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A. V. Pyatkin, I. D. Chernykh, On complexity of two-machine routing propotionate open shop,

*Сиб. электрон. матем. изв.*, 2022, том 19, выпуск 2, 528–539

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*Том 19, №2, стр. 528–539 (2022)*  
DOI 10.33048/semi.2022.19.044УДК 519.854.2  
MSC 90B35

Special issue: International conference «Mathematical Optimization Theory  
and Operations Research» (MOTOR 2022),  
July 2–6, 2022, Petrozavodsk, Republic of Karelia, Russia

ON COMPLEXITY OF TWO-MACHINE ROUTING  
PROPORTIONATE OPEN SHOP

A.V. PYATKIN, I.D. CHERNYKH

**ABSTRACT.** In the routing open shop problem a fleet of mobile machines has to traverse the given transportation network, to process immovable jobs located at its nodes and to return back to the initial location in the shortest time possible. The problem is known to be NP-hard even for the simplest case with two machines and two nodes. We consider a special *proportionate* case of this problem, in which processing time for any job does not depend on a machine. We prove that the problem in this simplified setting is still NP-hard for the same simplest case. To that end, we introduce the new problem we call **2-Summing** and reduce it to the problem under consideration. We also suggest a  $\frac{7}{6}$ -approximation algorithm for the two-machine proportionate problem with at most three nodes.

**Keywords:** routing open shop, proportionate open shop, complexity, approximation algorithm, standard lower bound, optima localization.

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This research was carried out within the framework of the state contract of the Sobolev Institute of Mathematics (project FWNF-2022-0019), and supported by the Russian Foundation for Basic Research, projects 20-01-00045 and 20-07-00458.

*Received April, 21, 2022, published August, 25, 2022.*

## 1. INTRODUCTION

The object of investigation of this paper is a special *proportionate* case of so-called *routing open shop problem*, which is a natural combination of the classical open shop scheduling problem and the metric traveling salesman problem (TSP).

The open shop problem was introduced in [8] and can be described as follows. The set of jobs  $\mathcal{J} = \{J_1, \dots, J_n\}$  is given. Each machine  $M_i$  from the given set  $\mathcal{M} = \{M_1, \dots, M_m\}$  has to perform an *operation* on every job  $J_j$ , which takes predefined processing time  $p_{ji}$ . Operations can be performed in any sequence, providing that operations of the same machine or of the same job do not overlap. The goal is to construct a feasible schedule minimizing so-called *makespan*, which is defined as the completion time of the last operation. Following the traditional three-field notation for scheduling problems (see *e.g.* [13]) the open shop problem with  $m$  machines is denoted as  $Om||C_{\max}$ . Notation  $O||C_{\max}$  is used for the case with unbounded number of machines.

The  $O2||C_{\max}$  problem can be solved to the optimum in linear time [8]. The complete review of all five known algorithms for this problem can be found in [9]. The analysis of any of these algorithm shows the important property of any instance of the  $O2||C_{\max}$  problem: optimal makespan always coincides with the *standard lower bound*  $\bar{C} = \max_{i,j} \{\ell_i, d_j\}$ , where  $\ell_i = \sum_j p_{ji}$  is the *load* of the machine  $M_i$ , and  $d_j = \sum_i p_{ji}$  is the *duration* of the job  $J_j$ . Following [11] we refer to this property as *normality*: a feasible schedule is called *normal* if its makespan equals the standard lower bound, and an instance is normal if it admits constructing a normal schedule.

The  $O3||C_{\max}$  problem is NP-hard [8], however its exact complexity is still unknown. On the other hand, the problem  $O||C_{\max}$  is strongly NP-hard even in the case when integer processing times do not exceed 2 [19]. For  $m \geq 3$  normality cannot be guaranteed: the optimal makespan for  $O3||C_{\max}$  can be as high as  $\frac{4}{3}\bar{C}$  [18]. The NP-hardness of the  $Om||C_{\max}$  problem justifies the research of its special cases. One of such relaxations is so-called *proportionate* open shop, in which the processing times are machine-independent, *i.e.*  $p_{ji} = p_j$  for all  $i$ . This problem is usually denoted as  $Om|prpt|C_{\max}$ . However it is suggested in [17] to use clearer notation  $j-prpt$  to distinguish this problem from another special case with job-independent processing times, for which the name *proportionate* is also used in some papers (*e.g.* [15, 16]). In this paper we will follow the notation  $j-prpt$ .

The  $O3|j-prpt|C_{\max}$  problem is also NP-hard. This can be easily shown by the reduction from a well-known **Partition problem** [7]; the formal proof of this fact was first published in [14]. Approximation algorithms for  $Om|j-prpt|C_{\max}$  are suggested in [12, 17]. The latter paper [17] provides a pseudopolynomial algorithm for the  $O3|j-prpt|C_{\max}$  problem, which gives a partial answer to the open question on exact computational complexity of  $O3||C_{\max}$ .

The routing open shop problem [1, 2] generalizes the open shop problem in the following manner. In addition to the open shop instance, a transportation network  $G = \langle V; E \rangle$  is given. Immovable jobs are distributed among the nodes of network. One of the nodes  $v_0 \in V$ , called the *depot*, is the initial location of mobile machines. A weight of edge  $e \in E$  represents the travel time which takes any machine to traverse this edge. In order to perform an operation on some job, machine has to reach the correspondent node first. The restrictions inherited from the open shop are

in order: intervals of performing operations of the same job (same machine) cannot have common inner points. However machines can travel without any restrictions: any number of machines can traverse the same edge simultaneously in any direction and can visit any node multiple times, even without processing some job from that node. Therefore we assume, that machines take the shortest route while traveling between nodes  $u, v \in V$ , and the correspondent travel time is denoted as  $\text{dist}(u, v)$ . After performing all operations, machine has to come back to the depot. The goal is to minimize the makespan  $R_{\max}$ , which is the completion time of the last machine's activity (i.e. traveling to the depot or processing some job located at the depot). We denote the  $m$ -machine routing open shop problem as  $ROm||R_{\max}$ . Notation  $ROm|G = X|R_{\max}$  is used when we want to specify the structure of the network  $G$ . In this case  $X$  is substituted with some common notation from graph theory (e.g. *tree* or  $K_p$ ).

The following standard lower bound for the routing open shop was introduced in [1]:

$$(1) \quad \bar{R} = \max\{\ell_{\max} + T^*, \max_{v \in V}(d_{\max}(v) + 2\text{dist}(v_0, v))\}.$$

Here  $\ell_{\max} = \max_i \ell_i$  is the *maximum machine load*,  $T^*$  is the optimum for the underlying TSP (i.e. the weight of the shortest Hamiltonian walk in  $G$ ),  $d_{\max}(v)$  is the maximum duration of job located at node  $v$ .

The routing open shop is strongly NP-hard for any number of machines, as it contains the metric TSP as a special case. In the contrast to  $O2||C_{\max}$ , the two-machine routing open shop is NP-hard even for the simplest case with two nodes ( $G = K_2$ ) [2]. On the other hand, this problem  $RO2|G = K_2|R_{\max}$  is not strongly NP-hard as it admits constructing of FPTAS [10]. The best up to date approximation algorithm for  $RO2||R_{\max}$  is proposed in [4]. It constructs a feasible schedule with makespan from the interval  $[\bar{R}, \frac{13}{8}\bar{R}]$  and therefore is a  $\frac{13}{8}$ -approximation. For a special cases with small number of nodes better approximation algorithms are given in [1] for  $RO2|G = K_2|R_{\max}$  and in [5] for  $RO2|G = K_3|R_{\max}$ . Both algorithms construct schedules with makespan not greater than  $\frac{6}{5}\bar{R}$ , and this bound is tight: there exists an instance of  $RO2|G = K_2|R_{\max}$  for which optimal makespan is equal to  $\frac{6}{5}\bar{R}$ . Therefore the interval  $[\bar{R}, \frac{6}{5}\bar{R}]$  is the *tight optima localization interval* for both  $RO2|G = K_2|R_{\max}$  and  $RO2|G = K_3|R_{\max}$ . For the relevant review on optima localization we refer the reader to [6] for routing open shop and to [17] for proportionate open shop.

In this paper we investigate the two-machine routing proportionate open shop  $RO2|j\text{-prpt}|R_{\max}$ . To the best of our knowledge this special case has not been studied before. In section 2 we show that the  $RO2|j\text{-prpt}, G = K_2|R_{\max}$  problem is NP-hard, therefore generalizing the known result from [2]. The tight optima localization interval  $[\bar{R}, \frac{7}{6}\bar{R}]$  for the case with at most three nodes is established in section 3. Some concluding remarks and directions of future research are discussed in 4.

## 2. THE $RO2|j\text{-prpt}, G = K_2|R_{\max}$ IS NP-HARD

The proof of NP-hardness is organized as follows. We introduce a problem that we call **2-Summing**, prove its NP-completeness and then show that it can be polynomially reduced to the  $RO2|j\text{-prpt}, G = K_2|R_{\max}$  problem. We believe the

following problem to be new: although there are some mentions of similar problems, we could not find any proof of its NP-completeness.

**Problem 2-Summing.** Given a set  $A = \{a_1, \dots, a_N\}$  of non-negative integers and a positive integer  $B$  such that  $\sum_{i=1}^N a_i = W \geq 2B$ , are there two disjoint (maybe empty) subsets  $I_1, I_2 \subseteq \{1, \dots, N\}$  such that

$$S(I_1, I_2) \doteq \sum_{i \in I_1} a_i + 2 \sum_{i \in I_2} a_i = B?$$

The name 2-Summing goes back to the well-known NP-complete Subset Sum problem [7], where the condition  $W \geq 2B$  is not required and no  $I_2$  is allowed.

We need the following classic NP-complete problem [7].

**Problem 3D Matching.** Given three pairwise disjoint sets  $X, Y, Z$  each of size  $M$  and a family  $E$  of  $N$  triples from  $X \times Y \times Z$ , is there a subset  $E_0 \subseteq E$  of size  $M$  such that  $\bigcup_{e \in E_0} e = X \times Y \times Z$ ?

Note that 3D Matching remains NP-complete under the additional requirement that every element  $t \in X \cup Y \cup Z$  belongs to at least two triples from  $E$ . Indeed, if some  $t$  does not belong to any  $e \in E$  then the answer is NO. If  $t$  lies only in one triple  $e \in E$ , then  $e$  must be in  $E_0$ , and the instance can be reduced.

**Theorem 1.** *2-Summing problem is NP-complete.*

*Proof.* Note that the idea of the proof is the same as a classic NP-completeness proof of Partition problem [7]. Consider an instance of 3D Matching problem, where each element lies in at least two triples. For all  $i = 1, \dots, M$  denote by  $x_i, y_i$ , and  $z_i$  the  $i$ -th element of  $X, Y$ , and  $Z$ , respectively. Put  $p = \lceil \log(N + 1) \rceil + 1$ . Each number from  $A$  corresponds to a triple from  $E$ . Namely, if  $e_s = (x_i, y_j, z_k)$  then put

$$a_s = 2^{p(i-1)} + 2^{p(M+j-1)} + 2^{p(2M+k-1)};$$

here  $s = 1, \dots, N$ . Also put  $B = \sum_{j=0}^{3M-1} 2^{pj}$ . Since the total length of the constructed instance input is at most  $\mathcal{O}(NMp)$ , the reduction is polynomial.

If there is a subset  $E_0$  of size  $M$  such that  $\bigcup_{e \in E_0} e = X \times Y \times Z$ , then clearly the sum of the numbers, corresponding to the elements from  $E_0$  is equal to  $B$  (and we take  $I_2 = \emptyset$  here).

Assume that  $S(I_1, I_2) = B$  for some disjoint subsets  $I_1$  and  $I_2$ . In this case it is convenient to present the numbers in binary form. There are  $3M$  groups of  $p$  bits, where each group corresponds to one element from  $X \cup Y \cup Z$  (the rightmost group corresponds to  $x_1$ , the leftmost — to  $z_M$ ). Each  $a \in A$  assigned to a triple  $e \in E$  contains in binary form exactly three 1's in the rightmost bits of groups, corresponding to the elements of  $E$ . The number  $B$  contains 1's in the rightmost bits of all groups. Since each element from  $X \cup Y \cup Z$  lies in at least two triples, the condition  $W \geq 2B$  holds. Note that the number  $N$  needs exactly  $\lceil \log(N + 1) \rceil$  binary digits; so,  $2N$  needs  $\lceil \log(N + 1) \rceil + 1 = p$  digits. This means that even in the double sum of all numbers from  $A$  the sums of the bits in each group remains in the same group; i. e., since  $S(I_1, I_2) \leq 2W$ , no shift of 1's between the groups occurs in  $S(I_1, I_2)$ . Therefore, the equality  $S(I_1, I_2) = B$  can take place only if  $|I_1| = M$ ,  $I_2 = \emptyset$  and for each group exactly one addend in  $S(I_1, I_2)$  contains 1 in

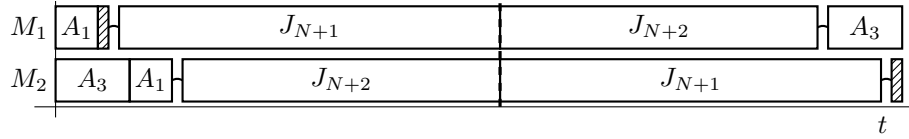


FIG. 1. A sketch of the feasible schedule in case of the positive answer to the 2-Summing problem. Hatched rectangles represent the set of jobs  $A_2$ .

the rightmost bit. But then the elements of  $E$ , corresponding to  $I_1$  form the desired  $E_0$ .  $\square$

*Remark 1.* Note that for any constant  $k$ , the presented proof can be generalized for the  $k$ -Summing problem where the input is the same and the question is whether there are  $k$  pairwise disjoint subsets  $I_1, \dots, I_k$  such that

$$\sum_{j=1}^k \left( j \sum_{i \in I_j} a_i \right) = B.$$

The only difference is that  $p = \lceil \log(N + 1) \rceil + \lceil \log k \rceil$  in this case.

*Remark 2.* Clearly,  $k$ -Summing problem is not NP-complete in the strong sense since the standard dynamic programming algorithm for the Subset Sum problem can be generalized for solving it: at each step the algorithm just has to choose the best of  $k + 1$  variants instead of 2.

Now we are ready to prove the main result of this section.

**Theorem 2.** *The  $RO2|j\text{-prpt}, G = K_2|R_{\max}$  problem is NP-complete.*

*Proof.* Consider an instance of 2-Summing and reduce it to an instance of  $RO2|j\text{-prpt}, G = K_2|R_{\max}$  in the following way. Put  $n = N + 2$  and let the depot  $v_0$  contain  $N$  jobs corresponding to the numbers  $a_i \in A$ , processing times of both operations of the job  $J_j$  equal to  $a_j$ ,  $j = 1, \dots, N$ . The second node  $v_1$  contains two auxiliary jobs  $J_{N+1}$  and  $J_{N+2}$  with processing times  $3W$  and  $2W + B$ , respectively. Choose an arbitrary  $\tau > 0$  and put  $\text{dist}(v_0, v_1) = \tau$ . Let us prove that this instance has a schedule of makespan  $\bar{R} = 6W + B + 2\tau$  if and only if in the instance of 2-Summing disjoint subsets  $I_1, I_2$  with  $S(I_1, I_2) = B$  exist.

*Necessity.* Assume that there are disjoint subsets  $I_1, I_2$  with  $S(I_1, I_2) = B$ . Denote by  $A_1, A_2$ , and  $A_3$  the sets of numbers with indices from  $I_1, I_2$ , and  $\{1, \dots, N\} \setminus (I_1 \cup I_2)$ , respectively, and let  $w(A_i)$  be the sum of the numbers from  $A_i$  for  $i = 1, 2, 3$ . Consider the following scheduling (see Fig. 1). Let the first machine process the jobs in the order  $A_1, A_2, J_{N+1}, J_{N+2}, A_3$ , while the second machine — in the order  $A_3, A_1, J_{N+2}, J_{N+1}, A_2$  (the order of jobs inside the sets  $A_i$  can be arbitrary for each machine). Clearly, the length of the schedule is  $\bar{R}$ . Let us verify its correctness. It is clear that the operations of any job from  $A_2 \cup A_3$  do not intersect. Since  $W = w(A_1) + w(A_2) + w(A_3)$  and  $B = w(A_1) + 2w(A_2)$ , we have  $w(A_1) + w(A_2) + 3W = w(A_3) + w(A_1) + 2W + B$ , i. e. the completion time of job  $J_{N+1}$  at the first machine coincides with the completion time of  $J_{N+2}$  at the second machine. Finally, note that

$$w(A_3) = W - w(A_1) - w(A_2) \geq 2B - w(A_1) - w(A_2) = w(A_1) + 3w(A_2) \geq w(A_1).$$

Therefore, processing any job from  $A_1$  at the second machine starts after finishing processing the last job from  $A_1$  at the first machine. So, the schedule is feasible.

*Sufficiency.* Assume that the instance of Problem 1 has a schedule with makespan  $\bar{R} = 6W + B + 2\tau$ . Then each machine has only one travel to the second node and back, and process jobs without idle. We may assume that the job  $J_{N+1}$  is processed first at the machine  $M_1$ . Since  $\bar{R} < 8W + B + 2\tau$ , the job  $J_{N+2}$  must be processed first at the machine  $M_2$ ; moreover, the absence of idle intervals implies that the machine  $M_1$  finishes processing the job  $J_{N+1}$  at the same time when  $M_2$  finishes processing  $J_{N+2}$ . Denote by  $X$  the set of all jobs in the depot that the machine  $M_1$  finished before processing the job  $J_1$ , and by  $Y$  the set of all jobs in the depot that the machine  $M_2$  started after processing the job  $J_2$ . Note that  $X$  and  $Y$  may intersect and one of them may be empty. By the above mentioned observation,  $w(X) + 3W = (W - w(Y)) + 2W + B$ , i. e.  $w(X) + w(Y) = B$ . Put  $A_2 = X \cap Y$  and  $A_1 = (X \cup Y) \setminus A_2$ . Then we have  $B = w(X) + w(Y) = w(A_1) + 2w(A_2)$ . Hence, the sets of indices  $I_1, I_2$  corresponding to the sets  $A_1, A_2$  satisfy the condition  $S(I_1, I_2) = B$ , as required.  $\square$

### 3. OPTIMA LOCALIZATION FOR SMALL NUMBER OF NODES

In [5] a reduction procedure for the  $RO2|R_{\max}$  problem is described, which transforms any instance  $I$  into simplified instance  $I'$  with the same value of the standard lower bound (1). This transformation has the following important properties:

- it is *reversible*, i. e. any feasible schedule for  $I'$  can be treated as a feasible schedule for the initial instance  $I$  with the same makespan,
- $I'$  contains a single job at each node of the transportation network except maybe one node that contains two or three jobs.

This procedure is based on a *job aggregation* operation (also known as *job grouping*), which basically is replacing a subset of jobs from the same node by a new one, with processing times equal to the total processing time of the replaced jobs.

The following approximation algorithm  $\mathcal{A}$  for  $RO2|G = K_3|R_{\max}$ , based on this procedure, is suggested:

- (1) Transform given instance  $I$  into a simplified one  $I'$ . (Note that it contains at most 5 jobs.)
- (2) Build an optimal schedule  $S'$  for  $I'$ . (Note that it can be done in constant time.)
- (3) Treat the schedule  $S'$  as a feasible schedule  $S$  for instance  $I$ . Output  $S$ .

It is shown in [5] by means of case analysis, that  $R_{\max}(S) \leq \frac{6}{5}\bar{R}$ , and the algorithm described is therefore a  $\frac{6}{5}$ -approximation.

In this section we use the same idea the  $RO2|j\text{-prpt}, G = K_3|R_{\max}$  problem. The following lemma establishes the lower bound on the performance guarantee of algorithm  $\mathcal{A}$  for the proportionate routing open shop.

**Lemma 1.** *There exists an instance  $\tilde{I}$  of  $RO2|j\text{-prpt}, G = K_2|R_{\max}$ , such that the optimal makespan for  $\tilde{I}$  equals  $\frac{7}{6}\bar{R}$ .*

*Proof.* Consider the following instance  $\tilde{I}$ : the depot  $v_0$  contains a single job  $J_1$  with processing times 2, the other node  $v_1$  contains two identical jobs  $J_2, J_3$  with processing times 4. The travel time between the nodes is  $\text{dist}(v_0, v_1) = 1$ . Note that

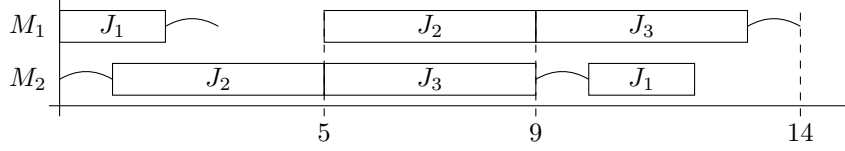


FIG. 2. An optimal schedule  $S$  for instance  $\tilde{I}$ .

machine loads are  $\ell_1 = \ell_2 = 10$ , maximum job duration in node  $v_1$  is  $d_{\max}(v_1) = 8$ , and  $T^* = 2$ , therefore  $\bar{R}(\tilde{I}) = 12$ . A feasible schedule  $S$  for  $\tilde{I}$  with makespan 14 is given in Fig. 2. Concave arcs represent travel times.

Let us prove that schedule  $S$  is optimal for  $\tilde{I}$ . Suppose otherwise and consider a schedule  $S'$  such that  $R_{\max}(S') < 14$ . Note that each machine in  $S'$  travels from  $v_0$  to  $v_1$  and back exactly once, otherwise the travel time would be at least 4 and the makespan at least 14. Without loss of generality assume that machine  $M_1$  processes jobs in order  $J_1, J_2, J_3$ . Idle time of any machine in  $S'$  is less than 2 (otherwise  $R_{\max}(S')$  is at least 14), therefore  $M_2$  cannot start with job  $J_1$ , and hence job  $J_1$  is performed last by machine  $M_2$ . If machine  $M_2$  processes job  $J_2$  after machine  $M_1$ , then it completes  $J_2$  not earlier than at 11. In this case  $R_{\max}(S') \geq 14$ . On the other hand, if machine  $M_2$  processed  $J_2$  before  $M_1$ , machine  $M_1$  idles for at least two time units before the processing of operation of job  $J_2$ . Therefore, by contradiction, such schedule  $S'$  does not exist.  $\square$

Now consider an instance  $I$  of  $RO2|G = K_3|R_{\max}$  with nodes  $\{v_0, v_1, v_2\}$  and corresponding simplified instance  $I'$ , obtained from  $I$  by the reduction procedure from [5]. Assume that  $I'$  is *irreducible*, *i.e.* no further job aggregation in  $I'$  is possible without violating the standard lower bound  $\bar{R}$ . If a node in  $I'$  contains more than one job, it is called *overloaded*. A node with three jobs in irreducible instance is referred to as *superoverloaded* (see [6] for details).

Our goal is to prove that there exists a feasible schedule for  $I'$  (and therefore for  $I$ ) with makespan of at most  $\frac{7}{6}\bar{R}$ . It is proved in [3] that any instance of  $RO2||R_{\max}$  with an overloaded depot admits constructing a normal schedule (*i.e.* feasible schedule of makespan  $\bar{R}$ ). It follows from [6] that any instance of  $RO2|G = K_3|R_{\max}$  with a superoverloaded node has the same property. Therefore we only need to consider a special case of irreducible instance  $I'$  with at most two jobs at node  $v_1$  and single jobs at nodes  $v_0$  and  $v_2$ .

**Lemma 2.** *Let  $I$  be an irreducible instance of  $RO2|j\text{-prpt}, G = K_3|R_{\max}$  with single jobs at  $v_0$  and  $v_2$  and at most two jobs at  $v_1$ . Then there exists a feasible schedule  $S$  for  $I$  such that  $R_{\max}(S) \leq \frac{7}{6}\bar{R}(I)$ .*

*Proof.* We consider two cases with different number of jobs at node  $v_1$  separately.

**Case 1.** Node  $v_1$  has two jobs.

Denote the processing times of four jobs  $J_1, J_2, J_3, J_4$  by  $a, b, c, d$  respectively,  $J_1$  at  $v_0, J_4$  at  $v_2$ . Denote travel times as follows:

$$\text{dist}(v_0, v_1) = \tau, \text{dist}(v_0, v_2) = \mu, \text{dist}(v_1, v_2) = \nu, T^* = \tau + \mu + \nu.$$



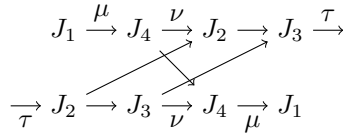


FIG. 3. Processing order for schedule  $S_1$ .

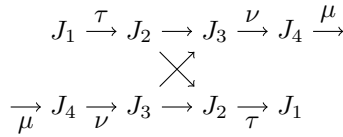


FIG. 4. Processing order for schedule  $S_2$ .

Note that, due to the irreducibility of instance  $I$ , aggregation of jobs  $J_2$  and  $J_3$  would violate the lower bound  $R$ , therefore

$$(2) \quad 2b + 2c + 2\text{dist}(v_0, v_1) > \bar{R}.$$

Without loss of generality assume

$$(3) \quad b \geq c.$$

Note that in this case

$$(4) \quad \bar{R}(I) = \max\{a + b + c + d + \tau + \mu + \nu, 2a, 2b + 2\tau, 2d + 2\mu\}.$$

We will describe a series of schedules for  $I$  and prove that at least one of the schedules constructed has a makespan of at most  $\frac{7}{6}\bar{R}$ . Each schedule will be described by specifying the linear order of operations on each machine and each job.

Let  $S_1$  be a schedule with sequence of jobs  $J_1 \rightarrow J_4 \rightarrow J_2 \rightarrow J_3$  on  $M_1$  and  $J_2 \rightarrow J_3 \rightarrow J_4 \rightarrow J_1$  on  $M_2$ , jobs  $J_1$  and  $J_4$  are processed first by  $M_1$ , and  $J_2, J_3$  processed first by  $M_2$  (Fig. 3).

Note that due to (2) we have

$$2a + 2d + 2\mu \leq 2\bar{R} - (2b + 2c + 2\tau) < \bar{R},$$

therefore by (3) and (4) it is sufficient to consider  $R_{\max}(S_1) = R_1 = 2b + c + 2\tau$  (otherwise the makespan the schedule  $S_1$  is normal).

Construct schedule  $S_2$  according to the partial order of operations from Fig. 4.

**Case 1.1.**  $R_{\max}(S_2) = R_2 = 2a + 2b + 2\tau$ .

Construct schedule  $S_3$  according to the partial order of operations from Fig. 5.

It is sufficient to consider the case  $R_{\max}(S_3) = R_3 = 2a + 2c + 2d + \tau + \mu + \nu$ .

Construct schedule  $S_4$  according to the partial order of operations from Fig. 6.

Consider three subcases.

**Case 1.1.1.**  $R_{\max}(S_4) = R_4 = a + 2d + 2c + 2\mu + 2\nu$ .

In this case

$$2R_1 + 3R_2 + 4R_4 = 10a + 10b + 10c + 8d + 10\tau + 8\mu + 8\nu \leq 10\bar{R},$$

therefore  $\min\{R_1, R_2, R_4\} \leq \frac{10}{9}\bar{R} < \frac{7}{6}\bar{R}$ .

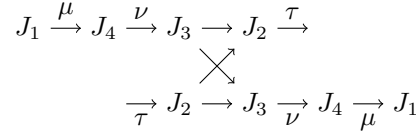


FIG. 5. Processing order for schedule  $S_3$ .

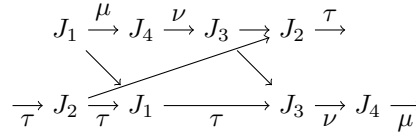


FIG. 6. Processing order for schedule  $S_4$ .

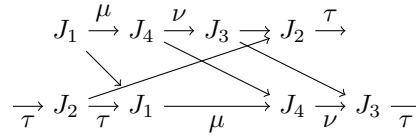


FIG. 7. Processing order for schedule  $S_5$ .

**Case 1.1.2.**  $R_{\max}(S_4) = R_4 = 2a + c + d + \tau + \mu + \nu$ .  
 In this case

$$R_1 + R_4 = 2a + 2b + 2c + d + 3\tau + \mu + \nu \leq 2\bar{R},$$

since by triangle inequality,  $\tau \leq \mu + \nu$ . So, at least one of schedules  $S_1$  and  $S_4$  is normal.

**Case 1.1.3.**  $R_{\max}(S_4) = R_4 = a + b + c + d + 3\tau + \mu + \nu$ .  
 In this case, due to (3) and  $\tau \leq \mu + \nu$

$$2R_1 + 3R_3 + R_4 = 5b + 9c + 7a + 7d + 10\tau + 4\mu + 4\nu \leq 7\bar{R},$$

and  $\min\{R_1, R_3, R_4\} \leq \frac{7}{6}\bar{R}$ .

**Case 1.2.**  $R_{\max}(S_2) = R_2 = 2c + 2d + 2\mu + 2\nu$ .

Construct schedule  $S_5$  with the partial order of operations as shown in Fig. 7.

Due to (3) it is sufficient to consider two cases.

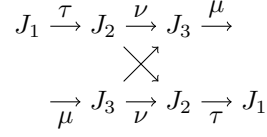
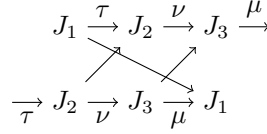
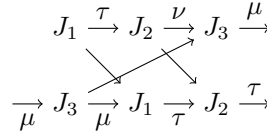
**Case 1.2.1.**  $R_{\max}(S_5) \leq R_5 = a + d + \max\{a, d\} + c + \tau + \mu + \nu$ .  
 In this case

$$R_1 + R_5 = 2b + 2c + a + d + \max\{a, d\} + 3\tau + \mu + \nu \leq 2\bar{R}$$

and the makespan of at least one of schedules  $S_1$  and  $S_5$  equals  $\bar{R}$ .

**Case 1.2.2.**  $R_{\max}(S_5) = R_5 = a + b + c + d + 3\tau + \mu + \nu$ .  
 In this case, due to (3)

$$2R_1 + 3R_2 + R_5 = 5b + 9c + 7d + a + 7\tau + 7\mu + 7\nu \leq 7\bar{R},$$


 FIG. 8. Processing order for schedule  $S_6$ .

 FIG. 9. Processing order for schedule  $S_7$ .

 FIG. 10. Processing order for schedule  $S_8$ .

and  $\min\{R_1, R_2, R_5\} \leq \frac{7}{6}\bar{R}$ .

**Case 2.** Each node contains exactly one job. In this case denote processing times of three jobs  $J_1, J_2, J_3$  by  $a, b, c$  respectively,  $J_1$  at  $v_0$ ,  $J_3$  at  $v_2$ . Travel time are denoted as in Case 1. In this case we have

$$(5) \quad \bar{R}(I) = \max\{a + b + c + \tau + \mu + \nu, 2a, 2b + 2\tau, 2c + 2\mu\}.$$

Without loss of generality we assume

$$(6) \quad b + \tau \leq c + \mu.$$

Construct three schedules  $S_6, S_7$  and  $S_8$  according to partial orders at Figures 8, 9, 10, respectively.

Using (5) and (6), and assuming each of the schedules built has makespan greater than  $\bar{R}$  we have

$$\begin{aligned}
 R_{\max}(S_6) &= R_6 = 2a + 2b + 2\tau, \\
 R_{\max}(S_7) &= R_7 = b + c + \max\{b, c\} + \tau + \mu + \nu, \\
 R_{\max}(S_8) &= R_8 = \max\{2a + b + 2\tau, a + b + c + 2\mu + 2\tau\}.
 \end{aligned}$$

Consider the following two subcases.

**Case 2.1.**  $R_8 = 2a + b + 2\tau$ .

In this case using (6),  $\mu + \tau \leq T^*$ , and  $2\tau \leq T^*$  we obtain

$$\begin{aligned}
 R_7 + R_8 &= 2a + 2b + c + \mu + \nu + 2\tau + \max\{b + \tau, c + \tau\} \leq \\
 &2a + 2b + 2c + \mu + \nu + 2\tau + \max\{\mu, \tau\} \leq 2a + 2b + 2c + 2T^* \leq 2\bar{R},
 \end{aligned}$$

therefore at least one of schedules  $S_7$  and  $S_8$  is normal.

**Case 2.2.**  $R_8 = a + b + c + 2\mu + 2\tau$ .

In this case using (6) and  $\tau \leq \mu + \nu$  we have

$$\begin{aligned}\Sigma &= 3R_6 + 2R_7 + R_8 = 7a + 9b + 3c + 2\max\{b, c\} + 10\tau + 4\mu + 2\nu \\ &\leq 7a + 7b + 5c + 2\max\{b, c\} + 8\tau + 6\mu + 2\nu \leq 7a + 7b + 5c + 2\max\{b, c\} + 7\tau + 7\mu + 3\nu.\end{aligned}$$

If  $b \leq c$  then  $\Sigma$  clearly does not exceed  $7\bar{R}$ . Assume  $b \geq c$ . Then

$$\begin{aligned}\Sigma &= 7a + 9b + 5c + 7\tau + 7\mu + 3\nu \leq 7a + 7b + 7c + 5\tau + 9\mu + 3\nu \\ &\leq 7a + 7b + 7c + 7\tau + 7\mu + 5\nu \leq 7\bar{R}.\end{aligned}$$

Anyway,  $3R_6 + 2R_7 + R_8 \leq 7\bar{R}$ ; so,

$$\min\{R_6, R_7, R_8\} \leq \frac{7}{6}\bar{R}.$$

We have considered all possible cases, and in each case the makespan of one of the schedules built does not exceed  $\frac{7}{6}\bar{R}$ . Lemma is proved.  $\square$

**Theorem 3.** *The tight optima localization interval for both  $RO2|G = K_2, j - prpt|R_{\max}$  and  $RO2|G = K_3, j - prpt|R_{\max}$  problems is  $[\bar{R}, \frac{7}{6}\bar{R}]$ .*

*Proof.* Straightforward from Lemmas 1 and 2.  $\square$

*Remark 3.* Theorem 3 together with algorithm  $\mathcal{A}$  imply the existence of  $\frac{7}{6}$ -approximation for the  $RO2|G = K_3, j - prpt|R_{\max}$  problem.

#### 4. CONCLUSION

The main result of this paper is the NP-hardness of a very restricted special case of  $RO2|R_{\max}$ . We presume, that the auxiliary problem 2-Summing or its variations might also be helpful in research of the computational complexity of various scheduling problems.

To our opinion, it makes sense to continue the investigation of the proportionate routing open shop in the following directions:

- (1) Approximation algorithms and optima localization for two-machine proportionate routing open shop with more complex structure of the transportation network, as well as with asymmetric distances.
- (2) Approximation algorithms for the problem with  $m \geq 3$ .
- (3) Approximation algorithms for the generalization of proportionate routing open shop with *proportional* processing times, which would finally justify the name *proportionate*.

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