[\mu]\xi_1\cdot\mathbf{N}, \quad x \in \mathcal{G}, \\ \frac{\partial\omega}{\partial n}\Big|_S &= -\xi_1\cdot\mathbf{n}. \end{aligned}$$

We pass to the estimates of the functions defined above. Since $\mathbf{a}^*\cdot\mathbf{N}|_{\mathcal{G}} = 0$. the problem (2.11) has a unique (up to a constant) solution and

$$\sum_{i=1}^{2}\|\nabla\psi\|_{W_2^{l+2}(\mathcal{F}_i)} \leq c\|\mathbf{a}^*\|_{W_2^{l+2}(\mathcal{F}_1)} \leq c\|\mathbf{a}\|_{W_2^{l+3/2}(\mathcal{G})}. \tag{2.13}$$

By the known estimates of the volume potentials (see [10]) it holds

$$\sum_{i=1}^{2} \|\xi_1\|_{W_2^{l+2}(\mathcal{F}_i)} \le c\Big(\sum_{i=1}^{2} \|\mathbf{j}^{(i)}\|_{W_2^{l+1}(\mathcal{F}_i)} + \|\mathbf{a}^*\|_{W_2^{l+2}(\mathcal{F}_1)}\Big). \tag{2.14}$$

The function ω satisfies the inequality

$$\begin{aligned}\sum_{i=1}^{2} \|\nabla\omega\|_{W_2^{l+2}(\mathcal{F}_i)} &\le c(\|\xi_1\cdot\mathbf{N}\|_{W_2^{l+3/2}(\mathcal{G})} + \|\xi_1\cdot\mathbf{n}\|_{W_2^{l+3/2}(S)}) \\ &\le c\Big(\sum_{i=1}^{2} \|\mathbf{j}^{(i)}\|_{W_2^{l+1}(\mathcal{F}_i)} + \|\mathbf{a}^*\|_{W_2^{l+1}(\mathcal{F}_1)}\Big).\end{aligned} \tag{2.15}$$

Inequalities (2.10), (2.13)–(2.15) imply (2.8).

The uniqueness follows from the fact that the difference of two solutions of (2.3) equals $\nabla\phi$ and

$$\nabla^2\phi(x) = 0, \quad x \in \mathcal{F}_1 \cup \mathcal{F}_2, \quad [\phi] = 0, \quad [\mu\frac{\partial\phi}{\partial N}] = 0, \quad \frac{\partial\phi}{\partial n}|_S = 0,$$

i.e., $\nabla\phi = 0$. The theorem is proved.

The next proposition concerns the estimates of some weak norms of $\mathbf{h}$.

Theorem 3. *Assume that* (2.7) *is satisfied and, in addition,*

$$\mathbf{a}(x) = [\mathbf{A}(x)], \quad x \in \mathcal{G}, \quad \mathbf{j}^{(i)}(x) = \nabla\cdot\mathbf{J}^{(i)}(x), \quad x \in \mathcal{F}_i, \quad i = 1, 2, \tag{2.16}$$

where $\mathbf{A}(x)$ *is given in* $\mathcal{F}_1 \cup \mathcal{F}_2$, $\mathbf{A}^{(i)} \in W_2^1(\mathcal{F}_i)$, $\mathbf{A}^{(i)}\cdot\mathbf{N}|_{\mathcal{G}} = 0$, $\mathbf{J}^{(i)} \in W_2^1(\mathcal{F}_i)$, *moreover,* $\mathbf{A}^{(2)}|_S = 0$, $\mathbf{J}^{(2)}|_S = 0$. *Then the solution* $\mathbf{h}$ *to (2.3) satisfies the inequality*

$$\begin{aligned}&\|\mathbf{h}\|_{L_2(\Omega)} + \sum_{i=1}^{2} \|\mathbf{h}^{(i)}\|_{W_2^{-1/2}(\mathcal{G})} \\ &\le c\sum_{i=1}^{2}\Big(\|\mathbf{A}^{(i)}\|_{W_2^{-1/2}(\mathcal{G})} + \|\mathbf{J}^{(i)}\|_{W_2^{-1/2}(\mathcal{G})} + \|\mathbf{A}^{(i)}\|_{L_2(\mathcal{F}_i)} + \|\mathbf{J}^{(i)}\|_{L_2(\mathcal{F}_i)}\Big).\end{aligned} \tag{2.17}$$

Proof. As above, we represent the solution in the form

$$\mathbf{h} = \mathbf{A} + \nabla\Psi + \mathbf{X},$$

where Ψ and $\mathbf{X}$ are solutions to the problems

$$\begin{aligned} &\nabla^2\Psi(x) = -\nabla\cdot\mathbf{A}(x), \quad x\in\mathcal{F}_1\cup\mathcal{F}_2,\\ &[\Psi]=0, \quad [\mu\frac{\partial\Psi}{\partial N}]=0, \quad x\in\mathcal{G}, \quad \frac{\partial\Psi}{\partial N}\Big|_S=0, \end{aligned} \tag{2.18}$$

$$\begin{aligned} &\operatorname{rot}\mathbf{X} = \mathbf{j}(x) - \operatorname{rot}\mathbf{A}(x), \quad \nabla\cdot\mathbf{X}=0, \quad x\in\mathcal{F}_1\cup\mathcal{F}_2,\\ &[\mu\mathbf{X}\cdot\mathbf{N}]=0, \quad [\mathbf{X}_\tau]=0, \quad x\in\mathcal{G}, \quad \mathbf{X}\cdot\mathbf{n}|_S=0. \end{aligned} \tag{2.19}$$

Since $\mathbf{A}|_S=0$, $[\mu\mathbf{A}\cdot\mathbf{N}]|_{\mathcal{G}}=0$, the problem (2.18) is solvable, and

$$\|\nabla\Psi\|_{L_2(\Omega)} \le c\|\mathbf{A}\|_{L_2(\Omega)}. \tag{2.20}$$

From the inequality

$$\|\mathbf{u}\cdot\mathbf{n}\|_{W_2^{-1/2}(\partial D)} \le c\|\mathbf{u}\|_{L_2(D)}$$

that is valid for arbitrary divergence free vector field $\mathbf{u}$ given in a bounded domain D we conclude that

$$\|\frac{\partial\Psi^{(i)}}{\partial N} + \mathbf{A}^{(i)}\cdot\mathbf{N}\|_{W_2^{-1/2}(\mathcal{G})} \le c\|\nabla\Psi^{(i)}+\mathbf{A}^{(i)}\|_{L_2(\mathcal{F}_i)}, \quad i=1,2.$$

Together with (2.20), this inequality implies

$$\|\frac{\partial\Psi^{(i)}}{\partial N}\|_{W_2^{-1/2}(\mathcal{G})} \le c\Big(\|\mathbf{A}\|_{L_2(\Omega)} + \sum_{i=1}^2 \|\mathbf{A}^{(i)}\cdot\mathbf{N}\|_{W_2^{-1/2}(\mathcal{G})}\Big).$$

Moreover,

$$\|\nabla_\tau\Psi^{(i)}\|_{W_2^{-1/2}(\mathcal{G})} \le c\|\Psi^{(i)}\|_{W_2^{1/2}(\mathcal{G})} \le c\|\nabla\Psi^{(i)}\|_{L_2(\mathcal{F}_i)},$$

hence

$$\|\nabla\Psi^{(i)}\|_{W_2^{-1/2}(\mathcal{G})} \le c\Big(\|\mathbf{A}\|_{L_2(\Omega)} + \sum_{i=1}^2 \|\mathbf{A}^{(i)}\cdot\mathbf{N}\|_{W_2^{-1/2}(\mathcal{G})}\Big). \tag{2.21}$$

Now we estimate the norms of $\mathbf{X}$. It can be represented in the form

$$\begin{aligned}
\mathbf{X}(x) &= \mathbf{X}_1(x) + \nabla U(x),\\
\mathbf{X}_1(x) &= \frac{1}{4\pi}\operatorname{rot}\int\limits_\Omega \frac{\mathbf{j}(y)-\operatorname{rot}\mathbf{A}(y)}{|x-y|}\,dy,\\
\nabla^2 U(x) &= 0, \quad x\in\mathcal{F}_1\cup\mathcal{F}_2,\\
[U(x)] &= 0, \quad [\mu\frac{\partial U}{\partial N}] = -[\mu]\mathbf{X}_1\cdot\mathbf{N}, \quad x\in\mathcal{G},\\
\frac{\partial U}{\partial n}|_S &= -\mathbf{X}_1\cdot\mathbf{n}.
\end{aligned}$$

We consider the integral $\int_\Omega \mathbf{X}_1(x)\cdot\mathbf{u}(x)\,dx$ with arbitrary $\mathbf{u}\in L_2(\Omega)$. We have

$$\int\limits_\Omega \mathbf{X}_1(x)\cdot\mathbf{u}(x)dx = \frac{1}{4\pi}\int\limits_\Omega (\mathbf{j}(y)-\operatorname{rot}\mathbf{A}(y))\cdot\mathbf{W}(y)\,dy, \tag{2.22}$$

where $\mathbf{W}(y) = \int\limits_\Omega \nabla\frac{1}{|x-y|}\times\mathbf{u}(x)\,dx$. It is clear that

$$\mathbf{j}(x)-\operatorname{rot}\mathbf{A}(x) = \nabla\cdot\mathbf{F}(x) = \sum_{k=1}^{3}\frac{\partial}{\partial x_k}F_{ki}(x)_{i=1,2,3}, \tag{2.23}$$

where F_{kj} are linear combinations of J_{im} and A_m, and

$$\begin{aligned}
&\sum_{i=1}^{2}\Big(\|\mathbf{F}^{(i)}\|_{L_2(\mathcal{F}_i)} + \|\mathbf{F}^{(i)}\|_{W_2^{1/2}(\mathcal{G})}\Big)\\
&\le c\sum_{i=1}^{2}\Big(\|\mathbf{J}^{(i)}\|_{L_2(\mathcal{F}_i)} + \|\mathbf{J}^{(i)}\|_{W_2^{-1/2}(\mathcal{G})} + \|\mathbf{A}^{(i)}\|_{L_2(\mathcal{F}_i)} + \|\mathbf{A}^{(i)}\|_{W_2^{-1/2}(\mathcal{G})}\Big).
\end{aligned}$$

We substitute (2.23) in (2.22) and integrate by parts. Since J_{im} and A_m vanish on S, this leads to

$$\begin{aligned}
&\int\limits_\Omega \mathbf{X}_1(x)\cdot\mathbf{u}(x)\,dx\\
&= \frac{1}{4\pi}\sum_{i=1}^{2}\int\limits_{\mathcal{F}_i}\sum_{k,j=1}^{3}F_{kj}(y)\frac{\partial}{\partial y_k}W_j(y)dy - \int\limits_{\mathcal{G}}\mathbf{N}(y)[\mathbf{F}(y)]\cdot\mathbf{W}(y)dS_y.
\end{aligned}$$

From the trace theorem and the Calderon–Zygmund theorem it follows that

$$\begin{aligned}
&|\int_\Omega \mathbf{X}_1 \cdot \mathbf{u}\, dx| \\
&\leq c\sum_{i=1}^{2}\Big(\|\mathbf{F}^{(i)}\|_{L_2(\mathcal{F}_i)}\|\mathbf{W}\|_{W_2^1(\mathcal{F}_i)} + \|\mathbf{F}^{(i)}\|_{W_2^{-1/2}(\mathcal{G})} + \|\mathbf{W}\|_{W_2^{1/2}(\mathcal{G})}\Big) \\
&\leq c\sum_{i=1}^{2}\Big(\|\mathbf{J}^{(i)}\|_{L_2(\mathcal{F}_i)} + \|\mathbf{J}^{(i)}\|_{W_2^{-1/2}(\mathcal{G})} + \|\mathbf{A}^{(i)}\|_{L_2(\mathcal{F}_i)} + \|\mathbf{A}^{(i)}\|_{W_2^{-1/2}(\mathcal{G})}\Big) \\
&\qquad \|\mathbf{u}\|_{L_2(\Omega)},
\end{aligned}$$

hence

$$\begin{aligned}
&\|\mathbf{X}_1\|_{L_2(\Omega)} \\
&\leq c\sum_{i=1}^{2}\Big(\|\mathbf{J}^{(i)}\|_{L_2(\mathcal{F}_i)} + \|\mathbf{J}^{(i)}\|_{W_2^{-1/2}(\mathcal{G})} + \|\mathbf{A}^{(i)}\|_{L_2(\mathcal{F}_i)} + \|\mathbf{A}^{(i)}\|_{W_2^{-1/2}(\mathcal{G})}\Big).
\end{aligned} \tag{2.24}$$

To estimate $\|\mathbf{X}_1\|_{W_2^{-1/2}(\mathcal{G})}$, we consider the integral

$$\int_{\mathcal{G}} \mathbf{X}_1(x)\cdot \mathbf{v}(x) dS_x = \frac{1}{4\pi}\int_\Omega (\mathbf{j}(y) - rot\mathbf{A}(y))\cdot \mathbf{V}(y)\, dy,$$

where $\mathbf{V}(y) = \int_{\mathcal{G}} \nabla \frac{1}{|x-y|} \times \mathbf{v}(x) dS$, $\mathbf{v} \in W_2^{1/2}(\mathcal{G})$. Integrating by parts we obtain

$$\begin{aligned}
&\int_{\mathcal{G}} \mathbf{X}_1(x)\cdot \mathbf{v}(x) dS_x \\
&= \frac{1}{4\pi}\sum_{i=1}^{2}\Big(\int_{\mathcal{F}_i}\sum_{k,j=1}^{3} F_{kj}(y)\frac{\partial}{\partial y_k}V_i(y)dy - \int_{\mathcal{G}} \mathbf{N}(y)[\mathbf{F}(y)]\cdot \mathbf{V}(y) dS_y.
\end{aligned}$$

Since

$$\|\mathbf{V}\|_{W_2^1(\mathcal{F}_i)} \leq c\|\mathbf{v}\|_{W_2^{1/2}(\mathcal{G})},$$

(see [10]), we have

$$|\int\limits_{\mathcal{G}} \mathbf{X}_1(x)\cdot\mathbf{v}(x)dS_x| \le c\|\mathbf{v}\|_{W_2^{1/2}\mathcal{G})}\sum_{i=1}^{2}\Big(\|\mathbf{F}^{(i)}\|_{L_2(\mathcal{F}_i)} + \|\mathbf{F}^{(i)}\|_{W_2^{-1/2}(\mathcal{G})}\Big),$$

which implies

$$\|\mathbf{X}_1\|_{W_2^{-1/2}(\mathcal{G})} \le c\sum_{i=1}^{2}\Big(\|\mathbf{J}^{(i)}\|_{L_2(\mathcal{F}_i)} + \|\mathbf{J}^{(i)}\|_{W_2^{-1/2}(\mathcal{G})} + \|\mathbf{A}^{(i)}\|_{L_2(\mathcal{F}_i)} + \|\mathbf{A}^{(i)}\|_{W_2^{-1/2}(\mathcal{G})}\Big). \tag{2.25}$$

Finally, ∇U can be estimated as follows:

$$\|\nabla U\|_{L_2(\Omega)} \le c\Big(\|\mathbf{X}_1\cdot\mathbf{N}\|_{W_2^{-1/2}(\mathcal{G})} + \|\mathbf{X}_1\cdot\mathbf{n}\|_{W_2^{-1/2}(S)}\Big) \le c\|\mathbf{X}_1\|_{L_2(\Omega)},$$

$$\|\nabla U\|_{W_2^{-1/2}(\mathcal{G})} \le \|\nabla_\tau U\|_{W_2^{-1/2}(\mathcal{G})} + \|\mathbf{N}\frac{\partial U}{\partial N}\|_{W_2^{-1/2}(\mathcal{G})}$$
$$\le c\Big(\|U\|_{W_2^{1/2}(\mathcal{G})} + \|\mathbf{X}_1\cdot\mathbf{N}\|_{W_2^{-1/2}(\mathcal{G})} + \|\mathbf{X}_1\cdot\mathbf{n}\|_{W_2^{-1/2}(S)}\Big) \le c\|\mathbf{X}_1\|_{L_2(\Omega)}. \tag{2.26}$$

Inequality (2.17) follows from (2.21), (2.24)–(2.26). The theorem is proved.

Corollary. *Assume that* $\mathbf{j}$, $\mathbf{a}$ *in* (2.3), *as well as* $\mathbf{A}$, $\mathbf{J}^{(i)}$ *in* (2.16) *depend on* $t\in(0,T)$, $\mathbf{j}^{(i)}\in W_2^{l+1,0}(Q_T^i)$, $\mathbf{a}\in W_2^{l+3/2,0}(G_T)$, $\mathbf{j}^{(i)}\in W_2^{l}(\mathcal{F}_i)$, $\mathbf{a}\in W_2^{l+1/2}(\mathcal{G})$, $\forall t<T$,

$$\mathbf{A}_t^{(i)} \in W_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G})), \quad \mathbf{J}^{(i)} \in W_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G})).$$

Then

$$\begin{aligned}
&\sum_{i=1}^{2}\Big(\|\mathbf{h}^{(i)}\|_{H^{l+2,l/2+1}(Q_T^i)} + \sup_{t<T}\|\mathbf{h}^{(i)}(\cdot,t)\|_{W_2^{l+1}(\mathcal{F}_i)} \\
&+ \|\mathbf{h}_t^{(i)}\|_{\widehat{W}_2^{l/2}(0,T:W_2^{-1/2}\mathcal{G}))}\Big) \\
&\le c\sum_{i=1}^{2}\Big(\|\mathbf{j}^{(i)}\|_{W_2^{l+1,0}(Q_T^i)} + \sup_{t<T}\|\mathbf{j}^{(i)}(\cdot,t)\|_{W_2^{l}(\mathcal{F}_i)} \\
&+ \|\mathbf{J}_t^{(i)}\|_{\widehat{W}_2^{0,l/2}(Q_T^i)} + \|\mathbf{A}_t^{(i)}\|_{\widehat{W}_2^{0,l/2}(Q_T^i)} \\
&+ \|\mathbf{J}_t^{(i)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))} + \|\mathbf{A}_t^{(i)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\Big) \\
&+ c\Big(\|\mathbf{a}\|_{W_2^{l+3/2,0}(G_T)} + \sup_{t<T}\|\mathbf{a}\|_{W_2^{l+1/2}(\mathcal{G})}\Big).
\end{aligned} \tag{2.27}$$

Indeed, by virtue of (2.8) we have

$$\sum_{i=1}^{2}\Big(\|\mathbf{h}^{(i)}\|_{W_2^{l+2,0}(Q_T^i)}+\sup_{t<T}\|\mathbf{h}^{(i)}(\cdot,t)\|_{W_2^{l+1}(\mathcal{F}_i)}\Big)$$
$$\le c\Big(\sum_{i=1}^{2}\big(\|\mathbf{j}^{(i)}\|_{W_2^{l+1,0}(Q_T^i)}+\sup_{t<T}\|\mathbf{j}^{(i)}(\cdot,t)\|_{W_2^{l}(\mathcal{F}_i)}\big)$$
$$+\|\mathbf{a}\|_{W_2^{l+3/2,0}(G_T)}+\sup_{t<T}\|\mathbf{a}(\cdot,t)\|_{W_2^{l+1/2}(\mathcal{G})}\Big).$$

Now we differentiate (2.3) and take the finite differences of both sides of the resulting equation with respect to time. Applying (2.17) to $\mathbf{h}_t$ and to $\mathbf{h}_t(x,t)-\mathbf{h}_t(x,t-h)$, $t>h$, we obtain

$$\sum_{i=1}^{2}\Big(\|\mathbf{h}_t^{(i)}\|_{\widehat{W}_2^{0,l/2}(Q_T^i)}+\|\mathbf{h}_t^{(i)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\le c\sum_{i=1}^{2}\Big(\|\mathbf{J}_t^{(i)}\|_{\widehat{W}_2^{0,l/2}(Q_T^i)}$$
$$+\|\mathbf{A}_t^{(i)}\|_{\widehat{W}_2^{0,l/2}(Q_T^i)}+\|\mathbf{J}_t^{(i)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))}+\|\mathbf{A}_t^{(i)}\|_{W_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\Big).$$

This completes the proof of (27).

Now we turn to the problem (2.2). At first we consider the case $\mathbf{j}=0$, $\mathbf{a}=0$.

Theorem 4. *Assume that the data of the problem* (2.2) *possess the following properties:* $\mathbf{j}=0$, $\mathbf{a}=0$, $\mathbf{G}\in W_2^{l,l/2}(Q_T^1)$, $\mathbf{H}_0\in W_2^{l+1}(\mathcal{F}_1)\cap W_2^{l+1}(\mathcal{F}_2)$ *and the compatibility conditions*

$$\nabla\cdot\mathbf{G}(x,t)=0,\quad x\in\mathcal{F}_1,\quad \nabla\cdot\mathbf{H}_0(x)=0,\quad x\in\mathcal{F}_1\cup\mathcal{F}_2,$$
$$\operatorname{rot}\mathbf{H}_0(x)=0,\ x\in\mathcal{F}_2,\ \ [\mu\mathbf{H}_0\cdot\mathbf{N}]=0,\ [\mathbf{H}_{0\tau}]=0,\ x\in\mathcal{G},\ \mathbf{H}_0\cdot\mathbf{n}|_S=0$$

are satisfied. Then the problem (2.2) *has a unique solution* $\mathbf{H}\in W_2^{l+2,l/2+1}(Q_T^1)\cap W_2^{l+2,l/2+1}(Q_T^2)$, *with* $\mathbf{H}_t^{(i)}\in W_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))$, $i=1,2$, *and*

$$\sum_{i=1}^{2}\Big(\|\mathbf{H}^{(i)}\|_{H^{l+2,l/2+1}(Q_T^i)}+\|\mathbf{H}_t^{(i)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\Big)$$
$$\le c\Big(\|\mathbf{G}\|_{\widehat{W}_2^{l,l/2}(Q_T^1)}+\|\mathbf{H}_0\|_{W_2^{l+1}(\mathcal{F}_1)}\Big). \tag{2.29}$$

Proof. The problem (2.2) has been already studied in [4] in the case $l=0$. We reproduce the proof of (2.29). Let $\mathcal{H}^{(k)}(\Omega)$, $k=1,2$, be the space of divergence free vector fields $\psi \in W_2^k(\mathcal{F}_i)$, $i = 1, 2$, satisfying the equation $rot\psi = 0$ in $\mathcal{F}_2$ and the boundary conditions

$$[\mu\psi \cdot \mathbf{N}]|_{\mathcal{G}} = 0, \quad [\psi_\tau]|_{\mathcal{G}} = 0, \quad \psi \cdot \mathbf{n}|_S = 0.$$

The space $\mathcal{H}^{(0)}(\Omega)$ we define as the closure of $\mathcal{H}^{(1)}(\Omega)$ in the norm $\|\psi\|_{L_2(\Omega)}$. Since $\mathcal{F}_1$ and Ω are simply connected, every element $\psi \in \mathcal{H}^{(1)}(\Omega)$ equals $\nabla\varphi$ in $\mathcal{F}_2$, where φ is a solution of

$$\nabla^2\varphi(x,t) = 0, \quad x \in \mathcal{F}_2, \quad \mu_2 \frac{\partial\varphi}{\partial N}\Big|_{\mathcal{G}} = \mu_1\psi^{(1)} \cdot \mathbf{N}, \quad \frac{\partial\varphi}{\partial n}\Big|_S = 0. \quad (2.30)$$

If $\psi \in \mathcal{H}^{(0)}(\Omega)$, then $\psi^{(1)}(x)$ is an arbitrary solenoidal vector field from $L_2(\mathcal{F}_1)$ and $\psi^{(2)} = \nabla\varphi(x)$ in $\mathcal{F}_2$ where φ is a weak solution of the problem (2.30), i.e.,

$$\mu_2 \int\limits_{\mathcal{F}_2} \nabla\varphi \cdot \nabla\eta \, dx = -\mu_1 \int\limits_{\mathcal{F}_1} \psi^{(1)}(x) \cdot \nabla\eta(x) dx, \quad \forall\eta \in W_2^1(\Omega).$$

The boundary condition $\nabla_\tau\varphi = \psi_\tau$ on $\mathcal{G}$ has no sense for $\psi \in \mathcal{H}^{(0)}(\Omega)$.

A weak solution of (2.2) can be defined as an element of

$$W_2^1(0,T; L_2(\Omega)) \cap L_2(0,T; \mathcal{H}^{(1)}(\Omega))$$

satisfying the integral identity

$$\int\limits_0^T \int\limits_\Omega \mu\mathbf{H}_t \cdot \psi \, dx \, dt + \alpha^{-1} \int\limits_0^T \int\limits_{\mathcal{F}_1} \mathrm{rot}\,\mathbf{H} \cdot \mathrm{rot}\,\psi \, dx \, dt = \int\limits_0^T \int\limits_\Omega \mathbf{G}^* \cdot \psi \, dx \, dt, \quad (2.31)$$

$\forall\psi \in L_2(0,T; \mathcal{H}^{(1)}(\Omega))$, and the initial condition $\mathbf{H}(x,0) = \mathbf{H}_0(x)$. By $\mathbf{G}^*$ we mean the extension of $\mathbf{G}$ into $\mathcal{F}_2$ equal to $\nabla\varphi$ where $\nabla\varphi$ is a weak solution of (2.30) with ψ replaced by $\mathbf{G}$.

In order to obtain (2.31), we associate to $\psi \in \mathcal{H}^{(1)}$ a function $\Phi(x)$, $x \in \mathcal{F}_1$, that is a solution of

$$\nabla^2\Phi(x) = 0, \quad x \in \mathcal{F}_1, \quad \Phi(x) = \varphi(x), \quad x \in \mathcal{G}. \quad (2.32)$$

Multiplying the first equation in (2.2) by $\psi - \nabla\Phi$ and integrating we obtain

$$\int_0^T \int_{\mathcal{F}_1} \mu_1 \mathbf{H}_t \cdot (\psi - \nabla\Phi)\, dx\, dt + \alpha^{-1} \int_0^T \int_{\mathcal{F}_1} \operatorname{rot} \mathbf{H} \cdot \operatorname{rot}(\psi - \nabla\Phi)\, dx\, dt$$

$$= \int_0^T \int_{\mathcal{F}_1} \mathbf{G} \cdot (\psi - \nabla\Phi)\, dx\, dt, \tag{2.33}$$

since $(\psi - \nabla\Phi)_\tau|_{\mathcal{G}} = 0$.

Let us verify that (2.33) implies (2.31). We consider the last integral in (2.33). By (2.30) and (2.32),

$$\int_{\mathcal{F}_1} \mathbf{G} \cdot (\psi - \nabla\Phi)\, dx = \int_{\mathcal{F}_1} \mathbf{G} \cdot \psi dx - \int_{\mathcal{G}} \mathbf{G} \cdot \mathbf{N}\Phi\, dS$$

$$= \int_{\mathcal{F}_1} \mathbf{G} \cdot \psi\, dx - \int_{\mathcal{G}} \mathbf{G} \cdot \mathbf{N}\varphi\, dS = \int_{\Omega} \mathbf{G}^* \cdot \psi\, dx.$$

In the same way the relation

$$\int_{\mathcal{F}_1} \mu_1 \mathbf{H}_t \cdot (\psi - \nabla\Phi)\, dx = \int_{\Omega} \mu \mathbf{H}_t \cdot \psi\, dx$$

is verified. Hence (2.33) is equivalent to (2.31).

The existence of a weak solution satisfying (2.31) can be proved by Galerkin's method (see [11]), and it is easily seen that

$$\|\mathbf{H}_t\|_{L_2(Q_T)} + \|\operatorname{rot} \mathbf{H}\|_{L_2(Q_T^1)} \le c\Big(\|\mathbf{G}^*\|_{L_2(Q_T)} + \|\operatorname{rot} \mathbf{H}_0\|_{L_2(\mathcal{F}_1)}\Big). \tag{2.34}$$

Since

$$\sum_{i=1}^{2} \|\mathbf{H}\|_{W_2^1(\mathcal{F}_i)} \le c\|\operatorname{rot} \mathbf{H}\|_{L_2(\mathcal{F}_1)},$$

inequality (2.34) furnishes the estimate

$$\|\mathbf{H}_t\|_{L_2(Q_T)} + \sum_{i=1}^{2} \|\mathbf{H}^{(i)}\|_{W_2^{1,0}(Q_T)} \le c\Big(\|\mathbf{G}\|_{L_2(Q_T^1)} + \|\mathbf{H}_0\|_{W_2^1(\mathcal{F}_1)}\Big).$$

From the continuity of the tangential component of $\mathbf{H} \in L_2(0,T;\mathcal{H}^{(1)}(\Omega))$ it follows that $[\operatorname{rot}\mathbf{H}]\cdot\mathbf{N}|_{x\in\mathcal{G}} = 0$ and $\operatorname{rot}\mathbf{H}\cdot\mathbf{N}|_{x\in\mathcal{G}} = 0$

To estimate the second derivatives of $\mathbf{H}$, we set $\xi = \operatorname{rot}\mathbf{H}$ and introduce $\xi' \in L_2(0,T;W_2^1(\mathcal{F}_1))$ as the solution of

$$\operatorname{rot}\xi' = \alpha(\mathbf{G}-\mu_1\mathbf{H}_t), \quad \nabla\cdot\xi' = 0, \quad x\in\mathcal{F}_1, \quad \xi'\cdot\mathbf{N} = 0 \quad x\in\mathcal{G}. \tag{2.35}$$

In view of (2.33), we have

$$\int_{\mathcal{F}_1} \operatorname{rot}\xi'\cdot(\psi-\nabla\Phi)\,dx = \alpha\int_{\mathcal{F}_1}(\mathbf{G}-\mu_1\mathbf{H}_t)\cdot(\psi-\nabla\Phi)\,dx = \int_{\mathcal{F}_1}\xi\cdot\operatorname{rot}\psi\,dx,$$

which implies $\int_0^T\int_{\mathcal{F}_1}(\xi-\xi')\cdot\operatorname{rot}\psi\,dx\,dt = 0$. By Theorem 7.3 in [4], arbitrary divergence free vector field $\mathbf{u}\in L_2(\mathcal{F}_1)$ with $\mathbf{u}\cdot\mathbf{N}|_{\mathcal{G}} = 0$ can be represented as $\operatorname{rot}\psi$, $\psi\in\mathcal{H}^{(1)}(\Omega)$, so we can conclude that $\xi-\xi' = 0$, which means that $\xi = \operatorname{rot}\mathbf{H}\in L_2(0,T;W_2^1(\mathcal{F}_1))$. Hence $\mathbf{H}$ can be regarded as a solution of the problem (2.3) with $\mathbf{j}^{(1)} = \xi$, $\mathbf{j}^{(2)} = 0$, $\mathbf{a} = 0$. By (2.8) (with $l = 0$),

$$\sum_{i=1}^2 \|\mathbf{H}\|_{W_2^{2,0}(Q_T^i)} \le c\|\operatorname{rot}\mathbf{H}\|_{W_2^{1,0}(Q_T)} \le c\Big(\|\mathbf{H}_0\|_{W_2^1(\mathcal{F}_1)} + \|\mathbf{G}\|_{L_2(Q_T^1)}\Big),$$

hence

$$\sum_{i=1}^2 \|\mathbf{H}\|_{W_2^{2,1}(Q_T^i)} \le c\Big(\|\mathbf{G}\|_{L_2(Q_T)} + \|\mathbf{H}_0\|_{W_2^1(\mathcal{F}_1)}\Big). \tag{2.36}$$

Now we pass to the proof of (2.29). We reduce our problem to a similar problem with zero initial data. We construct a solenoidal vector field $\mathbf{B}_1^{(1)}(x,t)$, $x\in\mathcal{F}_1$, $t>0$, such that

$$\mathbf{B}^{(1)}(x,0) = \mathbf{H}_0^{(1)}(x)$$

and

$$\|\mathbf{B}^{(1)}\|_{H^{2+l,1+l/2}(Q_T)} \le c\|\mathbf{B}^{(1)}\|_{W_2^{2+l,1+l/2}(Q_\infty^1)} \le c\|\mathbf{H}_0\|_{W_2^{l+1}(\mathcal{F}_1)}.$$

This can be done as follows: at first we extend $\mathbf{H}_0^{(1)}$ into $\mathbb{R}^3$ with preservation of class and solenoidality (to obtain a solenoidal extension, we use the result of Bogovskii [12]), and then we find a solenoidal $\mathbf{B}_1^{(1)}(x,t)$, $x \in \mathbb{R}^3$, $t > 0$, using the heat kernel and the cut-off function of t, as in [13]. The second step is to construct $\mathbf{B}_1^{(2)}(x,t) = \nabla\chi(x,t)$, $x \in \mathcal{F}_2$, as a solution of the problem

$$\nabla^2\chi(x,t) = 0,\ x \in \mathcal{F}_2,\ \mu_2\frac{\partial\chi}{\partial N} = \mu_1\mathbf{B}_1^{(1)} \cdot \mathbf{N}(x),\ x \in \mathcal{G},\ \frac{\partial\chi}{\partial n}\Big|_{x\in S} = 0.$$

The function χ satisfies the inequalities

$$\|\nabla\chi(\cdot,t)\|_{W_2^{2+l}(\mathcal{F}_2)} \le c\|\mathbf{B}_1^{(1)} \cdot \mathbf{N}\|_{W_2^{3/2+l}(\mathcal{G})},$$
$$\|\nabla\chi(\cdot,t)\|_{L_2(\mathcal{F}_2)} + \|\nabla\chi\|_{W_2^{-1/2}(\mathcal{G})} \le c\|\mathbf{B}_1^{(1)} \cdot \mathbf{N}\|_{W_2^{-1/2}(\mathcal{G})} \le c\|\mathbf{B}_1^{(1)}\|_{L_2(\mathcal{F}_1)},$$

which imply

$$\|\mathbf{B}_1^{(2)}(\cdot,t)\|_{H^{2+l,1+l/2}(Q_T^2)} + \|\mathbf{B}_{1t}^{(2)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))}$$
$$\le c\|\mathbf{B}_1^{(1)}\|_{H^{2+l,l+1/2}(Q_T^1)} \le c\|\mathbf{H}_0\|_{W_2^{l+1}(\mathcal{F}_1)}.$$

The solenoidal extension $\mathbf{B}_1$ of $\mathbf{H}_0$ obtained in this way does not satisfy the condition $[\mathbf{B}_{1\tau}] = 0$ on $\mathcal{G}$. Let $\mathbf{b} = [\mathbf{B}_{1\tau}]$. It is clear that $\mathbf{b}(x,0) = 0$ and $\mathbf{b} \in W_2^{l+3/2,l/2+3/4}(G_T)$. Using the inverse trace theorem for anisotropic Sobolev spaces, we can construct $\mathbf{B}_2^{(1)}(x,t)$ such that

$$\mathbf{B}_2^{(1)}(x,0) = 0 \quad \text{for} \quad t < 0, \quad \mathbf{B}_2^{(1)}(x,t)|_{x\in\mathcal{G}} = -\mathbf{b}(x,t) = -[\mathbf{B}_{1\tau}]$$

and

$$\|\mathbf{B}_2^{(1)}\|_{W_2^{2+l,1+l/2}(Q^1_{-\infty,T})} \le c\|\mathbf{b}\|_{W_2^{l+3/2,l/2+3/4}(G_{-\infty,T})}$$
$$\le c\|\mathbf{b}\|_{H^{l+3/2,l/2+3/4}(G_T)}$$
$$\le c\sum_{i-1}^{2}\|\mathbf{B}_1^{(i)}\|_{H^{2+l,1+l/2}(Q_T^i)} \le c\|\mathbf{B}_1^{(1)}\|_{H^{2+l,l+1/2}(Q_T^1)}.$$

We set $\mathbf{B}_2^{(2)}(x,t) = 0$.

Finally, we find $\mathbf{B}_3^{(1)}(x,t)$ satisfying the relations

$$\nabla \cdot \mathbf{B}_3^{(1)}(x,t) = -\nabla \cdot \mathbf{B}_2^{(1)}(x,t), \quad x \in \mathcal{F}_1, \quad \mathbf{B}_3^{(1)}(x,t)|_{x\in\mathcal{G}} = 0.$$

It is well known that there exists such $\mathbf{B}_3^{(1)}$ that

$$\|\mathbf{B}_3^{(1)}\|_{W_2^{2+l}(\mathcal{F}_1)} \leq c\|\mathbf{B}_2^{(1)}\|_{W_2^{2+l}(\mathcal{F}_1)}, \quad \|\mathbf{B}_3^{(1)}\|_{L_2(\mathcal{F}_1)} \leq c\|\mathbf{B}_2^{(1)}\|_{L_2(\mathcal{F}_1)},$$

hence

$$\|\mathbf{B}_3^{(1)}\|_{H^{2+l,1+l/2}(Q_T^1)} \leq c\|\mathbf{B}_2^{(1)}\|_{H^{2+l,1+l/2}(Q_T^1)}.$$

We set $\mathbf{B}_3^{(2)}(x,t) = 0$ for $x \in \mathcal{F}_2$.

It follows that $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3$ satisfies the conditions $\mathbf{B}(x,0) = \mathbf{H}_0$ for $x \in \mathcal{F}_1 \cup \mathcal{F}_2$,

$$[\mu\mathbf{B}\cdot\mathbf{N}]|_{x\in\mathcal{G}} = 0, \quad [\mathbf{B}_\tau]|_{x\in\mathcal{G}} = 0, \quad \mathbf{B}\cdot\mathbf{n}|_{x\in S} = 0,$$

$\nabla\cdot\mathbf{B} = 0$ in $\mathcal{F}_1 \cup \mathcal{F}_2$ and the inequality

$$\sum_{i=1}^2 \|\mathbf{B}^{(i)}\|_{H^{2+l,1+l/2}(Q_T^i)} \leq c\|\mathbf{H}_0\|_{W_2^{l+1}(\mathcal{F}_1)}.$$

Moreover, since $\mathbf{B}_\tau^{(1)} = \mathbf{B}_\tau^{(2)} = \nabla_\tau \chi$ on $\mathcal{G}$, we have

$$\begin{aligned}
&\|\mathbf{B}_t^{(i)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))} \leq \|\nabla_\tau\chi_t\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\\
&+\|\mathbf{B}_t^{(i)}\cdot\mathbf{N}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\\
&\leq c\big(\|\chi_t\|_{\widehat{W}_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))} + \|\mathbf{B}_t^{(i)}\|_{\widehat{W}_2^{0,l/2}(Q_t^i)}\big) \leq c\sum_{i=1}^2 \|\mathbf{B}^{(i)}\|_{H^{l+1,l/2+1}(Q_T^i)}\\
&\leq c\|\mathbf{H}_0\|_{W_2^{l+1}(\mathcal{F}_1)}.
\end{aligned}$$

The vector field $\mathbf{H}' = \mathbf{H} - \mathbf{B}$ is a solution of

$$\begin{aligned}
&\mu_1\mathbf{H}'_t + \alpha^{-1}\operatorname{rot}\operatorname{rot}\mathbf{H}' = \mathbf{G}', \quad \nabla\cdot\mathbf{H}' = 0, \quad x\in\mathcal{F}_1,\\
&\operatorname{rot}\mathbf{H}' = 0, \quad \nabla\cdot\mathbf{H}' = 0, \quad x\in\mathcal{F}_2,\\
&[\mu\mathbf{H}'\cdot\mathbf{N}] = 0, \quad [\mathbf{H}'_\tau] = 0, \quad \mathbf{H}'\cdot\mathbf{n}|_{x\in S} = 0,\\
&\mathbf{H}'(x,0) = 0, \quad x\in\mathcal{F}_1,
\end{aligned} \tag{2.37}$$

with $\mathbf{G}' = \mathbf{G} - \mu_1\mathbf{B}_t - \alpha^{-1}\operatorname{rot}\operatorname{rot}\mathbf{B}$; hence the proof of (2.29) reduces to the proof of

$$\sum_{i=1}^2 \Big(\|\mathbf{H}'^{(i)}\|_{H^{l+2,l/2+1}(Q_T^i)} + \|\mathbf{H}_t'^{(i)}\|_{\widehat{W}_2^{l/2}(0,T);W_2^{-1/2}(\mathcal{G}))}\Big) \leq c\|\mathbf{G}'\|_{\widehat{W}_2^{l,l/2}(Q_T^1)}. \tag{2.38}$$

From (2.36) it follows that $\mathbf{H}'$ satisfies (2.29) with $l = 0$. We extend $\mathbf{H}'$ and $\mathbf{G}'$ by zero into the domain $t < 0$, and we take the first finite difference with respect to time $\Delta_t(-h)$ of the first equation in (2.37). Then we multiply the equation by $\Delta_t(-h)(\mathbf{H}'_t - \nabla\Phi_t)$ and integrate it over $Q^1_{-\infty,T}$. This leads to

$$\int_{-\infty}^{T}\int_{\Omega} \mu|\Delta_t(-h)\mathbf{H}'_t|^2\,dx\,dt + \alpha^{-1}\int_{-\infty}^{T}\int_{\mathcal{F}_1} \operatorname{rot}\Delta_t(-h)\mathbf{H}'\cdot\operatorname{rot}\Delta_t(-h)\mathbf{H}'_t\,dx\,dt$$
$$= \int_{-\infty}^{T}\int_{\Omega} \mu|\Delta_t(-h)\mathbf{H}'_t|^2\,dx\,dt + \frac{1}{2\alpha}\int_{\mathcal{F}_1} |\operatorname{rot}\Delta_t(-h)\mathbf{H}'|^2|_{t=T}\,dx$$
$$= \int_{-\infty}^{T}\int_{\Omega} \Delta_t(-h)\mathbf{G}'^{*}\cdot\Delta_t(-h)\mathbf{H}'_t\,dx\,dt. \tag{2.39}$$

From this equation and from (2.31) we conclude that

$$\|\mathbf{H}'_t\|_{\widehat{W}_2^{0,l/2}(Q^1_T)} \le c\|\mathbf{G}'^{*}\|_{\widehat{W}_2^{0,l/2}(Q_T)} \le c\|\mathbf{G}'\|_{\widehat{W}_2^{0,l/2}(Q^1_T)}; \tag{2.40}$$

moreover, using the boundary conditions for $\mathbf{H}'$ on $\mathcal{G}$ we easily obtain

$$\|\mathbf{H}'_t\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))} \le c\|\mathbf{H}'_t\|_{\widehat{W}_2^{0,l/2}(Q^1_T)} \le c\|\mathbf{G}'\|_{\widehat{W}_2^{0,l/2}(Q^1_T)}. \tag{2.41}$$

Now we can conclude the proof of (2.29). We restrict ourselves with formal calculations. Since

$$\operatorname{rotrot}\mathbf{H}' = \alpha(\mathbf{G}' - \mu_1\mathbf{H}'_t), \quad x\in\mathcal{F}_1, \quad \operatorname{rot}\mathbf{H}'\cdot\mathbf{N}|_{\mathcal{G}} = 0,$$

we have

$$\sum_{i=1}^{2}\|\mathbf{H}'^{(i)}\|_{W_2^{2+l,0}(Q^i_{-\infty,T})} \le c\|\operatorname{rot}\mathbf{H}'\|_{W_2^{l+1,0}(Q^1_{-\infty,T})}$$
$$\le c\Big(\|\mathbf{G}'\|_{W_2^{l,0}(Q^1_{-\infty,T})} + \|\mathbf{H}'_t\|_{W_2^{l,0}(Q^1_{-\infty,T})}\Big).$$

We estimate the norm of $\mathbf{H}'_t$ by the interpolation inequality

$$\|\mathbf{H}'_t\|_{W_2^{l,0}(Q^1_{-\infty,T})} \le \epsilon\|\mathbf{H}'\|_{W_2^{l+2,0}(Q^1_{-\infty,T})} + c(\epsilon)\|\mathbf{H}'\|_{W_2^{0,1+l/2}(Q^1_{-\infty,T})}$$

with a small $\epsilon > 0$ (see [14]). Taking (2.40) into account, we obtain

$$\|\mathbf{H}'\|_{H^{l+2,l/2+1}(Q^1_T)} \le c\|\mathbf{G}'\|_{\widehat{W}_2^{l,l/2}(Q_T)}.$$

Together with (2.41), this inequality implies (2.38). In the general case the following proposition holds:

Theorem 5. *Assume that the data of the problem* (2.2) *possess the following properties:* $\mathbf{G} \in W_2^{l,l/2}(Q_T^1)$, $\mathbf{H}_0 \in W_2^{l+1}(\mathcal{F}_1) \cap W_2^{l+1}(\mathcal{F}_2)$, $\mathbf{j} = \mathbf{j}^{(2)} \in W_2^{l+1,(l+1)/2}(Q_T^2)$, $\mathbf{a} \in W_2^{l+3/2,l/2+3/4}(G_T)$, *moreover, assume that*

$$\mathbf{j}^{(2)} = \operatorname{rot} \mathbf{J}^{(2)}, \quad \mathbf{a} = [\mathbf{A}]$$

with $\mathbf{J}^{(2)} \in W_2^{l+2,l/2+1}(Q_T^2)$, $\mathbf{J}_t^{(2)} \in W_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))$, $\mathbf{A}^{(i)} \in W_2^{l+2,l/2+1}(Q_T^i)$, $\mathbf{A}_t^{(i)} \in W_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))$, $i = 1,2$, *and that the compatibility conditions*

$$\begin{aligned} &\nabla \cdot \mathbf{G}(x,t) = 0, \quad x \in \mathcal{F}_1, \quad \mathbf{a}(x,t) \cdot \mathbf{N}(x) = \mathbf{A}^{(i)} \cdot \mathbf{N} = 0, \quad x \in \mathcal{G}, \\ &\nabla \cdot \mathbf{H}_0(x) = 0, \quad x \in \mathcal{F}_1 \cup \mathcal{F}_2, \quad \operatorname{rot} \mathbf{H}_0(x) = \mathbf{j}^{(2)}(x,0), \quad x \in \mathcal{F}_2, \\ &[\mu \mathbf{H}_0 \cdot \mathbf{N}] = 0, \quad [\mathbf{H}_{0\tau}] = \mathbf{a}(x,0), \quad x \in \mathcal{G}, \quad \mathbf{H}_0 \cdot \mathbf{n}|_S = 0 \end{aligned} \tag{2.42}$$

are satisfied. Then the problem (2.2) *has a unique solution such that*

$$\begin{aligned} \mathbf{H} &\in W_2^{l+2,l/2+1}(Q_T^1) \cap W_2^{l+2,l/2+1}(Q_T^2), \\ \mathbf{H}_t^{(i)} &\in W_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G})), \quad i = 1,2, \end{aligned}$$

and

$$\begin{aligned} &\sum_{i=1}^{2} \Big(\|\mathbf{H}^{(i)}\|_{H^{l+2,l/2+1}(Q_T^i)} + \|\mathbf{H}_t^{(i)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))} \Big) \\ &\leq c \Big(\|\mathbf{G}\|_{\widehat{W}_2^{l,l/2}(Q_T^1)} + \|\mathbf{H}_0\|_{W_2^{l+1}(\mathcal{F}_1)} \\ &+ \|\mathbf{a}\|_{W_2^{l+3/2,0}(G_T)} + \sup_{t<T} \|\mathbf{a}(\cdot,t)\|_{W_2^{l+1/2}(\mathcal{G})} + \|\mathbf{j}\|_{W_2^{l+1,0}(Q_T^2)} \\ &+ \sup_{t<T} \|\mathbf{j}\|_{W_2^l(\mathcal{F}_2)} + \|\mathbf{J}_t^{(2)}\|_{\widehat{W}_2^{0,l/2}(Q_T^2)} \\ &+ \|\mathbf{J}_t^{(2)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))} + \|\mathbf{A}_t\|_{\widehat{W}_2^{0,l/2}(Q_T)} \\ &+ \sum_{i=1}^{2} \|\mathbf{A}_t^{(i)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}\mathcal{G}))} \Big) \end{aligned} \tag{2.43}$$

with the constant independent of T, *when* T *is bounded.*

Proof. We reduce the problem (2.2) to a similar problem with $\mathbf{j} = 0$, $\mathbf{a} = 0$. To this end, we construct the auxiliary vector field $\mathbf{h}$ as a solution of the problem (2.3). We find $\mathbf{j}^{(1)}$ satisfying the necessary compatibility conditions (2.7). We set $\mathbf{j}^{(1)}\cdot\mathbf{N}=\mathbf{j}^{(2)}\cdot\mathbf{N}-\mathbf{N}\cdot rot\mathbf{a}$ on $\mathcal{G}$ (the normal component of $rot\mathbf{a}$ depends only on $\mathbf{a}|_{x\in\mathcal{G}}$) and $\mathbf{j}^{(1)}=\nabla J^{(1)}$, where $J^{(1)}$ is a solution of

$$\nabla^2 J^{(1)}(x,t)=0,\quad x\in\mathcal{F}_1,\quad \frac{\partial}{\partial N}J^{(1)}|_{x\in\mathcal{G}}=\mathbf{j}^{(1)}\cdot\mathbf{N}.$$

The necessary compatibility condition is satisfied, and we require that $\int\limits_{\mathcal{F}_1} J^{(1)}\,dx=0$. We have

$$\begin{aligned}&\|\nabla J^{(1)}(\cdot,t)\|_{W_2^{l+1}(\mathcal{F}_1)}\le c\|(\mathbf{j}^{(2)}-\operatorname{rot}\mathbf{a})\cdot\mathbf{N}\|_{W_2^{l+1/2}(\mathcal{G})}\\ &\le c\Big(\|\mathbf{j}^{(2)}\|_{W_2^{l+1}(\mathcal{F}_2)}+\|\mathbf{a}\|_{W_2^{l+3/2}(\mathcal{G})}\Big).\end{aligned}$$

Now we show that

$$\|J^{(1)}\|_{L_2(\mathcal{F}_1)}+\|J^{(1)}\|_{W_2^{-1/2}(\mathcal{G})}\le c\Big(\|\mathbf{J}^{(2)}\|_{W_2^{-1/2}(\mathcal{G})}+\|\mathbf{a}\|_{W_2^{-1/2}(\mathcal{G})}\Big).\quad(2.44)$$

We consider the problems

$$\begin{aligned}\nabla^2 u(x)=J^{(1)}(x,t),\quad x\in\mathcal{F}_1,\quad \frac{\partial u}{\partial N}\Big|_{\mathcal{G}}=0,\\ \nabla^2 v(x)=0,\quad x\in\mathcal{F}_1,\quad \frac{\partial v}{\partial N}\Big|_{\mathcal{G}}=g\end{aligned}$$

with $g\in W_2^{1/2}(\mathcal{G})$, $\int_{\mathcal{G}} g dS=0$. By the Green identity,

$$\begin{aligned}\int\limits_{\mathcal{F}_1}(J^{(1)})^2(x,t)\,dx&=-\int\limits_{\mathcal{G}}\frac{\partial J^{(1)}}{\partial N}u dS=-\int\limits_{\mathcal{G}}u(x)\mathbf{N}\cdot rot(\mathbf{J}^{(2)}-\mathbf{a})\,dS\\ &=\int\limits_{\mathcal{G}}\mathbf{N}\cdot(\nabla u\times(\mathbf{J}^{(2)}-\mathbf{a}))\,dS,\\ \int\limits_{\mathcal{G}}J^{(1)}(x,t)g(x)\,dS&=\int\limits_{\mathcal{G}}\frac{\partial}{\partial N}J^{(1)}(x,t)v(x)\,dS\\ &=-\int\limits_{\mathcal{G}}\mathbf{N}\cdot(\nabla v\times(\mathbf{J}^{(2)}-\mathbf{a}))\,dS.\end{aligned}$$

From these relations and from the coercive estimates of u and v we conclude that

$$\Big|\int_{\mathcal{F}_1}(J^{(1)})^2(x,t)dx\Big| \le c\|u\|_{W_2^2(\mathcal{F}_1)}\Big(\|\mathbf{J}^{(2)}\|_{W_2^{-1/2}(\mathcal{G})}+\|\mathbf{a}\|_{W_2^{-1/2}(\mathcal{G})}\Big)$$

$$\le c\|J^{(1)}\|_{L_2(\mathcal{F}_1)}\Big(\|\mathbf{J}^{(2)}\|_{W_2^{-1/2}(\mathcal{G})}+\|\mathbf{a}\|_{W_2^{-1/2}(\mathcal{G})}\Big),$$

$$\Big|\int_{\mathcal{F}_1}J^{(1)}(x,t)g(x)dS\Big| \le c\|v\|_{W_2^2(\mathcal{F}_1)}\Big(\|\mathbf{J}^{(2)}\|_{W_2^{-1/2}(\mathcal{G})}+\|\mathbf{a}\|_{W_2^{-1/2}(\mathcal{G})}\Big)$$

$$\le c\|g\|_{W_2^{1/2}(\mathcal{G})}\Big(\|\mathbf{J}^{(2)}\|_{W_2^{-1/2}(\mathcal{G})}+\|\mathbf{a}\|_{W_2^{-1/2}(\mathcal{G})}\Big),$$

which imply (2.44).

Hence $\mathbf{h}$ is a solution of (2.3) with $\mathbf{j}^{(1)}=\nabla J^{(1)}, \mathbf{j}^{(2)}=\operatorname{rot}\mathbf{J}^{2)}$. Applying estimates (2.8) and (2.17), we obtain

$$\begin{aligned}
&\sum_{i=1}^{2}\Big(\|\mathbf{h}^{(i)}\|_{H^{l+2,l/2+1}(Q_T^i)}+\|\mathbf{h}_t^{(i)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\Big)\\
&\le c\Big(\|\mathbf{a}\|_{W_2^{l+3/2,0}(G_T)}+\sup_{t<T}\|\mathbf{a}\|_{W_2^{l+1/2}(\mathcal{G})}\\
&+\sum_{i=1}^{2}\big(\|\mathbf{j}^{(i)}\|_{W_2^{l+1,0}(Q_T^i)}+\sup_{t<T}\|\mathbf{j}^{(i)}\|_{W_2^{l}(\mathcal{F}_i)}\big)+\|J_t^{(1)}\|_{\widehat{W}_2^{0,l/2}(Q_T^1)}\\
&+\|J_t^{(1)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))}+\|\mathbf{J}_t^{(2)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\\
&+\|\mathbf{J}_t^{(2)}\|_{\widehat{W}_2^{0,l/2}(Q_T^2)}+\|\mathbf{A}_t\|_{\widehat{W}_2^{0,l/2}(Q_T)}\\
&+\sum_{i=1}^{2}\|\mathbf{A}_t^{(i)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\Big)\le c\Big(\|\mathbf{j}^{(2)}\|_{W_2^{l+1,0}(Q_T^2)}\\
&+\sup_{t<T}\|\mathbf{j}^{(2)}\|_{W_2^{l}(\mathcal{F}_2)}+\|\mathbf{a}\|_{W_2^{l+3/2,0}(G_T)}\\
&+\sup_{t<T}\|\mathbf{a}\|_{W_2^{l+1/2}(\mathcal{G})}+\|\mathbf{J}_t^{(2)}\|_{\widehat{W}_2^{0,l/2}(Q_T^2)}+\|\mathbf{J}_t^{(2)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\\
&+\sum_{i=1}^{2}\big(\|\mathbf{A}_t^{(i)}\|_{\widehat{W}_2^{0,l/2}(Q_T^i)}+\|\mathbf{A}_t^{(i)}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\big)\Big),
\end{aligned}\tag{2.45}$$

For the difference $\mathbf{H}'=\mathbf{H}-\mathbf{h}$ we have the problem (2.2) with $\mathbf{G}'=\mathbf{G}-(\mu_1\mathbf{h}_t+\alpha^{-1}\operatorname{rot}\operatorname{rot}\mathbf{h})$, $\mathbf{H}_0'=\mathbf{H}_0-\mathbf{h}(x,0)$, $\mathbf{j}=0$, $\mathbf{a}=0$, hence $\mathbf{H}'$

satisfies (2.29). Taking (2.45) into account, we obtain (2.43). The theorem is proved.

3. On the solvability of the problem (1.9)

As pointed out in Sec.1, our aim is to prove local in time solvability of the problem (1.9), (1.10).

Theorem 6. *Let* $\mathbf{u}_0 \in W_2^{l+1}(\mathcal{F})$, $\rho_0 \in W_2^{l+2}(\mathcal{G})$, $\mathbf{h}_0 \in W_2^{l'+1}(\mathcal{F})$ *with* $1/2 < l' < l < 1$ *and let the compatibility conditions*

$$\begin{aligned}
&\nabla \cdot \mathbf{u}_0 = l_2(\mathbf{u}_0, \rho_0), \quad y \in \mathcal{F}, \quad \Pi_{\mathcal{G}} S(\mathbf{u}_0)\mathbf{N}(y) = \mathbf{l}_3(\mathbf{u}_0, \rho_0),\\
&\nabla \cdot \mathbf{h}_0^{(1)} = 0, \quad \nabla \cdot \mathbf{h}_0^{(2)} = 0, \quad \operatorname{rot} \mathbf{h}_0^{(2)} = \operatorname{rot} \mathbf{l}_8(\mathbf{h}_0, \rho_0),\\
&[\mu \mathbf{h}_0 \cdot \mathbf{N}] = 0, \quad [\mathbf{h}_{0\tau}] = \mathbf{l}_9(\mathbf{h}_0, \rho_0), \quad x \in \mathcal{G}, \quad \mathbf{h}_0 \cdot \mathbf{n}|_S = 0,
\end{aligned}$$

and the smallness condition

$$\begin{aligned}
\|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} &\le \epsilon \ll 1\\
\|\mathbf{V} - \mathbf{u}_0\|_{W_2^{l+1/2}(\mathcal{G})} &\le \varepsilon_1 \ll 1
\end{aligned} \tag{3.2}$$

be satisfied. Then the problem (1.9), (1.10) *has a unique solution with the following regularity properties:*

$$\begin{aligned}
&\mathbf{u} \in W_2^{2+l.1+l/2}(Q_T^1), \quad \nabla q \in W_2^{l,l/2}(Q_T^1),\\
&q \in W_2^{l+1/2,0}(G_T) \cap W_2^{l/2}(0,T; W_2^{1/2}(\mathcal{G})),\\
&\rho \in W_2^{l+5/2,0}(G_T) \cap W_2^{l/2}(0,T; W_2^{5/2}(\mathcal{G})), \quad \rho_t \in W_2^{l+3/2,l/2+3/4}(G_T),\\
&\mathbf{h}^{(i)} \in W_2^{l'+2,l'/2+1}(Q_T^i),
\end{aligned}$$

where $Q_T^i = \mathcal{F}_i \times (0,T)$, $G_T = \mathcal{G} \times (0,T)$, $\mathbf{h}^{(i)} = \mathbf{h}|_{Q_T^i}$, $i = 1, 2$. *The solution is defined on a certain (small) time interval* $(0,T)$ *and satisfies the inequality*

$$\begin{aligned}
X \equiv\ & \|\mathbf{u}\|_{H^{l+2,l/2+1}(Q_T)} + \|\nabla q\|_{\widehat{W}_2^{l,l/2}(Q_T)} + \|q\|_{W_2^{l+1/2,0}(G_T)} + \|q\|_{\widehat{W}_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))}\\
&+ \|\rho\|_{W_2^{l+5/2,0}(G_T)} + \|\rho\|_{\widehat{W}_2^{l/2}(0,T;W_2^{5/2}(\mathcal{G}))} + \|\rho_t\|_{H^{l+3/2,l/2+3/4}(G_T)}\\
&+ \sup_{t<T} \|\rho(\cdot,t)\|_{W_2^{l+2}(\mathcal{G})} + \sum_{i=1}^{2} \big(\|\mathbf{h}^{(i)}\|_{H^{l'+2,l'/2+1}(Q_T^i)} + \|\mathbf{h}_t^{(i)}\|_{\widehat{W}^{l'/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\big)\\
&\le c\big(\|\mathbf{u}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\mathcal{H}\|_{W_2^{l+1/2}(\mathcal{G})} + \|\mathbf{h}_0\|_{W_2^{l'+1}(\mathcal{F}_1)}\big).
\end{aligned} \tag{3.3}$$

Remark. It is easily seen that addition of an appropriate constant to $q(x,t)$ permits to replace the norm $\|\mathcal{H}\|_{W_2^{l+1/2}(\mathcal{G})}$ by $\|\bar{\mathcal{H}}\|_{W_2^{l+1/2}(\mathcal{G})}$ in the inequality (3.3), where $\bar{\mathcal{H}}(y) = \mathcal{H}(y) - |\mathcal{G}|^{-1}\int_{\mathcal{G}} \mathcal{H}(x)\,dS$ (for a sphere, $\bar{\mathcal{H}} = 0$.

In this section we outline the main ideas of the proof. It is based on (2.6), (2.39) and on the estimates of the nonlinear terms (1.10). The solvability of the problem (1.9), (1.10) is proved by successive approximations, according to a standard scheme:

$$\begin{cases} \mathbf{u}_{m+1,t}(y,t) - \nu\nabla^2\mathbf{u}_{m+1} + \nabla q_{m+1} = \mathbf{l}_1(\mathbf{u}_m, q_m, \mathbf{h}_m, \rho_m), \\ \nabla\cdot\mathbf{u}_{m+1} = l_2(\mathbf{u}_m,\rho_m) = \nabla\cdot\mathbf{L}(\mathbf{u}_m,\rho_m), \quad y\in\mathcal{F}_1,\ t>0, \\ \Pi_{\mathcal{G}} S(\mathbf{u}_{m+1})\mathbf{N} = \mathbf{l}_3(\mathbf{u}_m,\rho_m), \\ -q_{m+1} + \nu\mathbf{N}\cdot S(\mathbf{u}_{m+1})\mathbf{N}(y) + \sigma\mathfrak{L}(\rho_{m+1}) = l_4(\mathbf{u}_m,\rho_m) + l_5(\rho_m) + \sigma\mathcal{H}(y), \\ \rho_{m+1,t} + \mathbf{V}(y)\cdot\nabla_\tau\rho_{m+1} - \mathbf{u}_{m+1}\cdot\mathbf{N}(y) = l_6(\mathbf{u}_m,\rho_m), \quad y\in\mathcal{G}, \\ \mu_1\mathbf{h}_{m+1,t} + \alpha^{-1}\,\mathrm{rot}\,\mathrm{rot}\,\mathbf{h}_{m+1} = \mathbf{l}_7(\mathbf{h}_m,\mathbf{u}_m,\rho_m), \\ \nabla\cdot\mathbf{h}_{m+1} = 0, \quad y\in\mathcal{F}_1, \\ \mathrm{rot}\,\mathbf{h}_{m+1} = \mathrm{rot}\,\mathbf{l}_8(\mathbf{h}_m,\rho_m), \quad \nabla\cdot\mathbf{h}_{m+1} = 0, \quad y\in\mathcal{F}_2, \\ [\mu\mathbf{h}_{m+1}\cdot\mathbf{N}] = 0, \quad [\mathbf{h}_{m+1\tau}] = \mathbf{l}_9(\mathbf{h}_m,\rho_m), \quad y\in\mathcal{G}, \\ \mathbf{h}_{m+1}\cdot\mathbf{n} = 0, \quad y\in S, \\ \mathbf{u}_{m+1}(y,0) = \mathbf{u}_0(y), \quad y\in\mathcal{F}_1, \quad \mathbf{h}_{m+1}(y,0) = \mathbf{h}_0(y), \quad y\in\mathcal{F}_1\cup\mathcal{F}_2, \\ \rho_{m+1}(y,0) = \rho_0(y), \quad y\in\mathcal{G}, \quad m = 1,2,\dots \end{cases} \tag{3.4}$$

The first approximation, $(\mathbf{u}_1, q_1, \rho_1, \mathbf{h}_1)$, is defined in the following way: $q_1 = 0$, $\mathbf{u}_1$ and ρ_1 satisfy the initial conditions

$$\mathbf{u}_1(y,0) = \mathbf{u}_0(y), \quad y\in\mathcal{F}_1, \quad \rho_1(y,0) = \rho_0(y), \quad y\in\mathcal{G}$$

and the inequalities

$$\begin{aligned} &\|\mathbf{u}_1\|_{H^{l+2,l/2+1}(Q_T^1)} \le c\|\mathbf{u}_1\|_{W_2^{l+2,l/2+1}(Q_\infty^1)} \le c\|\mathbf{u}_0\|_{W_2^{l+1}(\mathcal{F}_1)}, \\ &\|\rho_1\|_{W_2^{l+5/2,0}(G_\infty)} + \|\rho_{1,t}\|_{W_2^{l+3/2,l/2+3/4}(G_\infty)} \le c\|\rho_0\|_{W_2^{l+2}(\mathcal{G})} \end{aligned} \tag{3.5}$$

(see Proposition 4.1 in [8]), $\mathbf{h}_1$ is defined as a divergence free vector field belonging to $\mathcal{H}^{(2)}(\Omega)$ and satisfying the initial conditions

$$\mathbf{h}_1(y,0) = \mathbf{h}_0(y), \quad y\in\mathcal{F}_1\cup\mathcal{F}_2,$$

and the inequality

$$\sum_{i=1}^{2}\Big(\|\mathbf{h}_1^{(i)}\|_{W_2^{l+'2,1+l'/2}(Q_T^i)}+\|\mathbf{h}_1^{(i)}\|_{\widehat{W}_2^{l'/2}(0.T;W_2^{-1/2}(\mathcal{G}))}\Big)\leq c\|\mathbf{h}_0^{(1)}\|_{W_2^{l'+1}(\mathcal{F}_1)}. \tag{3.6}$$

The existence of such $\mathbf{u}_1$, ρ_1 follows from the inverse trace theorems, and $\mathbf{h}_1$ can be constructed exactly as $\mathbf{B}$ in the proof of Theorem 2.4.

In view of (3.1) and (1.7), the necessary compatibility conditions are satisfied in (3.4), and all the successive approximations are defined in a certain time interval $t\in(0,T_0)$. We show that they are uniformly bounded for $t\in(0,T)$, $T\leq T_0$. In the next section the following proposition is proved.

Theorem 7. *If* $\mathbf{V}(x)$ *satisfies* (3.2) *and*

$$\sup_{t<T}\|\rho_m(\cdot,t)\|_{W_2^{l+3/2}(\mathcal{G})}\leq\delta\ll 1, \tag{3.7}$$

then

$$\begin{aligned}
Z_m &\equiv \|\mathbf{l}_1(\mathbf{u}_m,q_m,\mathbf{h}_m,\rho)\|_{H^{l,l/2}(Q_T^1)}+\|l_2(\mathbf{u}_m,\rho_m)\|_{W_2^{l+1,0}(Q_T^1)}\\
&+\sup_{t<T}\|l_2(\mathbf{u}_m,\rho_m)\|_{W_2^{l}(\mathcal{F}_1)}\\
&+\|\mathbf{L}_t(\mathbf{u}_m,\rho_m)\|_{\widehat{W}_2^{0,l/2}(Q_T^1)}+\|\mathbf{l}_3(\mathbf{u}_m,\rho_m)\|_{H^{l+1/2,l/2+1/4}(G_T)}\\
&+\|l_4(\mathbf{u}_m,\mathbf{h}_m,\rho_m)\|_{W_2^{l+1/2,0}(G_T)}\\
&+\|l_5(\rho_m)\|_{W_2^{l+1/2,0}(G_T)}+\|l_4(\mathbf{u}_m,\mathbf{h}_m,\rho_m)\|_{\widehat{W}_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))}\\
&+\|l_5(\rho_m)\|_{\widehat{W}_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))}\\
&+\|l_6(\mathbf{u}_m,\rho_m)\|_{H^{l+3/2,l/2+3/4}(G_T)}+\|\mathbf{l}_7\|_{\widehat{W}_2^{l',l'/2}(Q_T^1)}\\
&+\|\operatorname{rot}\mathbf{l}_8(\mathbf{h}_m,\rho_m)\|_{W_2^{l'+1,0}(Q_T^2)}\\
&+\sup_{t<T}\|\operatorname{rot}\mathbf{l}_8(\mathbf{h}_m,\rho_m)\|_{W_2^{l'}(\mathcal{F}_2)}+\|\mathbf{l}_{8,t}(\mathbf{h}_m,\rho_m)\|_{\widehat{W}_2^{0,l'/2}(Q_T^2)}\\
&+\|\mathbf{l}_{8,t}(\mathbf{h}_m,\rho_m)\|_{\widehat{W}_2^{l'/2}(0,T;W_2^{-1/2}\mathcal{G}))}\\
&+\|\mathbf{l}_9(\mathbf{h}_m,\rho_m)\|_{W_2^{l'+3/2,0}(G_T)}+\sup_{t<T}\|\mathbf{l}_9(\mathbf{h}_m,\rho_m)\|_{W_2^{l'+1/2}(\mathcal{G})}\\
&+\sum_{i=1}^{2}\big(\|\mathbf{A}_t^{(i)}(\mathbf{h}_m,\rho_m)\|_{\widehat{W}_2^{0,l'/2}(Q_t^i)}+\|\mathbf{A}_t^{(i)}\|_{\widehat{W}_2^{l'/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\Big)
\end{aligned}$$

satisfies the inequality

$$Z_m \leq \delta_1 \sum_{j=1}^{3} X_m^j, \tag{3.8}$$

where X_m is defined in (3.3) (with $(\mathbf{u}_m, q_m, \mathbf{h}_m, \rho_m)$ instead of $(\mathbf{u}, q, \mathbf{h}, \rho)$) and δ_0 is a small constant depending on δ and T.

The condition $l' < l$ is explained by technical reasons connected with the estimate of one of the terms in $\mathbf{l}_7$ (see (4.16), (4.17)).

Using (2.6), (2.39) and (3.8), we obtain

$$X_{m+1} \leq c_1\delta_1 \sum_{j=1}^{3} X_m^j + c_2 N, \tag{3.9}$$

where

$$N = \|\mathbf{v}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\mathcal{H}\|_{W_2^{l+1/2}(\mathcal{G})} + \|\mathbf{h}_0\|_{W_2^{l'+1}(\mathcal{F}_1)}.$$

If δ_0 is sufficiently small, then (3.9) yields a uniform estimate

$$X_m \leq 2c_2 N, \tag{3.10}$$

and the condition (3.7) can be recovered from

$$\begin{aligned} \|\rho_m(\cdot,t)\|_{W_2^{l+3/2}(\mathcal{G})} &\leq \|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} + \int_0^t \|\rho_{m,\tau}(\cdot,t)\|_{W_2^{l+3/2}(\mathcal{G})}\, d\tau \\ &\leq \|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} + 2c_2 N\sqrt{T}. \end{aligned} \tag{3.11}$$

Inequalities (3.10, (3.11) guarantee the boundedness of all the successive approximations $(\mathbf{u}_m, q_m, \rho_m, \mathbf{h}_m)$ for $t \leq T$, if ε, ε_1 in (3.2) and T are sufficiently small (cf. [9, Theorem 2.1]). In order to prove the convergence of the sequence $(\mathbf{u}_m, q_m, \rho_m, \mathbf{h}_m)$, we should estimate the differences

$$\begin{aligned} \mathbf{w}_{m+1} &= \mathbf{u}_{m+1} - \mathbf{u}_m, \ s_{m+1} = q_{m+1} - q_m, \\ r_{m+1} &= \rho_{m+1} - \rho_m, \ \mathbf{k}_{m+1} = \mathbf{h}_{m+1} - \mathbf{h}_m. \end{aligned}$$

They satisfy the relations

$$\begin{cases} \mathbf{w}_{m+1,t}(y,t) - \nu\nabla^2\mathbf{w}_{m+1} + \nabla s_{m+1} \\ = \mathbf{l}_1(\mathbf{u}_m, q_m, \mathbf{h}_m, \rho_m) - \mathbf{l}_1(\mathbf{u}_{m-1}, q_{m-1}, \mathbf{h}_{m-1}, \rho_{m-1}), \\ \nabla\cdot\mathbf{w}_{m+1} = l_2(\mathbf{u}_m,\rho_m) - l_2(\mathbf{u}_{m-1},\rho_{m-1}), \quad y\in\mathcal{F}_1,\ t>0, \\ \Pi_{\mathcal{G}} S(\mathbf{w}_{m+1})\mathbf{N} = \mathbf{l}_3(\mathbf{u}_m,\rho_m) - \mathbf{l}_3(\mathbf{u}_{m-1},\rho_{m-1}), \\ -s_{m+1} + \nu\mathbf{N}\cdot S(\mathbf{w}_{m+1})\mathbf{N}(y) + \sigma\mathfrak{L}(r_{m+1}) \\ = l_4(\mathbf{u}_m,\rho_m) + l_5(\rho_m) - l_4(\mathbf{u}_{m-1},\rho_{m-1}) - l_5(\rho_{m-1}), \\ r_{m+1,t} + \mathbf{V}(y)\cdot\nabla_\tau r_{m+1} - \mathbf{w}_{m+1}\cdot\mathbf{N}(y) \\ = l_6(\mathbf{u}_m,\rho_m) - l_6(\mathbf{u}_{m-1},\rho_{m-1}), \quad y\in\mathcal{G}, \\ \mu_1\mathbf{k}_{m+1,t} + \alpha^{-1}\,\mathrm{rot}\mathrm{rot}\,\mathbf{k}_{m+1} = \mathbf{l}_7(\mathbf{h}_m,\mathbf{u}_m,\rho_m) - \mathbf{l}_7(\mathbf{h}_{m-1},\mathbf{u}_{m-1},\rho_{m-1}), \\ \nabla\cdot\mathbf{k}_{m+1} = 0, \quad y\in\mathcal{F}_1, \\ \mathrm{rot}\,\mathbf{k}_{m+1} = \mathrm{rot}\,\mathbf{l}_8(\mathbf{h}_m,\rho_m) - \mathrm{rot}\,\mathbf{l}_8(\mathbf{h}_{m-1},\rho_{m-1}), \nabla\cdot\mathbf{k}_{m+1} = 0,\ y\in\mathcal{F}_2, \\ [\mu\mathbf{k}_{m+1}\cdot\mathbf{N}] = 0, \quad [\mathbf{k}_{m+1\tau}] = \mathbf{l}_9(\mathbf{h}_m,\rho_m) - \mathbf{l}_9(\mathbf{h}_{m-1},\rho_{m-1}), \quad y\in\mathcal{G}, \\ \mathbf{k}_{m+1}\cdot\mathbf{n} = 0, \quad y\in S, \\ \mathbf{w}_{m+1}(y,0) = 0, \quad y\in\mathcal{F}_1, \quad \mathbf{k}_{m+1}(y,0) = 0, \quad y\in\mathcal{F}_1\cup\mathcal{F}_2, \\ r_{m+1}(y,0) = 0, \quad y\in\mathcal{G}, \qquad m = 2,3,\ldots \end{cases} \tag{3.12}$$

Theorem 8. *If* (3.7) *and* (3.10) *are satisfied, then*

$$\begin{aligned} &\|\mathbf{l}_1(\mathbf{u}_m,q_m,\mathbf{h}_m\rho_m) - \mathbf{l}_1(\mathbf{u}_{m-1},q_{m-1},\mathbf{h}_{m-1},\rho_{m-1})\|_{\widehat{W}_2^{l,l/2}(Q_T)} \\ &+ \|l_2(\mathbf{u}_m,\rho_m) - l_2(\mathbf{u}_{m-1},\rho_{m-1})\|_{W_2^{l+1,0}(Q_T)} \\ &+ \sup_{t<T}\|l_2(\mathbf{u}_m,\rho_m) - l_2(\mathbf{u}_{m-1},\rho_{m-1})\|_{W_2^{l}(Q_T)} \\ &+ \|\mathbf{L}_t(\mathbf{u}_m,\rho_m) - \mathbf{L}_t(\mathbf{u}_{m-1},\rho_{m-1})\|_{\widehat{W}_2^{0,l/2}(Q_T)} \\ &+ \|\mathbf{l}_3(\mathbf{u}_m,\rho_m) - \mathbf{l}_3(\mathbf{u}_{m-1},\rho_{m-1})\|_{H^{l+1/2,l/2+1/4}(G_T)} \\ &+ \|l_4(\mathbf{u}_m,\rho_m) - l_4(\mathbf{u}_{m-1},\rho_{m-1})\|_{W_2^{l+1/2,0}(G_T)} \\ &+ \|l_4(\mathbf{u}_m,\rho_m) - l_4(\mathbf{u}_{m-1},\rho_{m-1})\|_{\widehat{W}_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))} \\ &+ \|l_5(\rho_m) - l_5(\rho_{m-1})\|_{W_2^{l+1/2,0}(G_T)} \\ &+ \|l_5(\rho_m) - l_5(\rho_{m-1})\|_{\widehat{W}_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))} \\ &+ \|l_6(\mathbf{u}_m,\rho_m) - l_6(\mathbf{u}_{m-1},\rho_{m-1})\|_{H^{l+3/2,l/2+3/4}(G_T))} \end{aligned}$$

$$
\begin{aligned}
&+\|\mathbf{l}_7(\mathbf{h}_m,\mathbf{u}_m,\rho_m)-\mathbf{l}_7(\mathbf{h}_{m-1},\mathbf{u}_{m-1},\rho_{m-1})\|_{\widehat{W}_2^{l',l'/2}(Q_T^1)}\\
&+\|\operatorname{rot}(\mathbf{l}_8(\mathbf{h}_m,\rho_m)-\mathbf{l}_8(\mathbf{h}_{m-1},\rho_{m-1}))\|_{W_2^{l'+1,0}(Q_T^2)}\\
&+\sup_{t<T}\|rot(\mathbf{l}_8(\mathbf{h}_m,\rho_m)-\mathbf{l}_8(\mathbf{h}_{m-1},\rho_{m-1}))\|_{W_2^{l'}(\mathcal{F}_2)}\\
&+\|\mathbf{l}_{8,t}(\mathbf{h}_m,\rho_m)-\mathbf{l}_{8,t}(\mathbf{h}_{m-1},\rho_{m-1})\|_{\widehat{W}_2^{0,l/2}(Q_T^2)}\\
&+\|\mathbf{l}_{8,t}(\mathbf{h}_m,\rho_m)-\mathbf{l}_{8,t}(\mathbf{h}_{m-1},\rho_{m-1})\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\\
&+\|\mathbf{l}_9(\mathbf{h}_m,\rho_m)-\mathbf{l}_9(\mathbf{h}_{m-1},\rho_{m-1})\|_{W_2^{l'+3/2,0}(G_T)}\\
&+\sup_{t<T}\|\mathbf{l}_9(\mathbf{h}_m,\rho_m)-\mathbf{l}_9(\mathbf{h}_{m-1},\rho_{m-1}))\|_{W_2^{l'+1/2}(\mathcal{G})}\\
&+\sum_{i=1}^{2}\|\mathbf{A}_t^{(i)}(\mathbf{h}_m,\rho_m)-\mathbf{A}_t^{(i)}(\mathbf{h}_{m-1},\rho_{m-1})\|_{\widehat{W}_2^{0,l'/2}(Q_T^i)}\\
&+\sum_{i=1}^{2}\|\mathbf{A}_t^{(i)}(\mathbf{h}_m,\rho_m)-\mathbf{A}_t^{(i)}(\mathbf{h}_{m-1},\rho_{m-1})\|_{\widehat{W}_2^{l'/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\\
&\le c\vartheta(\epsilon,T)\Big(\|\mathbf{u}_m-\mathbf{u}_{m-1}\|_{H^{l+2,l/2+1}(Q_T)}+\|\nabla(q_m-q_{m-1})\|_{\widehat{W}_2^{l,l/2}(Q_T)}\\
&+\|\rho_m-\rho_{m-1}\|_{W_2^{l+5/2,0}(G_T)}+\|\rho_m-\rho_{m-1}\|_{\widehat{W}_2^{l/2}(0,T;W_2^{5/2}(\mathcal{G}))}\\
&+\|\rho_{m,t}-\rho_{m-1,t}\|_{H^{l+3/2,l/2+3/4}(G_T)}+\sum_{i=1}^{2}\|\mathbf{h}_m^{(i)}-\mathbf{h}_{m-1}^{(i)}\|_{H^{l'+2,l'/2+1}(Q_T^{(i)})}\\
&+\sum_{i=1}^{2}\|\mathbf{h}_{m,t}^{(i)}-\mathbf{h}_{m-1,t}^{(i)}\|_{\widehat{W}_2^{0,l'/2}(Q_T^i)}\\
&+\sum_{i=1}^{2}\|\mathbf{h}_{m,t}^{(i)}-\mathbf{h}_{m-1,t}^{(i)}\|_{\widehat{W}_2^{l'/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\Big).
\end{aligned}
\tag{3.13}
$$

with a small ϑ.

The inequality (3.13) is obtained by the same arguments as (3.8) (see [9], Sec. 2, 3 and Sec. 4 of the present paper). In the proof the following inequality for $\rho_m-\rho_{m-1}$ should be used instead of (3.11):

$$
\begin{aligned}
\|\rho_m(\cdot,t)-\rho_{m-1}(\cdot,t)\|_{W_2^{l+3/2}(\mathcal{G})}&\le\int_0^t\|\rho_{m,\tau}(\cdot,\tau)-\rho_{m-1,\tau}(\cdot,\tau)\|_{W_2^{l+3/2}(G)}d\tau\\
&\le 4c_2N\sqrt{T}.
\end{aligned}
$$

If $\vartheta(\epsilon, T)$ is sufficiently small, then (2.6) and (3.13) guarantee the convergence of $(\mathbf{u}_m, q_m, \mathbf{h}_m, \rho_m)$ to the solution of the problem (1.9), (1.10) (cf. [9]. Sec. 2). The uniqueness of the solution is also proved by similar arguments.

4. Proof of Theorem 7

We recall some auxiliary propositions from [9].

Proposition 1. *Arbitrary functions $u(x), v(x)$ given in a domain $\Omega \subset \mathbb{R}^n$ satisfy the inequalities*

$$\|uv\|_{W_2^l(\Omega)} \le c\|u\|_{W_2^l(\Omega)}\Big(\sup_\Omega |v(x)| + \|v\|_{W_2^{n/2}(\Omega)}\Big)$$
$$\le c\|u\|_{W_2^l(\Omega)}\|v\|_{W_2^s(\Omega)}, \quad s > n/2, \tag{4.1}$$
$$\|uv\|_{L_2(\Omega)} \le c\|u\|_{W_2^l(\Omega)}\|v\|_{W_2^{n/2-l}(\Omega)}, \tag{4.2}$$

if $l < n/2$,

$$\|uv\|_{W_2^l(\Omega)} \le c\|u\|_{W_2^l(\Omega)}\|v\|_{W_2^s(\Omega)}, \quad s > n/2, \tag{4.3}$$

if $l = n/2$,

$$\|uv\|_{W_2^l(\Omega)} \le c\|u\|_{W_2^l(\Omega)}\Big(\sup_\Omega |v(x)| + \|v\|_{W_2^{n/2}(\Omega)}\Big)$$
$$+ c\|v\|_{W_2^l(\Omega)}\Big(\sup_\Omega |u(x)| + \|u\|_{W_2^{n/2}(\Omega)}\Big)$$
$$\le c\Big(\|u\|_{W_2^l(\Omega)}\|v\|_{W_2^s(\Omega)} + \|v\|_{W_2^l(\Omega)}\|u\|_{W_2^s(\Omega)}\Big), \quad s > n/2, \tag{4.4}$$

if $l > n/2$.

Proposition 2. *For arbitrary $u(x)$ and $v(x)$, $x \in \mathcal{G}$, the following inequality holds*:

$$\|uv\|_{W_2^{-1/2}(\mathcal{G})} \le c\|u\|_{W_2^{-1/2}(\mathcal{G})}(\sup_{\mathcal{G}} |v(x)| + \|v\|_{W_2^1(\mathcal{G})})$$
$$\le c\|u\|_{W_2^{-1/2}(\mathcal{G})}\|v\|_{W_2^s(\mathcal{G})}, \qquad s > 1. \tag{4.5}$$

Proof. According to standard definition of the norm $\|u\|_{W_2^{-1/2}(\mathcal{G})}$, we should estimate the integral $\int\limits_{\mathcal{G}} u(x)v(x)w(x)dS$ with $w \in W_2^{1/2}(\mathcal{G})$. By (4.1),

$$\Big|\int\limits_{\mathcal{G}} u(x)v(x)w(x)\,dS\Big| \le \|u\|_{W_2^{-1/2}(\mathcal{G})}\|vw\|_{W_2^{1/2}(\mathcal{G})}$$
$$\le c\|u\|_{W_2^{-1/2}(\mathcal{G})}\|w\|_{W_2^{1/2}(\mathcal{G})}\big(\sup_{\mathcal{G}}|v(x)| + \|v\|_{W_2^1(\mathcal{G})}\big),$$

which implies (4.5).

Proposition 3. *Let* $\mathbf{R}(x,t) = \Big(\frac{\partial N_i^*\rho^*}{\partial x_j}\Big)_{i,j=1,2,3}$ *and let the function* ρ *satisfy the condition* (3.7). *Arbitrary smooth function* $f(\mathbf{R})$ *defined for* $|\mathbf{R}| \le \delta_0$, $\delta_0 > \delta$, *satisfies the inequalities*

$$\|f(\mathbf{R})\|_{W_2^{l+1}(\mathcal{F}_i)} \le c, \tag{4.6}$$
$$\|\mathbf{R}f(\mathbf{R})\|_{W_2^r(\mathcal{F}_i)} \le c\|\mathbf{R}\|_{W_2^r(\mathcal{F}_1)} \le c\|\rho\|_{W_2^{r+1/2}\mathcal{G})} \le \delta,\ r \in [0, l+1],\ i = 1,2. \tag{4.7}$$

We add to (4.6), (4.7) the estimate of $\nabla(\nabla\mathbf{R}f(\mathbf{R}))$ that is a linear combination of the expressions $D^2\mathbf{R}f(\mathbf{R})$ and $\nabla\mathbf{R}\nabla\mathbf{R}f_1(\mathbf{R})$ with f and f_1 also satisfying the assumptions of Proposition 3. By (4.1), (4.6), (1.5),

$$\|D^2\mathbf{R}f(\mathbf{R})\|_{W_2^l(\mathcal{F}_i)} \le c\|D^2\mathbf{R}\|_{W_2^l(\mathcal{F}_i)} \le c\|\rho^*\|_{W_2^{3+l}(\mathcal{F}_i)} \le c\|\rho\|_{W_2^{5/2+l}(\mathcal{G})},$$
$$\|\nabla\mathbf{R}\nabla\mathbf{R}f_2(\mathbf{R})\|_{W_2^l(\mathcal{F}_1)} \le c\|\nabla\mathbf{R}\|_{W_2^l(\mathcal{F}_1)}\|\nabla\mathbf{R}\|_{W_2^{3/2+\eta}(\mathcal{F}_1)}$$
$$\le c\|\rho\|_{W_2^{l+3/2}(\mathcal{F}_1)}\|\rho\|_{W_2^{l+5/2}(\mathcal{F}_1)}$$

with $\eta \in (0, l-1/2)$, hence

$$\|\mathbf{R}f(\mathbf{R})\|_{W_2^{l+1}(\mathcal{F}_i)} \le c\|\rho\|_{W_2^{5/2+l}(\mathcal{G})}. \tag{4.8}$$

Examples of functions satisfying the assumptions of Proposition 5 are provided by the elements of the matrices $\mathcal{L}(x,\rho^*)$, $\mathcal{L}^{-1}(x,\rho^*)$, $\widehat{\mathcal{L}}(x,\rho^*)$, $\mathcal{P} = \frac{1}{L(x,\rho^*)}\mathcal{L}^T(x,\rho)\mathcal{L}(x,\rho)$, whereas the elements of $\widehat{\mathcal{L}}\widehat{\mathcal{L}}^T|\widehat{\mathcal{L}}^T\mathbf{N}^*|^{-2}$ depend also on x. They also satisfy the inequalities (4.6)–(4.8).

We also cite Proposition 2.1 in [9] containing estimates of some nonlinear terms in (1.10).

Proposition 4. *If* $\mathbf{V}(x)$ *satisfies* (3.2) *and*

$$\sup_{t<T} \|\rho(\cdot,t)\|_{W_2^{l+3/2}(\mathcal{G})} \leq \delta \ll 1, \tag{4.9}$$

then

$$Z' \leq \delta_1' \sum_{j=1}^{3} X'^{j}, \tag{4.10}$$

where δ_1' *is a small constant depending on* δ *and* T,

$$Z' = \|\mathbf{l}_1'(\mathbf{u},q,\rho)\|_{\widehat{W}_2^{l,l/2}(Q_T)} + \|l_2(\mathbf{u},\rho)\|_{W_2^{l+1,0}(Q_T)} + \sup_{t<T}\|l_2(\mathbf{u},\rho)\|_{W_2^{l}(\mathcal{F})}$$
$$+ \|\mathbf{L}_t(u,\rho)\|_{\widehat{W}_2^{0,l/2}(Q_T)} + \|\mathbf{l}_3(u,\rho)\|_{H^{l+1/2,l/2+1/4}(G_T)}$$
$$+ \|l_4(u,\rho)\|_{W_2^{l+1/2,0}(G_T)}$$
$$+ \|l_5(\rho)\|_{W_2^{l+1/2,0}(G_T)} + \|l_4(\mathbf{u},\rho)\|_{\widehat{W}_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))}$$
$$+ \|l_5(\mathbf{u},\rho)\|_{\widehat{W}_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))}$$
$$+ \|l_6(\mathbf{u},\rho)\|_{H^{l+3/2,l/2+3/4}(G_T)},$$
$$X' = \|\mathbf{u}\|_{H^{l+2,l/2+1}(Q_T)} + \|\nabla q\|_{\widehat{W}_2^{l,l/2}(Q_T)} + \|q\|_{W_2^{l+1/2,0}(G_T)}$$
$$+ \|q\|_{\widehat{W}_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))} + \|\rho\|_{W_2^{l+5/2,0}(G_T)} + \|\rho\|_{\widehat{W}_2^{l/2}(0,T;W_2^{5/2}(\mathcal{G}))}$$
$$+ \sup_{t<T}\|\rho(\cdot,t)\|_{W_2^{l+2}(\mathcal{G})} + \|\rho_t\|_{H^{l+3/2,l/2+3/4}(G_T)},$$

$$\mathbf{l}_1'(\mathbf{u},q,\rho) = \nu(\widetilde{\nabla}^2 - \nabla^2)\mathbf{u} + (\nabla - \widetilde{\nabla})q + \rho_t^*(\mathcal{L}^{-1}\mathbf{N}^*\cdot\nabla)\mathbf{u} - (\mathcal{L}^{-1}\mathbf{u}\cdot\nabla)\mathbf{u}.$$

We do not reproduce the estimate of $\mathbf{l}_1'$ and pass to the estimates of the remaining terms in (1.10). Here and in what follows we write the norm $\|f\|_{\widehat{W}_2^{\mu}(0,T)}$, $\mu \in (0,1)$, in the form

$$\|f\|^2_{\widehat{W}_2^{\mu}(0,T)} = \frac{1}{T^{2\mu}}\|f\|^2_{L_2(0,T)} + \int_0^T \frac{dh}{h^{1+2\mu}} \int_h^T |\Delta_t(-h)f(t)|^2\,dt,$$

where $\Delta_t(-h)f(t) = f(t-h) - f(t)$. We set

$$|u|_{s,\lambda,Q_T} = \Big(\int_0^T \frac{dh}{h^{1+2\lambda}} \int_h^T \|\Delta_t(-h)u\|^2_{W_2^s(\mathcal{F})}\,dt\Big)^{1/2},$$
$$|u|_{s,\lambda,G_T} = \Big(\int_0^T \frac{dh}{h^{1+2\lambda}} \int_h^T \|\Delta_t(-h)u\|^2_{W_2^s(\mathcal{G})}\,dt\Big)^{1/2}$$

Using the elementary formula $\Delta_t(-h)u(x,t) = -\int\limits_0^h u_t(x,t-\tau)\,d\tau$ it is easy to prove that

$$
\begin{aligned}
&|u|_{s,\lambda,Q_T^i} \le cT^{1-\lambda}\|u_t\|_{W_2^{s,0}(Q_T^i)}, \quad i=1,2, \quad \text{if} \quad \lambda<1,\\
&\int\limits_0^T \frac{dh}{h^{1+2\lambda}} \int\limits_h^T V^2(t)\|\Delta_t(-h)u(\cdot,t)\|^2_{W_2^s(\mathcal{F}_i)}\,dt \qquad (4.11)\\
&\le cT^{1-2\lambda}\int\limits_0^T V^2(t)\,dt\int\limits_0^t \|u_\tau(\cdot,t)\|^2_{W_2^s(\mathcal{F}_i)}d\tau, \quad \text{if} \quad \lambda<1/2,
\end{aligned}
$$

with $\forall\, V \in L_2(0,T)$. Moreover, by Proposition 1.2 in [9],

$$
|u|_{s,\lambda,Q_T^i} \le c\|u\|_{H^{s+2\lambda,s/2+\lambda}(Q_T^i)}, \quad |u|_{s,\lambda,G_T} \le c\|u\|_{H^{s+2\lambda,s/2+\lambda}(G_T)}. \qquad (4.12)
$$

From now on, $\eta \in (0, l'-1/2)$.

1. *Estimate of* $\widetilde{\nabla}\cdot T_M(\frac{1}{L}\mathcal{L}\mathbf{h}) = \mathcal{L}^{-T}\nabla\cdot T_M(\widetilde{\mathbf{H}})$.

By (4.1),

$$
\begin{aligned}
\|\widetilde{\nabla}\cdot T_M(\widetilde{\mathbf{H}})\|_{W_2^l(\mathcal{F}_1)} &\le c\|\nabla\widetilde{\mathbf{H}}\|_{W_2^l(\mathcal{F}_1)}\|\widetilde{\mathbf{H}}\|_{W_2^{3/2+\eta}(\mathcal{F}_1)}\\
&\le c\|\widetilde{\mathbf{H}}\|_{W_2^{l+1}(\mathcal{F}_1)}\|\widetilde{\mathbf{H}}\|_{W_2^{l'+1}(\mathcal{F}_1)}.
\end{aligned}
$$

From (4.4), (4.6) and from the well known Ehrling inequality (see [15], p.79) it follows that the right hand side does not exceed

$$
c\|\mathbf{h}\|_{W_2^{l+1}(\mathcal{F}_1)}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)} \le c\|\mathbf{h}\|^{\alpha}_{W_2^{l'+2}(\mathcal{F}_1)}\|\mathbf{h}\|^{2-\alpha}_{W_2^{l'+1}(\mathcal{F}_1)},
$$

where $\alpha = l - l'$. Hence

$$
\begin{aligned}
&\|\widetilde{\nabla}\cdot T_M(\widetilde{\mathbf{H}})\|_{W_2^{l,0}(Q_T^1)} \le c\sup_{t<T}\|\mathbf{h}(\cdot,t)\|^{2-\alpha}_{W_2^{l'+1}(\mathcal{F}_1)}\left(\int\limits_0^T\|\mathbf{h}\|^{2\alpha}_{W_2^{2+l'}(\mathcal{F}_1)}\,dt\right)^{1/2}\\
&\le cT^{(1-\alpha)/2}\sup_{t<T}\|\mathbf{h}(\cdot,t)\|^{2-\alpha}_{W_2^{l'+1}(\mathcal{F}_1)}\|\mathbf{h}\|^{\alpha}_{W_2^{l'+2,0}(Q_T^1)}\\
&\le cT^{(1-l+l')/2}\|\mathbf{h}\|^2_{H^{2+l',1+l'/2}(Q_T^1)}.
\end{aligned} \qquad (4.13)
$$

Now we estimate $\|\widetilde{\nabla}\cdot T_M(\widetilde{\mathbf{H}})\|_{\widehat{W}_2^{0,l/2}(Q_T^1)}$. By (4.2) and (4.4), we have

$$\frac{1}{T^{l/2}}\|\widetilde{\nabla}\cdot T_M(\widetilde{\mathbf{H}})\|_{L_2(Q_T^1)}\le\frac{c}{T^{l/2}}\left(\int\limits_0^T\|\nabla\widetilde{\mathbf{H}}\|^2_{W_2^{l'}(\mathcal{F}_1)}\|\widetilde{\mathbf{H}}\|^2_{W_2^{3/2-l'}(\mathcal{F}_1)}\,dt\right)^{1/2}$$
$$\le cT^{(1-l)/2}\sup_{t<T}\|\mathbf{h}\|^2_{W_2^{l'+1}(\mathcal{F}_1)}.$$

Since

$$\|\Delta_t(-h)(\widetilde{\nabla}\cdot T_M(\widetilde{\mathbf{H}}))\|_{L_2(\mathcal{F}_1)}\le\|(\Delta_t(-h)\mathcal{L}^{-T})\nabla\cdot T_M(\widetilde{\mathbf{H}})\|_{L_2(\mathcal{F}_1)}$$
$$+\,c\big(\||\Delta_t(-h)\nabla\widetilde{\mathbf{H}}||\widetilde{\mathbf{H}}|\|_{L_2(\mathcal{F}_1)}+\||\nabla\widetilde{\mathbf{H}}||\Delta_t(-h)\widetilde{\mathbf{H}}|\|_{L_2(\mathcal{F}_1)}\big),$$
$$\|(\Delta_t(-h)\mathcal{L}^{-T})\nabla\cdot T_M(\widetilde{\mathbf{H}})\|_{L_2(\mathcal{F}_1)}$$
$$\le c\sup_{\mathcal{F}_1}|\widetilde{\mathbf{H}}(x,t)|\|\nabla\widetilde{\mathbf{H}}\|_{W_2^{l'}(\mathcal{F}_1)}\|\Delta_t(-h)\mathcal{L}^{-1}\|_{W_2^{3/2-l'}(\mathcal{F}_1)},$$
$$\||\Delta_t(-h)\nabla\widetilde{\mathbf{H}}||\widetilde{\mathbf{H}}|\|_{L_2(\mathcal{F}_1)}\le c\sup_{\mathcal{F}_1}|\widetilde{\mathbf{H}}(x,t)|\|\Delta_t(-h)\nabla\widetilde{\mathbf{H}}\|_{L_2(\mathcal{F}_1)},$$
$$\||\nabla\widetilde{\mathbf{H}}||\Delta_t(-h)\widetilde{\mathbf{H}}|\|_{L_2(\mathcal{F}_1)}\le c\|\nabla\widetilde{\mathbf{H}}\|_{W_2^{l'}(\mathcal{F}_1)}\|\Delta_t(-h)\widetilde{\mathbf{H}}\|_{W_2^{3/2-l'}(\mathcal{F}_1)},$$

we have

$$\left(\int\limits_0^T\frac{dh}{h^{1+l}}\int\limits_h^T\||\Delta_t(-h)\nabla\widetilde{\mathbf{H}}||\widetilde{\mathbf{H}}|\|^2_{L_2(\mathcal{F}_1)}\,dt\right)^{1/2}$$
$$+\left(\int\limits_0^T\frac{dh}{h^{1+l}}\int\limits_h^T\||\nabla\widetilde{\mathbf{H}}||\Delta_t(-h)\widetilde{\mathbf{H}}|\|^2_{L_2(\mathcal{F}_1)}\right)^{1/2}$$
$$\le c\sup_{t<T}\|\mathbf{h}(\cdot,t)\|_{W_2^{l'+1}(\mathcal{F}_1)}|\widetilde{\mathbf{H}}|_{1,l/2,Q_T},$$
$$\left(\int\limits_0^T\frac{dh}{h^{1+l}}\int\limits_h^T\|(\Delta_t(-h)\mathcal{L}^{-T})\nabla\cdot T_M(\widetilde{\mathbf{H}})\|^2_{L_2(\mathcal{F}_1)}\right)^{1/2}$$
$$\le c\sup_{t<T}\|\mathbf{h}(\cdot,t)\|^2_{W_2^{l'+1}(\mathcal{F}_1)}|\mathbf{R}|_{3/2-l',l/2,Q_T^1}$$
$$\le cT^{1-l/2}\sup_{t<T}\|\mathbf{h}(\cdot,t)\|^2_{W_2^{l'+1}(\mathcal{F}_1)}\|\rho t\|_{W_2^{l+1,0}(G_T)}.$$

It remains to estimate $|\widetilde{\mathbf{H}}|_{1,l/2,Q_T^1}$. Since $\widetilde{\mathbf{H}} = \frac{\mathcal{L}}{L}\mathbf{h}$, inequalities (4.1), (4.6) imply

$$\|\Delta_t(-h)\widetilde{\mathbf{H}}\|_{W_2^1(Q_T^1)}$$
$$\leq c\Big(\|\Delta_t(-h)\widetilde{\mathbf{h}}\|_{W_2^1(Q_T^1)} + \|\mathbf{h}\|_{W_2^{3/2+\eta}(\mathcal{F}_1)}\|\Delta_t(-h)\frac{\mathcal{L}}{L}\|_{W_2^1(\mathcal{F}_1)}\Big),$$

from which it follows (due to (1.5), (4.11)) that

$$|\widetilde{\mathbf{H}}|_{1,l/2,Q_T^1} \leq c\big(|\mathbf{h}|_{1,l/2,Q_T^1} + \sup_{t<T}\|\mathbf{h}(\cdot,t)\|_{W_2^{l'+1}(\mathcal{F}_1)}|\mathbf{R}|_{1,l/2,Q_T^1}$$
$$\leq c\big(T^{(1-l+l')/2}|\mathbf{h}|_{1,(l'+1)/2,Q_T^1}$$
$$+ T^{1-l/2}\sup_{t<T}\|\mathbf{h}(\cdot,t)\|_{W_2^{l'+1}(\mathcal{F}_1)}\|\rho_t\|_{W_2^{3/2,0}(G_T)}\big).$$

Hence collecting estimates and making use of (4.12) we obtain

$$\|\widetilde{\nabla}\cdot T_M(\widetilde{\mathbf{H}})\|_{\widehat{W}_2^{l,l/2}(Q_T^1)}$$
$$\leq c\|\mathbf{h}\|^2_{H^{l'+2,l'/2+1}(Q_T^1)}\big(T^{(1-l)/2} + T^{1-l/2}\|\rho_t\|_{W_2^{3/2,0}(G_T)}\big). \tag{4.14}$$

This completes the estimate of $\mathbf{l}_1(\mathbf{u}, q, \mathbf{h}, \rho)$.

2. *Estimate of* $\mathbf{l}_7' = \operatorname{rot}(rot\mathbf{h} - \mathcal{P}\operatorname{rot}\mathcal{P}\mathbf{h})$, $\mathcal{P} = L^{-1}\widehat{\mathcal{L}}^T\mathcal{L}$.

We represent $\mathbf{l}_7'$ in the form

$$\mathbf{l}_7' = \operatorname{rot}(I-\mathcal{P})\operatorname{rot}\mathbf{h} + \operatorname{rot}\mathcal{P}\operatorname{rot}(1-\mathcal{P})\mathbf{h}. \tag{4.15}$$

By (4.4) and (4.7),

$$\|\operatorname{rot}(I-\mathcal{P})\operatorname{rot}\mathbf{h}\|_{W_2^{l'}(\mathcal{F}_1)} \leq c\|(I-\mathcal{P})\operatorname{rot}\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)} \leq c\delta\|\mathbf{h}\|_{W_2^{l'+2}(\mathcal{F}_1)},$$
$$\|\operatorname{rot}\mathcal{P}\operatorname{rot}(I-\mathcal{P})\mathbf{h}\|_{W_2^{l'}(\mathcal{F}_1)} \leq c\|\operatorname{rot}(I-\mathcal{P})\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}$$
$$\leq c\|(I-\mathcal{P})\mathbf{h}\|_{W_2^{l'+2}(\mathcal{F}_1)}$$
$$\leq c\delta\|\mathbf{h}\|_{W_2^{l'+2}(\mathcal{F}_1)} + c\|\mathbf{h}\|_{W_2^{3/2+\eta}(\mathcal{F}_1)}\|I-\mathcal{P}\|_{W_2^{l'+2}(\mathcal{F}_1)},$$

hence

$$\|\mathbf{l}_7'\|_{W_2^{l',0}(Q_T^1)} \leq c\delta\|\mathbf{h}\|_{W_2^{l'+2,0}(Q_T^1)} + \sup_{t<T}\|\mathbf{h}(\cdot,t)\|_{W_2^{l'+1}(\mathcal{F}_1)}\|I-\mathcal{P}\|_{W_2^{l'+2,0}(Q_T^1)}.$$

We estimate the norm of $I-\mathcal{P}$ by the Ehrling inequality

$$\|I-\mathcal{P}\|_{W_2^{l'+2}(\mathcal{F}_1)} \le c\|I-\mathcal{P}\|^{\beta}_{W_2^{l+2}(\mathcal{F}_1)}\|I-\mathcal{P}\|^{1-\beta}_{W_2^{l+1}(\mathcal{F}_1)}$$

where $\beta = 1-l+l'$. Since

$$\|I-\mathcal{P}\|_{W_2^{l+1}(\mathcal{F}_1)} \le c\|\mathbf{R}\|_{W_2^{l+1}(\mathcal{F}_1)} \le c\|\rho\|_{W_2^{3/2+l}(\mathcal{G})},$$
$$\|I-\mathcal{P}\|_{W_2^{l+2}(\mathcal{F}_1)} \le c\|\mathbf{R}\|_{W_2^{l+2}(\mathcal{F}_1)} \le c\|\rho\|_{W_2^{5/2+l}(\mathcal{G})},$$

(by virtue of (4.6), (4.7), (4.9)), we obtain

$$\|I-\mathcal{P}\|_{W_2^{l'+2,0}(Q_T^1)} \le c\sup_{t<T}\|\rho(\cdot,t)\|^{1-\beta}_{W_2^{l+3/2}(\mathcal{G})}\left(\int_0^T\|\rho(\cdot,t)\|^{2\beta}_{W_2^{l+5/2}(\mathcal{G})}\,dt\right)^{1/2}$$
$$\le cT^{(l-l')/2}\sup_{t<T}\|\rho(\cdot,t)\|^{1-\beta}_{W_2^{l+3/2}(\mathcal{G})}\|\rho\|^{\beta}_{W_2^{l+5/2,0}(Q_T^1)}$$

and

$$\|\mathbf{l}_7'\|_{W_2^{l',0}(Q_T^1)} \le c\|\mathbf{h}\|_{H_2^{l'+2,l'/2+1}(Q_T^1)}$$
$$\times\big(\delta + T^{(l-l')/2}\sup_{t<T}\|\rho(\cdot,t)\|^{l-l'}_{W_2^{l+3/2}(\mathcal{G})}\|\rho\|^{1-l+l'}_{W_2^{l+5/2,0}(Q_T^1)}\big). \tag{4.16}$$

Now we pass to the estimate of $\|\mathbf{l}_7'\|_{\widehat{W}_2^{0,l'/2}(Q_T^1)}$. We have

$$\|\Delta_t(-h)\operatorname{rot}(I-\mathcal{P})\operatorname{rot}\mathbf{h}\|_{L_2(\mathcal{F}_1)} \le c\|\Delta_t(-h)(I-\mathcal{P})\operatorname{rot}\mathbf{h}\|_{W_2^1(\mathcal{F}_1)}$$
$$\le c\big(\|(\Delta_t(-h)\mathcal{P})\operatorname{rot}\mathbf{h}\|_{W_2^1(\mathcal{F}_1)} + \|(I-\mathcal{P})\Delta_t(-h)\operatorname{rot}\mathbf{h}\|_{W_2^1(\mathcal{F}_1)}\big),$$
$$\frac{1}{T^{l'/2}}\Big(\int_0^T\|(I-\mathcal{P})\operatorname{rot}\mathbf{h}\|^2_{W_2^1(\mathcal{F}_1)}\,dt\Big)^{1/2}$$
$$+\Big(\int_0^T\frac{dh}{h^{1+l'}}\int_h^T\|(I-\mathcal{P})\Delta_t(-h)\operatorname{rot}\mathbf{h}\|^2_{W_2^1(\mathcal{F}_1)}dt\Big)^{1/2}$$
$$\le c\delta\|\operatorname{rot}\mathbf{h}\|_{\widehat{W}_2^{l'/2}(0,T;W_2^1(\mathcal{F}_1))} \le c\delta\|\mathbf{h}\|_{\widehat{W}_2^{l'/2}(0,T;W_2^2(\mathcal{F}_1))},$$

$$\Big(\int_0^T \frac{dh}{h^{1+l'}} \int_h^T \|(\Delta_t(-h)\mathcal{P})\operatorname{rot}\mathbf{h}\|^2_{W_2^1(\mathcal{F}_1)}dt\Big)^{1/2}$$
$$\le c\Big(\int_0^T \frac{dh}{h^{1+l'}} \int_h^T \|\operatorname{rot}\mathbf{h}\|^2_{W_2^{3/2+\eta}(\mathcal{F}_1)}\|\Delta_t(-h)\mathcal{P}\|^2_{W_2^1(\mathcal{F}_1)}\,dt\Big)^{1/2}$$
$$\le cT^{(1-l')/2}\Big(\int_0^T \|\operatorname{rot}\mathbf{h}\|^2_{W_2^{l'+1}(\mathcal{F}_1)}\,dt\int_0^t \|\rho_\tau\|^2_{W_2^{3/2}(\mathcal{G})}d\tau\Big)^{1/2},$$

by virtue of (4.11).

The last term in (4.15) is estimated in a similar way:

$$\|\operatorname{rot}\mathcal{P}\operatorname{rot}(I-\mathcal{P})\mathbf{h}\|_{L_2(\mathcal{F}_1)} \le c\|\operatorname{rot}(I-\mathcal{P})\mathbf{h}\|_{W_2^1(\mathcal{F}_1)},$$
$$\|\Delta_t(-h)\operatorname{rot}\mathcal{P}\operatorname{rot}(I-\mathcal{P})\mathbf{h}\|_{L_2(\mathcal{F}_1)} \le c\|\Delta_t(-h)\mathcal{P}\operatorname{rot}(I-\mathcal{P})\mathbf{h}\|_{W_2^1(\mathcal{F}_1)}$$
$$\le c\big(\|(\Delta_t(-h)\mathcal{P})\operatorname{rot}(I-\mathcal{P})\mathbf{h}\|_{W_2^1(\mathcal{F}_1)} + \|\mathcal{P}\Delta_t(-h)\operatorname{rot}(I-\mathcal{P})\mathbf{h}\|_{W_2^1(\mathcal{F}_1)}\big),$$
$$\|(\Delta_t(-h)\mathcal{P})\operatorname{rot}(I-\mathcal{P})\mathbf{h}\|_{W_2^1(\mathcal{F}_1)}$$
$$\le c\|\Delta_t(-h)\mathcal{P}\|_{W_2^1(\mathcal{F}_1)}\|\operatorname{rot}(I-\mathcal{P})\mathbf{h}\|_{W_2^{3/2+\eta}(\mathcal{F}_1)}$$
$$\le c\|\Delta_t(-h)\rho\|_{W_2^{3/2}(\mathcal{G})}\|\operatorname{rot}(I-\mathcal{P})\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)},$$
$$\frac{1}{T^{l'/2}}\Big(\int_0^T \|\mathcal{P}\operatorname{rot}(I-\mathcal{P})\mathbf{h}\|^2_{W_2^1(\mathcal{F}_1)}dt\Big)^{1/2}$$
$$+\Big(\int_0^T \frac{dh}{h^{1+l'}}\int_h^T \|\mathcal{P}\Delta_t(-h)\operatorname{rot}(I-\mathcal{P})\mathbf{h}\|^2_{W_2^1(\mathcal{F}_1)}\,dt\Big)^{1/2}$$
$$\le c\|\operatorname{rot}(I-\mathcal{P})\mathbf{h}\|_{\widehat{W}_2^{l'/2}(0,T;W_2^1(\mathcal{F}_1))},$$
$$\Big(\int_0^T \frac{dh}{h^{1+l'}}\int_h^T \|(\Delta_t(-h)\mathcal{P})\operatorname{rot}(I-\mathcal{P})\mathbf{h}\|^2_{W_2^1(\mathcal{F}_1)}dt\Big)^{1/2}$$
$$\le cT^{(1-l')/2}\Big(\int_0^T \|\operatorname{rot}(I-\mathcal{P})\mathbf{h}\|^2_{W_2^{l'+1}(\mathcal{F}_1)}\,dt\int_0^t \|\rho_\tau\|^2_{W_2^{3/2}(\mathcal{G})}d\tau\Big)^{1/2}$$

Now we estimate the norms of $\operatorname{rot}(1-\mathcal{P})\mathbf{h}$. As we have seen above,

$$\|\operatorname{rot}(I-\mathcal{P})\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)} \le c\delta\|\mathbf{h}\|_{W_2^{l'+2}(\mathcal{F}_1)} + c\|\mathbf{h}\|_{W_2^{3/2+\eta}(\mathcal{F}_1)}\|\rho\|_{W_2^{l'+5/2}(\mathcal{G})},$$

moreover,

$$\|\operatorname{rot}(I-\mathcal{P})\mathbf{h}\|_{W_2^1(\mathcal{F}_1)} \le c\delta\|\mathbf{h}\|_{W_2^2(\mathcal{F}_1)} + c\|\mathbf{h}\|_{W_2^{3/2+\eta}(\mathcal{F}_1)}\|\rho\|_{W_2^{5/2}(\mathcal{G})},$$

$$\|\Delta_t(-h)(\operatorname{rot}(I-\mathcal{P})\mathbf{h})\|_{W_2^1(\mathcal{F}_1)} \le c\big(\delta\|\Delta_t(-h)\mathbf{h}\|_{W_2^2(\mathcal{F}_1)}$$
$$+ \|\Delta_t(-h)\mathbf{h}\|_{W_2^{3/2+\eta}(\mathcal{F}_1)}\|\rho\|_{W_2^{5/2}(\mathcal{G})}$$
$$+ \|\mathbf{h}\|_{W_2^2(\mathcal{F}_1)}\|\Delta_t(-h)\mathcal{P}\|_{W_2^{3/2+\eta}(\mathcal{F}_1)} + \|\mathbf{h}\|_{W_2^{3/2+\eta}(\mathcal{F}_1)}\|\Delta_t(-h)\mathcal{P}\|_{W_2^2(\mathcal{F}_1)}\big).$$

It follows that

$$\|\operatorname{rot}(I-\mathcal{P})\mathbf{h}\|_{\widehat{W}_2^{l'/2}(0,T;W_2^1(\mathcal{F}_1))} \le c\frac{1}{T^{l'/2}}\|(I-\mathcal{P})\mathbf{h}\|_{W_2^{2,0}(Q_T^1)}$$
$$+\left(\int_0^T \frac{dh}{h^{1+l'}}\int_h^T \|(I-\mathcal{P})\Delta_t(-h)\mathbf{h})\|^2_{W_2^2(\mathcal{F}_1)}\,dt\right)^{1/2}$$
$$\le c\delta\|\mathbf{h}\|_{\widehat{W}^{l'/2}(0,T;W_2^2(\mathcal{F}_1))} + cT^{(1-l')/2}\sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}\sup_{t<T}\|\rho\|_{W_2^{2+l}(\mathcal{G})}$$
$$+ c\sup_{t<T}\|\rho\|_{W_2^{2+l'}(\mathcal{G})}|\mathbf{h}|_{3/2+\eta,l'/2,Q_T^1}$$
$$\le c\delta\|\mathbf{h}\|_{\widehat{W}^{l'/2}(0,T;W_2^2(\mathcal{F}_1))} + cT^{(1-l')/2}\sup_{t<T}\|\rho\|_{W_2^{2+l}(\mathcal{G})}\|\mathbf{h}\|_{H^{l'+2,l'/2+1}(Q_T^1)},$$

$$\Big(\int_0^T \frac{dh}{h^{1+l'}}\int_h^T \|(\Delta_t(-h)\mathcal{P})\mathbf{h})\|^2_{W_2^2(\mathcal{F}_1)}dt\Big)^{1/2}$$
$$\le cT^{(1-l')/2}\int_0^T \|\mathbf{h}\|^2_{W_2^2(\mathcal{F}_1)}\,dt\int_0^t \|\rho_\tau\|^2_{W_2^{3/2+l}(\mathcal{G})}d\tau\Big)^{1/2}$$
$$+ cT^{(l-l')/2}\sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}\|\rho\|_{W_2^{l/2}(0,T;W_2^{5/2}(\mathcal{G}))}.$$

Collecting the estimates we obtain

$$\|\mathbf{l}_7'\|_{\widehat{W}_2^{0,l'/2}(Q_T^1)} \le c\delta\|\mathbf{h}\|_{H^{2+l',1+l'/2}(Q_T^1)}$$
$$+ cT^{(1-l')/2}\|\mathbf{h}\|_{H^{2+l',1+l'/2}(Q_T^1)}\big(\sup_{t<T}\|\rho\|_{W_2^{l+2}(\mathcal{G})} + \|\rho_t\|_{W_2^{3/2+l,0}(G_T)}\big)$$
$$+ cT^{(l-l')/2}\sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{G})}\big(\|\rho\|_{W_2^{5/2+l}(G_T)}$$
$$+ \sup_{t<T}\|\rho\|^{l-l'}_{W_2^{l'+3/2}(\mathcal{G})}\|\rho\|^{1-l+l'}_{W_2^{5/2+l,0}(G_T)}\big). \tag{4.17}$$

3. *Estimate of* $\frac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\mathbf{h}$.
By (4.1),

$$\|\frac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\mathbf{h}\|_{W_2^{l'}(\mathcal{F}_1)} \leq c\|\widehat{\mathcal{L}}_t\|_{W_2^{l'}(\mathcal{F}_1)}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}$$
$$\leq c\|\rho_t\|_{W_2^{l'+1/2}(\mathcal{G})}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)},$$

which implies

$$\|\frac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\mathbf{h}\|_{W_2^{l',0}(Q_T^1)} \leq cT^{1/2}\sup_{t<T}\|\rho_t(\cdot,t)\|_{W_2^{l'+1/2}(\mathcal{G})}\sup_{t<T}\|\mathbf{h}(\cdot,t)\|_{W_2^{l'+1}(\mathcal{F}_1)}.$$

Further we have

$$\frac{1}{T^{l'/2}}\|\frac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\mathbf{h}\|_{L_2(Q_T^1)} \leq cT^{(1-l')/2}\sup_{t<T}\|\widehat{\mathcal{L}}_t\|_{L_2(\mathcal{F}_1)}\sup_{Q_T^1}|\mathbf{h}(x,t)|$$
$$\leq cT^{(1-l')/2}\sup_{t<T}\|\rho_t\|_{W_2^{1/2}(\mathcal{G})}\sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)},$$
$$\|(\Delta_t(-h)\widehat{\mathcal{L}}_t)\frac{1}{L}\mathcal{L}\mathbf{h}\|_{L_2(\mathcal{F}_1)} \leq c\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}\|\Delta_t(-h)\rho_t\|_{W_2^{1/2}(\mathcal{G})},$$
$$\|\widehat{\mathcal{L}}_t(\Delta_t(-h)\frac{1}{L}\mathcal{L})\mathbf{h}\|_{L_2(\mathcal{F}_1)}$$
$$\leq c\|\widehat{\mathcal{L}}_t\|_{W_2^{l}(\mathcal{F}_1)}\sup_{\mathcal{F}_1}|\mathbf{h}(x,t)|\|\Delta_t(-h)\frac{1}{L}\mathcal{L}\|_{W_2^{3/2-l}(\mathcal{F}_1)}$$
$$\leq c\|\rho_t\|_{W_2^{l+1/2}(\mathcal{G})}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}\|\Delta_t(-h)\rho\|_{W_2^{2-l}(\mathcal{G})},$$
$$\|\widehat{\mathcal{L}}_t\frac{1}{L}\mathcal{L}\Delta_t(-h)\mathbf{h}\|_{L_2(\mathcal{F}_1)} \leq c\|\widehat{\mathcal{L}}_t\|_{W_2^{l}(\mathcal{F}_1)}\|\Delta_t(-h)\mathbf{h}\|_{W_2^{3/2-l}(\mathcal{F}_1)}$$
$$\leq c\|\rho_t\|_{W_2^{l+1/2}(\mathcal{G})}\|\Delta_t(-h)\mathbf{h}\|_{W_2^{1}(\mathcal{F}_1)},$$

and, as a consequence,

$$\Big(\int_0^T\frac{dh}{h^{l'+1}}\int_h^T\|(\Delta_t(-h)\widehat{\mathcal{L}}_t)\frac{1}{L}\mathcal{L}\mathbf{h}\|^2_{L_2(\mathcal{F}_1)}\,dt\Big)^{1/2}$$
$$\leq c\sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}|\rho_t|_{1/2,l'/2,G_T}$$

$$
\begin{aligned}
&\le cT^{1/2}\sup_{t<T}\|\mathbf{h}(\cdot,t)\|_{W_2^{l'+1}(\mathcal{G})}\|\rho_t\|_{H^{l+3/2,l/2+3/4}(G_T)},\\
&\Big(\int_0^T\frac{dh}{h^{l'+1}}\int_h^T\|\widehat{\mathcal{L}}_t(\Delta_t(-h)\frac{1}{L}\mathcal{L})\mathbf{h}\|^2_{L_2(\mathcal{F}_1)}\Big)^{1/2}\\
&\le cT^{1-l'/2}\sup_{t<T}\|\rho_t\|_{W_2^{l+1/2}(\mathcal{G})}\sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}\|\rho_t\|_{W_2^{l+1,0}(G_T)},\\
&\Big(\int_0^T\frac{dh}{h^{l'+1}}\int_h^T\|\widehat{\mathcal{L}}_t\frac{1}{L}\mathcal{L}\Delta_t(-h)\mathbf{h}\|^2_{L_2(\mathcal{F}_1)}\Big)^{1/2}\le c\sup_{t<T}\|\rho_t\|_{W_2^{l+1/2}(\mathcal{G})}|\mathbf{h}|_{1,l'/2,Q_T^1}\\
&\le cT^{1/2}\sup_{t<T}\|\rho_t\|_{W_2^{l+1/2}(\mathcal{G})}\|\mathbf{h}\|_{H^{l'+2,l'/2+1}(Q_T^1)}.
\end{aligned}
$$

From the above estimates it follows that

$$
\begin{aligned}
\|\frac{1}{L}\widehat{\mathcal{L}}_t\mathcal{L}\mathbf{h}\|_{\widehat{W}_2^{l',l'/2}(Q_T^1)}\le cT^{(1-l')/2}\|\mathbf{h}\|_{H^{2+l',1+l'/2}(Q_T^1)}\|\rho_t\|_{W_2^{3/2+l,3/4+l/2}(G_T)}\\
\big(1+T^{1/2}\|\rho_t\|_{W_2^{l+1,0}(G_T)}\big). \qquad (4.18)
\end{aligned}
$$

4. *Estimate of* $\operatorname{rot}(\mathcal{L}^{-1}\mathbf{u}\times\mathbf{h})$.

This expression is estimated in the same way as $(\mathcal{L}^{-1}\mathbf{u}\cdot\nabla)\mathbf{u}$ (see [9], Subsec. 3.3). We have

$$
\begin{aligned}
&\|\operatorname{rot}(\mathcal{L}^{-1}\mathbf{u}\cdot\mathbf{h})\|_{W_2^{l'}(\mathcal{F}_1)}\\
&\le c\big(\|\nabla\mathcal{L}^{-1}\mathbf{u}\|_{W_2^{l'}(\mathcal{F}_1)}\|\mathbf{h}\|_{W_2^{3/2-l'}(\mathcal{F}_1)}+\|\nabla\mathbf{h}\|_{W_2^{l'}(\mathcal{F}_1)}\|\mathcal{L}^{-1}\mathbf{u}\|_{W_2^{3/2-l'}(\mathcal{F}_1)}\big)\\
&\le c\|\mathcal{L}^{-1}\mathbf{u}\|_{W_2^{l'+1}(\mathcal{F}_1)}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}\le c\|\mathbf{u}\|_{W_2^{l'+1}(\mathcal{F}_1)}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)},
\end{aligned}
$$

which implies

$$
\|\operatorname{rot}(\mathcal{L}^{-1}\mathbf{u}\cdot\mathbf{h})\|_{W_2^{l',0}(Q_T^1)}\le cT^{1/2}\sup_{t<T}\|\mathbf{u}\|_{W_2^{l+1}(\mathcal{F}_1)}\sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}.
$$

Moreover,

$$
\begin{aligned}
&\frac{1}{T^{l'/2}}\|\operatorname{rot}(\mathcal{L}^{-1}\mathbf{u}\times\mathbf{h})\|_{L_2(Q_T^1)}\\
&\le cT^{(1-l')/2}\sup_{t<T}\|\mathbf{u}\|_{W_2^{l'+1}(\mathcal{F}_1)}\sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)},\\
&\|\operatorname{rot}(\Delta_t(-h)\mathcal{L}^{-1}\mathbf{u}\times\mathbf{h})\|_{L_2(\mathcal{F}_1)}\le c\big(\sup_{\mathcal{F}_1}|\mathbf{h}(x,t)|\|\Delta_t(-h)\mathcal{L}^{-1}\mathbf{u}\|_{W_2^1(\mathcal{F}_1)}\\
&+\|\nabla\mathbf{h}\|_{W_2^{l'}(\mathcal{F}_1)}\|\Delta_t(-h)\mathcal{L}^{-1}\mathbf{u}\|_{W_2^{3/2-l'}(\mathcal{F}_1)}\big)\\
&\le c\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}\|\Delta_t(-h)\mathcal{L}^{-1}\mathbf{u}\|_{W_2^1(\mathcal{F}_1)}\\
&\le c\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}\big(\|\Delta_t(-h)\mathbf{u}\|_{W_2^1\mathcal{F}_1)}\\
&+\|\mathbf{u}\|_{W_2^{3/2+\eta}(\mathcal{F}_1)}\|\Delta_t(-h)\mathcal{L}^{-1}\|_{W_2^1(\mathcal{F}_1)}\big),\\
&\|\operatorname{rot}(\mathcal{L}^{-1}\mathbf{u}\times\Delta_t(-h)\mathbf{h})\|_{L_2(\mathcal{F}_1)}\le c\big(\sup_{\mathcal{F}_1}|\mathbf{u}(x,t)|\|\Delta_t(-h)\mathbf{h}\|_{W_2^1(\mathcal{F}_1)}\\
&+\|\nabla\mathcal{L}^{-1}\mathbf{u}\|_{W_2^{l'}(\mathcal{F}_1)}\|\Delta_t(-h)\mathbf{h}\|_{W_2^{3/2-l'}(\mathcal{F}_1)}\\
&\le c\|\mathbf{u}\|_{W_2^{l'+1}(\mathcal{F}_1)}\|\Delta_t(-h)\mathbf{h}\|_{W_2^1(\mathcal{F}_1)},
\end{aligned}
$$

and, as a consequence,

$$
\begin{aligned}
&\Big(\int_0^T\frac{dh}{h^{1+l'}}\int_h^T\|\operatorname{rot}(\mathcal{L}^{-1}\mathbf{u}\times\Delta_t(-h)\mathbf{h})\|^2_{L_2(\mathcal{F}_1)}dt\Big)^{1/2}\\
&\le c\sup_{t<T}\|\mathbf{u}\|_{W_2^{l+1}(\mathcal{F}_1)}|\mathbf{h}|_{1,l'/2,Q_T^1}\\
&\le cT^{1/2}\|\mathbf{h}\|_{H^{l'+2,l'/2+1}(Q_T^1)}\|\mathbf{u}\|_{H^{l+2,l/2+1}(Q_T^1)},\\
&\Big(\int_0^T\frac{dh}{h^{1+l'}}\int_h^T\|rot(\Delta_t(-h)\mathcal{L}^{-1}\mathbf{u}\times\mathbf{h})\|^2_{L_2(\mathcal{F}_1)}\,dt\Big)^{1/2}\\
&\le cT^{1/2}\|\mathbf{h}\|_{H^{l'+2,l'/2+1}(Q_T^1)}\|\mathbf{u}\|_{H^{l+2,l/2+1}(Q_T^1)}\\
&+c\sup_{t<T}\|\mathbf{u}\|_{W_2^{l+1}(\mathcal{F}_1)}\sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}|\mathbf{R}|_{1,l'/2,Q_T^1}\\
&\le c\big(T^{1/2}+T^{1-l/2}\|\rho_t\|_{W_2^{3/2,0}(G_T)}\big)\|\mathbf{h}\|_{H^{l'+2,l'/2+1}(Q_T^1)}\|\mathbf{u}\|_{H^{l+2,l/2+1}(Q_T^1)}.
\end{aligned}
$$

Putting all the above estimates together, we obtain

$$\|\operatorname{rot}(\mathcal{L}^{-1}\mathbf{u}\times\mathbf{h})\|_{\widehat{W}_2^{l',l'/2}(Q_T^1)} \\ \le cT^{(1-l')/2}\|\mathbf{h}\|_{H^{2+l',1+l'/2}(Q_T^1)}\|\mathbf{u}\|_{H^{2+l,1+l/2}(Q_T^1)}\big(1+T^{1/2}\|\rho_t\|_{W_2^{3/2,0}(G_T)}\big) \tag{4.19}$$

5. *Estimate of* $\rho_t^*(\mathcal{L}^{-1}\mathbf{N}\cdot\nabla)\frac{1}{L}\mathcal{L}\mathbf{h}$.

We follow the arguments in [9], Subsec. 3.4. We recall that $\frac{1}{L}\mathcal{L}\mathbf{h}=\widetilde{\mathbf{H}}$. By (4.1), (4.6),

$$\begin{aligned}\|\rho_t^*(\mathcal{L}^{-1}\mathbf{N}\cdot\nabla)\widetilde{\mathbf{H}}\|_{W_2^{l'}(\mathcal{F}_1)} &\le c\|\nabla\widetilde{\mathbf{H}}\|_{W_2^{l'}(\mathcal{F}_1)}\|\rho_t^*\|_{W_2^{3/2-l'}(\mathcal{F}_1)}\\ &\le c\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}\|\rho_t\|_{W_2^{l'+1/2}(\mathcal{G})},\end{aligned}$$

which implies

$$\|\rho_t^*(\mathcal{L}^{-1}\ N\cdot\nabla)\widetilde{\mathbf{H}}\|_{W_2^{l',0}(Q_T^1)}\le cT^{1/2}\sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}\sup_{t<T}\|\rho_t\|_{W_2^{l'+1/2}(\mathcal{G})}.$$

In addition, we have

$$\begin{aligned}&\frac{1}{T^{l'/2}}\|\rho_t^*(\mathcal{L}^{-1}\mathbf{N}\cdot\nabla)\widetilde{\mathbf{H}}\|_{L_2(Q_T^1)}\\ &\le cT^{(1-l')/2}\sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}\sup_{t<T}\|\rho_t\|_{W_2^{l'+1/2}(\mathcal{G})},\\ &\|(\Delta_t(-h)\rho_t^*)(\mathcal{L}^{-1}\mathbf{N}\cdot\nabla)\widetilde{\mathbf{H}}\|_{L_2(\mathcal{F}_1)}\le c\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}\|\Delta_t(-h)\rho_t^*\|_{W_2^{3/2-l'}(\mathcal{F}_1)}\\ &\le c\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}\|\Delta_t(-h)\rho_t\|_{W_2^{1-l'}(\mathcal{G})},\\ &\|\rho_t^*(\mathcal{L}^{-1}\mathbf{N}\cdot\nabla)\Delta_t(-h)\widetilde{\mathbf{H}}\|_{L_2(\mathcal{F}_1)}\le c\sup_{\mathcal{F}_1}|\rho_t^*(x,t)|\|\Delta_t(-h)\nabla\widetilde{\mathbf{H}}\|_{L_2(\mathcal{F}_1)}\\ &\le c\|\rho_t\|_{W_2^{l+1/2}(\mathcal{G})}\big(\|\Delta_t(-h)\mathbf{h}\|_{W_2^1(\mathcal{F}_1)}\\ &+\|\Delta_t(-h)\frac{1}{L}\mathcal{L}\|_{W_2^1(\mathcal{F}_1)}\|\mathbf{h}\|_{W_2^{3/2+\eta}(\mathcal{F}_1)}\big).\end{aligned}$$

It follows that

$$\Big(\int\limits_0^T \frac{dh}{h^{1+l'}} \int\limits_h^T \|(\Delta_t(-h)\rho_t^*)(\mathcal{L}^{-1}\mathbf{N}\cdot\nabla)\widetilde{\mathbf{H}}\|^2_{L_2(\mathcal{F}_1)}dt\Big)^{1/2}$$
$$\le \sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}|\rho_t|_{1/2,l'/2,G_T}$$
$$\le cT^{1/2}\sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}\|\rho_t\|_{H^{l+3/2,l/2+3/4}(G_T)},$$
$$\Big(\int\limits_0^T \frac{dh}{h^{1+l'}} \int\limits_h^T \|\rho_t^*(\mathcal{L}^{-1}\mathbf{N}\cdot\nabla)\Delta_t(-h)\widetilde{\mathbf{H}}\|^2_{L_2(\mathcal{F}_1)}\,dt\Big)^{1/2}$$
$$\le c\sup_{t<T}\|\rho_t\|_{W_2^{l'+1/2}(\mathcal{G})}\big(T^{1/2}\|\mathbf{h}\|_{H^{l'+2,l'/2+1}(Q_T^1)}$$
$$+\,T^{1-l'/2}\sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_1)}\|\rho_t\|_{W_2^{3/2,0}(G_T)}\big).$$

Collecting the estimates we obtain

$$\|\rho_t^*(\mathcal{L}^{-1}\mathbf{N}\cdot\nabla)\widetilde{\mathbf{H}}\|_{\widehat{W}_2^{l',l'/2}(Q_T^1)}$$
$$\le cT^{(1-l')/2}\|\mathbf{h}\|_{H^{2+l',1+l'/2}(Q_T^1)}\|\rho_t\|_{H^{3/2+l,3/4+l/2}(G_T)}\big(1+\|\rho_t\|_{W_2^{3/2,0}(G_T)}\big). \tag{4.20}$$

6. *Estimates of* $\operatorname{rot}\mathbf{l}_8(\mathbf{h},\rho)$ *and* $\mathbf{l}_{8t}(\mathbf{h},\rho)$.
As we have already seen above,

$$\|\operatorname{rot}\mathbf{l}_8\|_{W_2^{l'}(\mathcal{F}_2)}\le c\|(I-\mathcal{P})\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_2)}\le c\delta\|\mathbf{h}\|_{W_2^{l'+1,0}(\mathcal{F}_2)},$$
$$\|\operatorname{rot}\mathbf{l}_8\|_{W_2^{l'+1}(\mathcal{F}_2)}\le c\|(I-\mathcal{P})\mathbf{h}\|_{W_2^{l'+2}(\mathcal{F}_2)}\le c\big(\delta\|\mathbf{h}\|_{W_2^{l'+2}(\mathcal{F}_2)}$$
$$+\,\|\mathbf{h}\|_{W_2^{3/2+\eta}(\mathcal{F}_2)}\|I-\mathcal{P}\|_{W_2^{l'+2}(\mathcal{F}_2)}\big),$$

which implies

$$\|\operatorname{rot}\mathbf{l}_8\|_{W_2^{l'+1,0}(Q_T^2)}+\sup_{t<T}\|\operatorname{rot}\mathbf{l}_8\|_{W_2^{l'}(\mathcal{F}_2)}$$
$$\le c\big(\delta+T^{(l-l')/2}\sup_{t<T}\|\rho\|^{l-l'}_{W_2^{l+3/2}(\mathcal{G})}\|\rho\|^{1-l+l'}_{W_2^{l+5/2,0}(G_T)}\big)\|\mathbf{h}\|_{H^{l'+2,l'/2+1}(Q_T^2)}. \tag{4.21}$$

Now we consider the time derivative $\mathbf{l}_{8t} = (I-\mathcal{P})\mathbf{h}_t - \mathcal{P}_t\mathbf{h}$ and the finite difference

$$\begin{aligned}&\Delta_t(-h)\mathbf{l}_{8t}(\mathbf{h},\rho)\\ &= (I-\mathcal{P})\Delta_t(-h)\mathbf{h}_t - (\Delta_t(-h)\mathcal{P})\mathbf{h}_t - (\Delta_t(-h)\mathcal{P}_t)\mathbf{h} - \mathcal{P}_t\Delta_t(-h)\mathbf{h}.\end{aligned}$$

We have

$$\begin{aligned}&\frac{1}{T^{l'/2}}\|(I-\mathcal{P})\mathbf{h}_t\|_{L_2(Q_T)} + \left(\int_0^T \frac{dh}{h^{1+l'}}\int_h^T \|(I-\mathcal{P})\Delta_t(-h)\mathbf{h}_t\|^2_{L_2(\mathcal{F}_2)}\,dt\right)^{1/2}\\
&\le c\delta\|\mathbf{h}_t\|_{\widehat{W}_2^{0,l'/2}(Q_T^2)},\\
&\frac{1}{T^{l'/2}}\|\mathcal{P}_t\mathbf{h}\|_{L_2(Q_T)} \le cT^{(1-l')/2}\sup_{t<T}\|\rho_t\|_{W_2^{l'+1/2}(\mathcal{G})}\sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_2)},\\
&\|\Delta_t(-h)\mathcal{P}\mathbf{h}_t\|_{L_2(\mathcal{F}_2)} \le c\|\mathbf{h}_t\|_{W_2^{l'}(\mathcal{F}_2)}\|\Delta_t(-h)\mathcal{P}\|_{W_2^{3/2-l'}(\mathcal{F}_2)},\\
&\|(\Delta_t(-h)\mathcal{P}_t)\mathbf{h}\|_{L_2(\mathcal{F}_2)} \le c\sup_{\mathcal{F}_2}|\mathbf{h}(x,t)|\|\Delta_t(-h)\mathcal{P}_t\|_{L_2(\mathcal{F}_2)},\\
&\|\mathcal{P}_t\Delta_t(-h)\mathbf{h}\|_{L_2(\mathcal{F}_2)} \le c\|\mathcal{P}_t\|_{W_2^{l'}(\mathcal{F}_2)}\|\Delta_t(-h)\mathbf{h}\|_{W_2^{3/2-l'}(\mathcal{F}_2)}.\end{aligned}$$

In view of (4.11) it follows that

$$\begin{aligned}&\Big(\int_0^T \frac{dh}{h^{1+l'}}\int_h^T \|(\Delta_t(-h)\mathcal{P})\mathbf{h}_t\|^2_{L_2(\mathcal{F}_2)}dt\Big)^{1/2}\\
&\le cT^{(1-l')/2}\left(\int_0^T \|\mathbf{h}_t\|^2_{W_2^{l'}(\mathcal{F}_2)}\,dt\int_0^t \|\rho_\tau\|^2_{W_2^{l'+3/2}(\mathcal{G})}d\tau\right)^{1/2},\\
&\left(\int_0^T \frac{dh}{h^{1+l'}}\int_h^T \|(\Delta_t(-h)\mathcal{P}_t)\mathbf{h}\|^2_{L_2(\mathcal{F}_2)}\,dt\right)^{1/2}\\
&\le c\sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_2)}|\rho_t|_{1/2,l'/2,G_T}\\
&\le cT^{1/2}\sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1}(\mathcal{F}_2)}\|\rho_t\|_{H^{l+3/2,l/2+3/4}(G_T)},\end{aligned}$$

$$\left(\int_0^T \frac{dh}{h^{1+l'}} \int_h^T \|\mathcal{P}_t \Delta_t(-h)\mathbf{h}\|^2_{L_2(\mathcal{F}_2)}\, dt\right)^{1/2}$$
$$\leq c \sup_{t<T} \|\rho_t\|_{W_2^{l'+1/2}(\mathcal{G})} |\mathbf{h}|_{1,l'/2,Q_T^2}$$
$$\leq cT^{1/2} \sup_{t<T} \|\rho_t\|_{W_2^{l'+1/2}(\mathcal{G})} \|\mathbf{h}\|_{H^{l'+2,l'/2+1}(Q_T^2)}.$$

Putting the estimates together, we obtain

$$\|\mathbf{l}_{8t}\|_{\widehat{W}_2^{0,l'/2}(\mathcal{F}_2)}$$
$$\leq c\delta\|\mathbf{h}_t\|_{\widehat{W}_2^{0,l'/2}(Q_T^2)} + cT^{(1-l')/2}\|\mathbf{h}\|_{H^{2+l',1+l'/2}(Q_T^2)}\|\rho\|_{H^{3/2+l,3/4+l/2}(G_T)}. \tag{4.22}$$

The next step is the estimate of the norm $\|\mathbf{l}_{8t}\|_{\widehat{W}_2^{l'/2}(0,T;W_2^{-1/2}(\mathcal{G}))}$. We repeat the above arguments making use of the inequality (4.5). Sometimes we replace the $W_2^{-1/2}(\mathcal{G})$- norm by a stronger $L_2(\mathcal{G})$-norm. We have

$$\frac{1}{T^{l'/2}}\left(\int_0^T \|(I-\mathcal{P})\mathbf{h}_t\|^2_{W_2^{-1/2}(\mathcal{G})}\, dt\right)^{1/2}$$
$$+\left(\int_0^T \frac{dh}{h^{1+l'}}\int_h^T \|(I-\mathcal{P})\Delta_t(-h)\mathbf{h}_t\|^2_{W_2^{-1/2}(\mathcal{G})}\, dt\right)^{1/2}$$
$$\leq c\delta\|\mathbf{h}_t\|_{\widehat{W}_2^{l'/2}(0,T;W_2^{-1/2}(\mathcal{G}))},$$
$$\frac{1}{T^{l'/2}}\left(\int_0^T \|\mathcal{P}_t\mathbf{h}\|^2_{W_2^{-1/2}(\mathcal{G})}\, dt\right)^{1/2} \leq \frac{1}{T^{l'/2}}\left(\int_0^T \|\mathcal{P}_t\mathbf{h}\|^2_{L_2(\mathcal{G})}\, dt\right)^{1/2}$$
$$\leq cT^{(1-l')/2}\sup_{t<T}\|\mathcal{P}_t\|_{W_2^{l'-1/2}(\mathcal{G})}\sup_{t<T}\|\mathbf{h}\|_{W_2^{3/2-l'}(\mathcal{G})}$$
$$\leq cT^{(1-l')/2}\sup_{t<T}\|\rho_t\|_{W_2^{l'+1/2}(\mathcal{G})}\sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1/2}(\mathcal{G})},$$
$$\|\mathcal{P}_t\Delta_t(-h)\mathbf{h}\|_{L_2(\mathcal{G})} \leq c\|\Delta_t(-h)\mathbf{h}\|_{W_2^{l'-1/2}(\mathcal{G})}\|\mathcal{P}_t\|_{W_2^{3/2-l'}(\mathcal{G})},$$
$$\|(\Delta_t(-h)\mathcal{P}_t)\mathbf{h}\|_{L_2(\mathcal{G})} \leq c\sup_{\mathcal{G}}|\mathbf{h}(x,t)|\|(\Delta_t(-h)\mathcal{P}_t)\|_{L_2(\mathcal{G})},$$
$$\|(\Delta_t(-h)\mathcal{P})\mathbf{h}_t\|_{L_2(\mathcal{G})} \leq c\|\mathbf{h}_t\|_{W_2^{l'-1/2}(\mathcal{G})}\|\Delta_t(-h)\mathcal{P}\|_{W_2^{3/2-l'}(\mathcal{G})}.$$

The last three inequalities imply

$$\left(\int_0^T \frac{dh}{h^{1+l'}} \int_h^T \|\mathcal{P}_t\Delta_t(-h)\mathbf{h}\|^2_{L_2(\mathcal{G})}\,dt\right)^{1/2}$$

$$\leq cT^{(1-l')/2}\left(\int_0^T \|\mathbf{h}_t\|^2_{W_2^{l'-1/2}(\mathcal{G})}\,dt \int_0^t \|\rho_\tau\|^2_{W_2^{l'+3/2}(\mathcal{G})}d\tau\right)^{1/2},$$

$$\left(\int_0^T \frac{dh}{h^{1+l'}} \int_h^T \|(\Delta_t(-h)\mathcal{P}_t)\mathbf{h}\|^2_{L_2(\mathcal{G})}\right)^{1/2}$$

$$\leq c\sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1/2}(\mathcal{G})}|\rho_t|_{1,l'/2,G_T}$$

$$\leq cT^{1/4}\sup_{t<T}\|\mathbf{h}\|_{W_2^{l'+1/2}(\mathcal{G})}\|\rho_t\|_{H^{l'+3/2,l'/2+3/4}(G_T)},$$

$$\Big(\int_0^T \frac{dh}{h^{1+l'}} \int_h^T \|(\Delta_t(-h)\mathcal{P})\mathbf{h}_t\|^2_{L_2(\mathcal{G})}\Big)^{1/2}$$

$$\leq cT^{(1-l')/2}\left(\int_0^T \|\mathbf{h}_t\|^2_{W_2^{l'-1/2}(\mathcal{G})}\,dt \int_0^t \|\rho_t\|^2_{W_2^{l'+3/2}(\mathcal{G})}\,d\tau\right)^{1/2}.$$

Thus,

$$\begin{aligned}&\|\mathbf{l}_{8t}\|_{\widehat{W}_2^{l'/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\\ &\leq c\delta\|\mathbf{h}_t\|_{\widehat{W}_2^{l'/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\\ &+ cT^{(1-l')/2}\|\mathbf{h}\|_{H^{2+l',1+l'/2}(Q_T^2)}\|\rho\|_{H^{3/2+l,3/4+l/2}(G_T)}.\end{aligned} \tag{4.23}$$

This completes the estimates of $\mathbf{l}_{8t}$.

7. *Estimates of* $l_9(\mathbf{h},\rho)$ *and* $\mathbf{A}^{(i)}(\mathbf{h},\rho)$.

In view of (1.11),

$$\mathbf{A}^{(i)}(\mathbf{h},\rho) = (\mathcal{Q}\mathbf{N}^*\otimes\mathbf{N}^*)\mathbf{h}^{(i)},$$

where

$$\mathcal{Q} = \frac{\widehat{\mathcal{L}}(y,\rho^*)\widehat{\mathcal{L}}^T(y,\rho^*)}{|\widehat{\mathcal{L}}^T\mathbf{N}^*|^2} - \frac{I}{|\mathbf{N}^*|^2}.$$

Although this matrix depends not only on $\mathbf{R}$, but also on x, it possesses all the properties of $I-\mathcal{P}$. Therefore in the same way as above one can prove the inequalities

$$\begin{aligned}
&\|\mathbf{A}_t^{(i)}(\mathbf{h},\rho)\|_{\widehat{W}_2^{0,l/2}(Q_T^i)}\\
&\leq c\delta\|\mathbf{h}_t^{(i)}\|_{\widehat{W}_2^{0,l'/2}(Q_T^i)}\\
&+cT^{(1-l')/2}\|\mathbf{h}^{(i)}\|_{H^{2+l',1+l'/2}(Q_T^i)}\|\rho\|_{H^{3/2+l,3/4+l/2}(G_T)},\\
&\|\mathbf{A}_t^{(i)}(\mathbf{h},\rho)\|_{\widehat{W}_2^{l/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\\
&\leq c\delta\|\mathbf{h}_t^{(i)}\|_{\widehat{W}_2^{l'/2}(0,T;W_2^{-1/2}(\mathcal{G}))}\\
&+cT^{(1-l')/2}\|\mathbf{h}^{(i)}\|_{H^{2+l',1+l'/2}(Q_T^i)}\|\rho\|_{H^{3/2+l,3/4+l/2}(G_T)},
\end{aligned}\tag{4.24}$$

$i=1,2$. In addition, since

$$l_9(\mathbf{h},\rho)=\mathbf{A}^{(1)}(\mathbf{h},\rho)-\mathbf{A}^{(2)}(\mathbf{h},\rho),$$

we can use the trace theorem for Sobolev–Slobodetskii spaces and obtain

$$\begin{aligned}
&\|l_9\|_{W_2^{l'+3/2,0}(G_T)}+\sup_{t<T}\|l_9\|_{W_2^{l'+1/2}(\mathcal{G})}\\
&\leq c\big(\delta+T^{(l-l')/2}\sup_{t<T}\|\rho\|^{l-l'}_{W_2^{l+3/2}(\mathcal{G})}\|\rho\|^{1-l+l'}_{W_2^{l+5/2,0}(G_T)}\big)\\
&\times\sum_{i=1}^{2}\|\mathbf{h}^{(i)}\|_{H^{l'+2,l'/2+1}(Q_T^i)}.
\end{aligned}\tag{4.25}$$

Estimates (4.10), (4.14), (4.16)–(4.25) imply (3.8). Theorem 7 is proved.

References

1. P. H. Roberts, *An introduction to magnetohydrodynamics.* London, Longmans, Green and Co ltd (1967).
2. M. Padula, *On nonlinear stability of MHD equilibrium figures.* — Adv. Math. Fluid Mech. (2009), 301–331.
3. M. Padula, *On nonlinear stability of nonlinear pinch.* — to appear in Appl. Anal.
4. O. A. Ladyzhenskaya and V. A. Solonnikov, *The linearization principle and invariant manifolds for problems of magnetohydrodynamics.* — J. Math. Sci. **8** (1977), 384–422.
5. G.Ströhmer. *About an initial-boundary value problem from magnetohydrodynamics*, Math. Z. **209** (1992), 345–362.

6. G. Ströhmer, *About the equations of non-stationary magnetohydrodynamics.* — Nonlin. Anal. **52** (2003), 1249–1273.
7. S. Mosconi and V. A. Solonnikov, *On a problem of magnetohydrodynamics in a multi-connected domain.* — Submitted to Nonlinear Analysis.
8. V. A. Solonnikov, *On the linear problem arising in the study of a free boundary problem for the Navier–Stokes equations.* — Submitted to Algebra Analiz.
9. M. Padula and V. A. Solonnikov, *On the local solvability of free boundary problem for the Navier-Stokes equations.* — Probl. Mat. Anal., Tamara Rozhkovskaya **50** (2010), 87–133.
10. V. A. Solonnikov, *On the estimates of volume and surface potentials in domains with boundaries of class* W_2^l. — Probl. Mat. Anal. **36** (2008), 93–111.
11. O. A. Ladyzhenskaya and V. A. Solonnikov, *Solution of some non-stationary problems of magnetohydrodynamics for a viscous incompressible fluid.* — Tr. Mat. Inst. Steklov **59** (1960), 115–173.
12. M. E. Bogovskii, *Resolution of some problems of the vector analysis connected with the operators* div *and* grad. — Trudy semin. S. L. Sobolev No. 1 (1980), 5–40.
13. V. A. Solonnikov, *On the linear problem related to the stability of uniformly rotating self-gravitating liquid.* — J. Math. Sci. **144**, No. 6 (2007), p.4671.
14. V. A. Solonnikov, *A-priori estimates for second order equations of parabolic type.* — Tr. Mat. Inst. Steklov **70** (1964), 133–212.
15. R. A. Adams, *Sobolev Spaces.* Acad. Press, N. Y. (1975).

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