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Parabolically connected subgroups

I. V. Netaĭ

Abstract. All reductive spherical subgroups of the group $\mathrm{SL}(n)$ are found for which the intersections with every parabolic subgroup of $\mathrm{SL}(n)$ are connected. This condition guarantees that open equivariant embeddings of the corresponding homogeneous spaces into Moishezon spaces are algebraic.

Bibliography: 6 titles.

Keywords: reductive group, parabolic subgroup, spherical subgroup, flag, Moishezon space.

§ 1. Introduction

Let G be a connected reductive algebraic group over the field of complex numbers $\mathbb{C}$.

Definition 1. A closed subgroup $H \subseteq G$ is called *parabolically connected* if for any parabolic subgroup $P \subseteq G$ the intersection $P \cap H$ is connected.

It is useful to note that if for a given algebraic subgroup H its intersection with any Borel subgroup $B \subset G$ is connected, then H is parabolically connected in G. Indeed, let $P \subseteq G$ be a parabolic subgroup, and $B \subseteq G$ a Borel subgroup contained in P. Then B is also a Borel subgroup of P. Every element of the connected algebraic subgroup P lies in some Borel subgroup of it (see [1], Ch. 8, § 22), and $H \cap P = \bigcup_{B \subseteq P} (H \cap B)$. In this union every element $H \cap B$ is connected and contains the identity element, so that $H \cap P$ is connected.

Since a subgroup of a unipotent group is connected, every unipotent subgroup $H \subset G$ is parabolically connected. It was shown in [2] that for a connected reductive group H the diagonal $\Delta H = \{(h,h) : h \in H\}$ is parabolically connected in the group $G = H \times H$ (Theorem 3).

Recall that an algebraic subgroup $H \subseteq G$ is called *spherical* if the induced action of a Borel subgroup B of the group G on the homogeneous space G/H has an open orbit. The main result of the present paper is the classification of parabolically connected reductive spherical subgroups of the group $\mathrm{SL}(n)$. Our task is to choose the parabolically connected subgroups in the list of connected reductive spherical subgroups obtained in [3]. We denote by $\mathrm{S}(\mathrm{GL}(m) \times \mathrm{GL}(n))$ the subgroup of $\mathrm{SL}(m+n)$ consisting of all block-diagonal matrices in which the sizes of the blocks are m and n. The group $\mathrm{SL}(m) \times \mathrm{SL}(n)$ is embedded into the group $\mathrm{SL}(m+n)$ in similar fashion. The subgroups $\mathrm{Sp}(2n) \subset \mathrm{SL}(2n)$ and $\mathrm{SO}(n) \subset \mathrm{SL}(n)$ are embedded in standard fashion, $\mathrm{Sp}(2n)$ is embedded into $\mathrm{SL}(2n+1)$ in the

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block-wise fashion where one block has size $2n$ and corresponds to the standard embedding into $\mathrm{SL}(2n)$, and the second block of size 1 is equal to 1. We denote by T^1 the one-dimensional algebraic subtorus $\{\mathrm{diag}(\lambda,\dots,\lambda,\lambda^{-2n})\}$ in the group $\mathrm{SL}(2n+1)$. The group $\mathrm{Sp}(2n)\cdot\mathrm{T}^1\subset\mathrm{SL}(2n+1)$ consists of matrices of the form

$$\left(\begin{array}{c|c}\lambda A & 0\\ \hline 0 & \lambda^{-2n}\end{array}\right),\qquad A\in\mathrm{Sp}(2n),\quad \lambda\in\mathbb{C}^\times.$$

Theorem 1. *The list of parabolically connected reductive spherical subgroups of the special linear group is exhausted by the subgroups*

$$\mathrm{SL}(m)\times\mathrm{SL}(n)\subset\mathrm{SL}(m+n)\qquad\forall\, m,n,$$
$$\mathrm{S}(\mathrm{GL}(m)\times\mathrm{GL}(n))\subset\mathrm{SL}(m+n),\qquad m\neq n,$$
$$\mathrm{Sp}(2n)\subset\mathrm{SL}(2n),\qquad \mathrm{Sp}(2n)\subset\mathrm{SL}(2n+1),\qquad \mathrm{Sp}(2n)\cdot\mathrm{T}^1\subset\mathrm{SL}(2n+1).$$

In turn, the list of reductive spherical subgroups that are not parabolically connected consists of the subgroups

$$\mathrm{SO}(n)\subset\mathrm{SL}(n),\qquad \mathrm{S}(\mathrm{GL}(n)\times\mathrm{GL}(n))\subset\mathrm{SL}(2n).$$

The interest in parabolically connected subgroups is related to problems of complex analysis. Let X be a compact Moishezon space, that is, a smooth complex-analytic compact manifold such that the transcendence degree of the field of meromorphic functions on it coincides with the dimension of the manifold X. It is known that the connected component of the identity $\mathrm{Aut}^\circ(X)$ of the automorphism group of the Moishezon space X carries the natural structure of an affine algebraic group. The action of a connected reductive group G on X is called *algebraic* if the homomorphism $G\to\mathrm{Aut}^\circ(X)$ defined by this action is a homomorphism of algebraic groups. It is natural to conjecture that if the group $\mathrm{Aut}^\circ(X)$ is sufficiently large, then the space X is an algebraic variety. One of the first results in this direction was obtained by Grauert and Remmert. In [4] they showed that a compact homogeneous Moishezon manifold is algebraic. Later Luna considered Moishezon manifolds with a locally transitive action of a torus.

Theorem 2 (see [5], Theorem 1). *Let X be a Moishezon space with a given algebraic action of a torus* T *for which there is an open dense orbit. Then X is an algebraic* T*-variety.*

The following result of Hausen generalizes Theorem 2.

Theorem 3 (see [2], Theorem 2). *Let X be a compact Moishezon space with a given algebraic action of a connected reductive group G. If for some Borel subgroup $B\subset G$ and some point $x_0\in X$ the orbit Bx_0 is open and dense in X and every closed G-orbit contains a point x such that, for a parabolic subgroup $Q\subset G$ that is opposite to the stabilizer G_x and contains B, the intersection $Q\cap G_{x_0}$ is connected, then X is an algebraic G-variety.*

Corollary 1. *Let $H\subset G$ be a spherical parabolically connected subgroup, and $G/H\to X$ an open equivariant embedding into a Moishezon G-space X. Then X is an algebraic G-variety.*

In many cases this corollary gives an affirmative answer to the question discussed in [5]: is it true that every Moishezon space with a locally transitive action of a semisimple simply connected group G for which the stabilizer of a point in an open orbit is connected is an algebraic G-variety? An example of a non-algebraic PSL(2)-quasihomogeneous Moishezon space was obtained in [6]. It is interesting to find out whether the homogeneous spaces $\mathrm{SL}(n)/H$, where H is one of the two reductive spherical subgroups of $\mathrm{SL}(n)$ that are not parabolically connected, admit an open equivariant embedding into a non-algebraic Moishezon space.

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§ 2. Lemmas on compatible bases

In the analysis of the intersections of a subgroup H with Borel subgroups we shall find useful results on the existence of bases compatible with flags and bilinear forms. Such results may also be of interest in their own right.

We denote a complete flag $\{0 \subset V_1 \subset \dots \subset V_n = V\}$ in a space V by the symbol $V_\bullet$.

Definition 2. A basis $\{e_1, \dots, e_n\}$ of a space V is said to be *compatible with a flag* $V_\bullet$ if each of the subspaces V_i is generated by some set of elements of this basis.

Definition 3. A basis $\{e_1, \dots, e_n\}$ of a space V is said to be *compatible with a decomposition* $V = U \oplus W$ if each vector e_i belongs either to U or to W.

Definition 4. A *hyperbolic basis with respect to a skew-symmetric form* ω is a basis $\{e_1, \dots, e_n\}$ such that for each e_i either $\omega(e_i, \cdot\,) \equiv 0$ or there exists a unique vector e_j such that $\omega(e_i, e_j) = \pm 1$ and $\omega(e_i, e_k) = 0$ for $k \neq j$.

Definition 5. Let $V_\bullet$ be a flag in a space V, and $W \subset V$ a subspace. Then the *quotient flag* $V_\bullet/W$ is the flag in V/W consisting of the images of subspaces in the flag $V_\bullet$.

Lemma 1. *Let $V = U \oplus W$ and let $V_\bullet$ be a complete flag in the space V. Then there exist bases $\{e_1, \dots, e_n\}$ and $\{v_1, \dots, v_n\}$ of the space V such that $\{e_1, \dots, e_n\}$ is compatible with the decomposition $V = U \oplus W$, $e_1, \dots, e_m \in U$, $e_{m+1}, \dots, e_{m+n} \in W$, the basis $\{v_1, \dots, v_n\}$ is compatible with the flag $V_\bullet$, and each v_i is equal to either some e_l or to the sum $e_j + e_k$ for some $e_j \in U$ and $e_k \in W$. Furthermore, in expressions of the form $v_i = e_j + e_k$ or $v_i = e_j$ for all v_i every e_j occurs either once or exactly twice in expressions of the form $v_i = e_j + e_k$, $v_{i'} = e_j$, where $i < i'$, and then e_k occurs in the expressions once.*

Proof. We construct the bases by induction. At the kth step we construct a basis $\{v_1, \dots, v_k\}$ of a space V_k and a basis $\{e_1, \dots, e_l\}$ of the space $\mathrm{pr}_U(V_k) \oplus \mathrm{pr}_W(V_k)$ that is compatible with the decomposition, where pr_U and pr_W are the projections onto U and W along W and U. Suppose that k steps of the construction have been made. We carry out the $(k+1)$st step. We observe that

$$\dim(V_i) = \dim(V_i \cap U) + \dim(\mathrm{pr}_W(V_i)) = \dim(V_i \cap W) + \dim(\mathrm{pr}_U(V_i)).$$

Exactly one of the following four cases holds.

Case 1.

$$\dim(\mathrm{pr}_U(V_{k+1})) = \dim(\mathrm{pr}_U(V_k)) + 1,$$
$$\dim(\mathrm{pr}_W(V_{k+1})) = \dim(\mathrm{pr}_W(V_k)) + 1.$$

Then there exist v_1 and v_2 such that $\mathrm{pr}_U(v_1) \notin \mathrm{pr}_U(V_k)$ and $\mathrm{pr}_W(v_2) \notin \mathrm{pr}_W(V_k)$. If $\mathrm{pr}_W(v_1) \notin \mathrm{pr}_W(V_k)$, then we set $v = v_1$. If $\mathrm{pr}_U(v_2) \notin \mathrm{pr}_U(V_k)$, then we set $v = v_2$. If both conditions do not hold, then we set $v = v_1 + v_2$. Thus, there exists $v \in V_{k+1}$ such that $\mathrm{pr}_U(v) \notin \mathrm{pr}_U(V_k)$ and $\mathrm{pr}_W(v) \notin \mathrm{pr}_W(V_k)$. We set $v_{k+1} = v$, $e_{l+1} = \mathrm{pr}_U(v)$, and $e_{l+2} = \mathrm{pr}_W(v)$.

Case 2.

$$\dim(\mathrm{pr}_U(V_{k+1})) = \dim(\mathrm{pr}_U(V_k)) + 1,$$
$$\dim(U \cap V_{k+1}) = \dim(U \cap V_k) + 1.$$

Here we set $v_{k+1} = e_{k+1} \in U \cap (V_{k+1} \setminus V_k)$.

Case 3.

$$\dim(\mathrm{pr}_W(V_{k+1})) = \dim(\mathrm{pr}_W(V_k)) + 1,$$
$$\dim(W \cap V_{k+1}) = \dim(W \cap V_k) + 1.$$

This case is similar to the preceding one.

Case 4.

$$\dim(U \cap V_{k+1}) = \dim(U \cap V_k) + 1,$$
$$\dim(W \cap V_{k+1}) = \dim(W \cap V_k) + 1.$$

Since $V_{k+1} \cap U \subset \mathrm{pr}_U(V_{k+1}) = \mathrm{pr}_U(V_k)$, there exists $u \in V_k$ such that $\mathrm{pr}_U(u) \in V_{k+1} \setminus V_k$. Similarly, there exists $w \in V_k$ such that $\mathrm{pr}_W(w) \in V_{k+1} \setminus V_k$. If $\mathrm{pr}_U(w) \in V_{k+1} \setminus V_k$, then we set $v = w$. If $\mathrm{pr}_W(u) \in V_{k+1} \setminus V_k$, then we set $v = u$. Otherwise we set $v = u + w$. Thus, $v \in V_k$, $\mathrm{pr}_U(v) \in V_{k+1} \setminus V_k$, and $\mathrm{pr}_W(v) \in V_{k+1} \setminus V_k$. As an element of the space V_k, the vector v is a linear combination of the basis vectors $v_1, \ldots, v_k$: $v = \sum_{i=1}^{k} \alpha_k v_k$. Let

$$v' = \sum_{\substack{i=1,\ldots,k \\ v_i \notin U \cup W}} \alpha_k v_k.$$

Then

$$\mathrm{pr}_U(v - v') = \sum_{\substack{i=1,\ldots,k \\ v_i \in U}} \alpha_i v_i \in V_k, \qquad \mathrm{pr}_W(v - v') = \sum_{\substack{j=1,\ldots,k \\ v_j \in W}} \alpha_j v_j \in V_k.$$

Therefore,

$$\mathrm{pr}_U(v') \in V_{k+1} \setminus V_k, \qquad \mathrm{pr}_W(v') \in V_{k+1} \setminus V_k,$$
$$v' = \sum_{\substack{i=1,\ldots,k \\ v_i \notin U \cup W}} \alpha_i v_i = \sum_{\substack{i=1,\ldots,q \\ v_i \notin U \cup W}} \alpha_i v_i, \qquad q \leqslant k, \quad \alpha_q \neq 0, \quad v_q \notin U \cap W.$$

Since $v_q \notin U \cup W$, by construction we have $v_q = e_s + e_t$ for some $e_s \in U$, $e_t \in W$. We replace v_q, e_s, e_t by v', $\mathrm{pr}_U(v')$, $\mathrm{pr}_W(v')$. This replacement is compatible with the flag and the decomposition, since $v_q \in V_q \setminus V_{q-1}$. Thus, the required bases for V_{k+1} and $\mathrm{pr}_U(V_{k+1}) \oplus \mathrm{pr}_W(V_{k+1})$ are constructed and $v_{k+1} = \mathrm{pr}_U(v')$.

We renumber the elements of the basis $\{e_1, \dots, e_{m+n}\}$ in such a way that $e_1, \dots, e_m \in U$ and $e_{m+1}, \dots, e_{m+n} \in W$, preserving the order of indices separately among the elements of U and among the elements of W.

Lemma 2. *Let V be a $2n$-dimensional vector space with a complete flag $V_\bullet$, and ω a nondegenerate skew-symmetric form in the space V. Then in V there exists a basis $\{e_1, \dots, e_{2n}\}$ that is compatible with the flag $V_\bullet$ and hyperbolic for ω.*

Proof. We conduct the proof by induction on n. The basis of induction for $n = 1$. We set e_1 to be such that $V_1 = \langle e_1 \rangle$. Since ω is nondegenerate, there exists $e_2 \in V_2 = V$ such that $\omega(e_1, e_2) = 1$. Since the form ω is skew-symmetric, it follows that $e_2 \notin V_1$.

Suppose that the induction assertion is proved for $m < n$. We choose arbitrarily $e_1 \in V_1 \setminus \{0\}$. Let $k = \min\{l : \omega(e_1, \cdot)|_{V_l} \not\equiv 0\}$. We choose a vector $v_k \in V_k$ such that $\omega(v_1, v_k) = 1$. Let $V'_\bullet$ be the complete flag in $\langle e_1, e_k \rangle^\perp$ obtained from $V_\bullet$ by taking the intersection of all its subspaces, except for V_1 and V_k, with $\langle e_1, e_k \rangle^\perp$. For $V'_\bullet$, in $\langle e_1, e_k \rangle^\perp$ there exists a basis $\{e_2, \dots, e_{k-1}, e_{k+1}, \dots, e_{2n}\}$ that is compatible with the flag $V'_\bullet$ and hyperbolic for the restriction of the form ω to $\langle e_1, e_k \rangle^\perp$. Then the basis $\{e_1, \dots, e_{2n}\}$ is the required one.

Lemma 3. *Let V be a vector space of dimension $2n + 1$, $U \subset V$ a hyperplane, $V_\bullet$ a complete flag in V, and ω a skew-symmetric form on V with nondegenerate restriction to U. Then in V there exists a basis $\{e_1, \dots, e_{2n+1}\}$ such that $e_{2n+1} \in \ker(\omega)$, $e_1, \dots, e_{2n+1} \in U$, each of the subspaces V_i is generated by V_{i-1} and some vector $v_i = e_l$ or $v_i = e_j + e_{2n+1}$, $j \neq 2n+1$, and the basis $\{e_1, \dots, e_{2n+1}\}$ is hyperbolic for the form ω.*

Proof. Since V is odd-dimensional, the form ω is degenerate. We define e_{2n+1} by the rule $\ker(\omega) = \langle e_{2n+1} \rangle$. Since $\omega|_U$ is nondegenerate, we have $e_{2n+1} \notin U$, so that $V = U \oplus \langle e_{2n+1} \rangle$. Let $U_\bullet$ be the complete flag obtained by the projection $\mathrm{pr}\colon V \to U$ along the space $\langle e_{2n+1} \rangle$ of the elements of the flag $V_\bullet$ such that $V_{k-1}, V_k \mapsto U_{k-1}$. By the preceding lemma, there exists a basis $\{u_1, \dots, u_{2n}\}$ that is compatible with the flag $U_\bullet$ and hyperbolic for $\omega|_U$. Let $v'_1, \dots, v'_{k-1}, v'_{k+1}, \dots, v'_{2n+1}$ be inverse images of the vectors $u_1, \dots, u_{2n}$ under the projection pr such that $v'_i \in V_i$ for $i \neq k$. We set $v_k = v'_k = e_{2n+1}$. The basis $v'_1, \dots, v'_{2n+1}$ is compatible with $V_\bullet$ and hyperbolic for ω. Since $\ker(\mathrm{pr}) = \langle e_{2n+1} \rangle$, we have $v - \mathrm{pr}(v) \in \langle e_{2n+1} \rangle$ for any v. Let $v'_i - \mathrm{pr}(v'_i) = \alpha_i e_{2n+1}$. We consider the $i < j$ such that $\omega(e_i, e_j) = 1$. The case $\omega(e_i, e_j) = -1$ does not hold for $i < j$, which follows from the proof of Lemma 2. One of the following four cases holds.

1) $\alpha_i = 0$ and $\alpha_j = 0$. We set $v_i = v'_i$ and $v_j = v'_j$.
2) $\alpha_i = 0$ and $\alpha_j \neq 0$. We set $v_i = \alpha_j v'_i$ and $v_j = \alpha_j^{-1} v'_j$.
3) $\alpha_i \neq 0$ and $\alpha_j = 0$. We set $v_i = \alpha_i^{-1} v'_i$ and $v_j = \alpha_i v'_j$.
4) $\alpha_i \neq 0$ and $\alpha_j \neq 0$. We set $v_i = \alpha_i^{-1} v'_i$ and $v_j = \alpha_i v'_j - \alpha_j v'_i$.

For $i \neq 2n + 1$ we now set $e_i = \mathrm{pr}(v_i) \in U$. Thus, the basis $\{e_1, \dots, e_{2n+1}\}$ is the required one.

§ 3. Cases $\mathbf{SL}(n) \times \mathbf{SL}(m) \subset \mathbf{SL}(m+n)$ and $\mathbf{S}(\mathbf{GL}(m) \times \mathbf{GL}(n)) \subset \mathbf{SL}(m+n)$

Proposition 1. *The subgroup* $\mathrm{GL}(m) \times \mathrm{GL}(n) \subset \mathrm{GL}(m+n)$ *is parabolically connected.*

Proof. We carry out the proof by induction on (m,n) assuming that $(m',n') \leqslant (m,n)$ if $m' \leqslant m$ and $n' \leqslant n$. The basis of induction for $m=0$ or $n=0$ is equivalent to the fact that a Borel subgroup of the full linear group is connected.

We use the following observation: if $\varphi\colon G_1 \to G_2$ is a surjective homomorphism of algebraic groups, the group G_2 is connected, and the kernel $\ker(\varphi)$ is contained in the connected component of the identity G_1°, then the group G_1 is connected.

Let $V = U \oplus W$, $\dim(U) = m$, $\dim(W) = n$, let $V_\bullet$ be a complete flag in the space V, let $H = \mathrm{GL}(U) \times \mathrm{GL}(W)$, and let $K = H \cap \mathrm{Stab}(V_\bullet)$, where $B = \mathrm{Stab}(V_\bullet)$ is a Borel subgroup. We claim that K is connected. In the space V we choose bases $\{e_1, \dots, e_{m+n}\}$ and $\{v_1, \dots, v_{m+n}\}$ by Lemma 1.

Let $v_1 = e_1 \in U$ (the case $v_1 = e_{m+1} \in W$ is considered in similar fashion), $U' = U/\langle e_1 \rangle$, and $V'_\bullet = V_\bullet/\langle e_1 \rangle$. Consider the projection

$$\varphi\colon K \to \mathrm{GL}(U' \oplus W).$$

In the basis $\{e_1, \dots, e_{m+n}\}$, the kernel $\ker(\varphi)$ consists of matrices of the form

$$\left(\begin{array}{cc|c} * & * & \\ 0 & E & 0 \\ \hline \multicolumn{2}{c|}{0} & E \end{array}\right);$$

therefore it is connected, so that the connectedness of K follows from the connectedness of the image $(\mathrm{GL}(U') \times \mathrm{GL}(W)) \cap \mathrm{Stab}(V'_\bullet)$, that is, by the induction hypothesis for $(m-1, n)$.

Let $v_1 = e_1 + e_{m+1}$, $e_1 \in U$, $e_{m+1} \in W$, $U' = U/\langle e_1 \rangle$, $W' = W/\langle e_{m+1} \rangle$, $V'_\bullet = V_\bullet/\langle e_1, e_{m+1} \rangle$. Consider the projection $\varphi\colon K \to \mathrm{GL}(U' \oplus W')$. The kernel $\ker(\varphi)$ consists of matrices of the form

$$\left(\begin{array}{cc|cc} \lambda & * & & \\ 0 & E & \multicolumn{2}{c}{0} \\ \hline & & \lambda & * \\ \multicolumn{2}{c|}{0} & 0 & E \end{array}\right), \qquad \lambda \in \mathbb{C}^\times,$$

and is therefore connected, the image is equal to $(\mathrm{GL}(U') \times \mathrm{GL}(W')) \cap \mathrm{Stab}(V'_\bullet)$. Therefore, the connectedness of K follows from the connectedness of the image, that is, from the induction hypothesis for $(m-1,\, n-1)$.

Proposition 2. *The subgroup* $\mathrm{SL}(m) \times \mathrm{GL}(n) \subset \mathrm{GL}(m+n)$ *is parabolically connected.*

Proof. We conduct the proof by the same method as the preceding one, using the following notation: $H = \mathrm{SL}(U) \times \mathrm{GL}(W) \subset \mathrm{GL}(U \oplus W)$, $K = H \cap \mathrm{Stab}(V_\bullet)$, where $(m,n) = (\dim(U), \dim(W))$, $V_\bullet$ is a complete flag in the space $V = U \oplus W$, and induction is conducted with respect to (m,n) with the same order relation. The basis of induction for $m = 0$ or $n = 0$ consists in the fact that Borel subgroups of $\mathrm{SL}(m)$ and $\mathrm{GL}(n)$ are connected. In the space V we choose bases $\{e_1, \dots, e_{m+n}\}$ and $\{v_1, \dots, v_{m+n}\}$ by Lemma 1.

Let $v_1 = e_1 \in U$, $U' = U/\langle e_1 \rangle$, $\varphi\colon K \to \mathrm{GL}(U' \oplus W)$, and $V'_\bullet = V/\langle e_1 \rangle$. The kernel $\ker(\varphi)$ consists of matrices of the form

$$\left(\begin{array}{cc|c} 1 & * & \multirow{2}{*}{0} \\ 0 & E & \\ \hline \multicolumn{2}{c|}{0} & E \end{array}\right).$$

The kernel is connected, the image is equal to $(\mathrm{GL}(U') \times \mathrm{GL}(W)) \cap \mathrm{Stab}(V'_\bullet)$ and is connected by Proposition 1.

Let $v_1 = e_{m+1} \in W$, $W' = W/\langle e_{m+1} \rangle$, $\varphi\colon K \to \mathrm{GL}(U \oplus W')$, $V'_\bullet = V_\bullet/\langle e_{m+1} \rangle$. The kernel $\ker(\varphi)$ consists of matrices of the form

$$\left(\begin{array}{c|cc} E & \multicolumn{2}{c}{0} \\ \hline \multirow{2}{*}{0} & * & * \\ & 0 & E \end{array}\right)$$

and is therefore connected, the image φ is equal to $(\mathrm{SL}(U) \times \mathrm{GL}(W')) \cap \mathrm{Stab}(V'_\bullet)$ and is connected by the induction hypothesis for $(m, n-1)$.

Let $v_1 = e_1 + e_{m+1}$, $e_1 \in U$, $e_{m+1} \in W$, $U' = U/\langle e_1 \rangle$, $W' = W/\langle e_{m+1} \rangle$, $\varphi\colon K \to \mathrm{GL}(U' \oplus W')$, and $V'_\bullet = V_\bullet/\langle e_1, e_{m+1} \rangle$. The kernel $\ker(\varphi)$ is connected, since it consists of matrices of the form

$$\left(\begin{array}{cc|cc} 1 & * & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & E & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1 & * \\ & & 0 & E \end{array}\right).$$

The image is equal to $(\mathrm{GL}(U') \times \mathrm{GL}(W')) \cap \mathrm{Stab}(V'_\bullet)$ and is connected by Proposition 1. Therefore the subgroup K is connected.

Proposition 3. *The subgroup*

$$\{(A, B) \in \mathrm{GL}(m) \times \mathrm{GL}(n) : \det(A) = \det(B)\} \subset \mathrm{GL}(m+n)$$

is parabolically connected.

Proof. We conduct the proof by the method similar to the preceding one, using the following notation:

$$H = \{(A, B) \in \mathrm{GL}(U) \times \mathrm{GL}(W) : \det(A) = \det(B)\} \subset \mathrm{GL}(U \oplus W),$$
$$K = H \cap \mathrm{Stab}(V_\bullet),$$

where $(m, n) = (\dim(U), \dim(W))$, $V_\bullet$ is a complete flag in the space $V = U \oplus W$, and induction is conducted with respect to (m, n) with the same order relation. The basis of induction for $m = 0$ or $n = 0$ consists in the fact that a Borel subgroup of the special linear group is connected. In the space V we choose bases $\{e_1, \dots, e_{m+n}\}$ and $\{v_1, \dots, v_{m+n}\}$ by Lemma 1.

Let $v_1 = e_1 \in U$ (the case $v_1 = e_{2n+1} \in W$ is considered in similar fashion), $\varphi\colon K \to \mathrm{GL}(U' \oplus W)$, $U' = U/\langle e_1\rangle$, and $V'_\bullet = V_\bullet/\langle e_1\rangle$. The kernel $\ker(\varphi)$ is connected, since it consists of elements of the form

$$\left(\begin{array}{cc|c} 1 & * & \multirow{2}{*}{0} \\ 0 & E & \\ \hline \multicolumn{2}{c|}{0} & E \end{array}\right).$$

The image $\mathrm{Im}(\varphi)$ is equal to $(\mathrm{GL}(U') \times \mathrm{GL}(W)) \cap \mathrm{Stab}(V'_\bullet)$ and is connected by Proposition 1.

Let $v_1 = e_1 + e_{m+1}$, $e_1 \in U$, $e_{m+1} \in W$, $\varphi\colon K \to \mathrm{GL}(U' \oplus W')$, $U' = U/\langle e_1\rangle$, $W' = W/\langle e_{m+1}\rangle$, and $V'_\bullet = V_\bullet/\langle e_1, e_{m+1}\rangle$. The kernel $\ker(\varphi)$ is connected, since it consists of elements of the form

$$\left(\begin{array}{cc|cc} \lambda & * & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & E & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \lambda & * \\ & & 0 & E \end{array}\right).$$

The image $\mathrm{Im}(\varphi)$ is equal to $(\{(A, B) \in \mathrm{GL}(U') \times \mathrm{GL}(W') : \det(A) = \det(B)\}) \cap \mathrm{Stab}(V'_\bullet)$ and is connected by the induction hypothesis for $(m-1, n-1)$. Thus, the subgroup K is connected.

Proposition 4. *The subgroup* $\mathrm{SL}(m) \times \mathrm{SL}(n) \subset \mathrm{SL}(m+n)$ *is parabolically connected.*

Proof. Let us prove that the subgroup $H = \mathrm{SL}(m) \times \mathrm{SL}(n) \subset \mathrm{GL}(m+n)$ is parabolically connected. Let $B \subset \mathrm{GL}(m+n)$ be some Borel subgroup. Then $B' = B \cap \mathrm{SL}(m+n)$ is a Borel subgroup of $\mathrm{SL}(m+n)$, and $H \cap B = H \cap B'$ by the condition $H \subset \mathrm{SL}(m+n)$, so that the parabolic connectedness of $\mathrm{SL}(n) \times \mathrm{SL}(n)$ as a subgroup of $\mathrm{SL}(m+n)$ is equivalent to its parabolic connectedness as a subgroup of $\mathrm{GL}(m+n)$.

Further the proof and notation are similar to the preceding ones:

$$H = \mathrm{SL}(U) \times \mathrm{SL}(W), \qquad K = H \cap \mathrm{Stab}(V_\bullet),$$

where $(m, n) = (\dim(U), \dim(W))$, $V_\bullet$ is a complete flag in the space $V = U \oplus W$, and induction is conducted with respect to (m, n) with the same order relation. The basis of induction for $m = 0$ or $n = 0$ consists in the fact that Borel subgroups of the special linear group are connected. In the space V we choose bases $\{e_1, \dots, e_{m+n}\}$ and $\{v_1, \dots, v_{m+n}\}$ by Lemma 1.

Let $v_1 = e_1 \in U$, $U' = U/\langle e_1\rangle$, $\varphi\colon K \to \mathrm{GL}(U' \oplus W)$, and $V'_\bullet = V_\bullet/\langle e_1\rangle$. One of the following three cases holds.

1) The set of indices $\{2, \dots, m+n\}$ is a union of disjoint pairs $\{i, j\}$ such that $e_i + e_j = v_k$ for some k. We denote by s a permutation of the elements of the set $\{2, \dots, m+n\}$ such that if there exist i, j, k for which $e_i + e_j = v_k$, then $s(i) = j$ and $s(j) = i$. Let an element $g \in H$ be written in the basis $\{e_1 \dots, e_{m+n}\}$ by a matrix (a_{ij}). We claim that $a_{ii} = a_{s(i)s(i)}$ for $i = 2, \dots, m+n$. We fix i and $v_k = e_i + e_{s(i)}$. For $g \in \mathrm{Stab}(V_\bullet)$ it is true that $gv_k = \lambda_k v_k + v$ for $v \in V_{k-1}$. Since the flag $V_\bullet$ is stabilized by the element g and is compatible with the basis $\{v_1, \dots, v_{m+n}\}$,

the matrix of g in this basis is upper-triangular, so that $\det(g) = \lambda_1 \cdots \lambda_{m+n}$. Furthermore, $V_k = V_{k-1} \oplus \langle v_k \rangle$ and $V_{k-1} \cap \langle e_i, e_{s(i)} \rangle = 0$ by the construction of the basis. Let $v_k = e_i + e_{s(i)}$, $v_{k'} = e_i$, $k' > k$. Then $a_{ii} = a_{s(i)s(i)} = \lambda_k = \lambda_{k'}$. We set $\{k : v_k \notin U \cup W\} = \{i_1, \dots, i_l\}$. Then $\det(g|_U) = a_{11}\lambda_{i_1} \cdots \lambda_{i_l} = 1$ and $\det(g|_W) = \lambda_{i_1} \cdots \lambda_{i_l}$, since $g \in \mathrm{SL}(U) \times \mathrm{SL}(W)$. Hence, $a_{11} = 1$. Therefore the kernel of φ consists of matrices of the form

$$\left(\begin{array}{cc|c} 1 & * & \multirow{2}{*}{0} \\ 0 & E & \\ \hline \multicolumn{2}{c|}{0} & E \end{array}\right)$$

and therefore is connected. The image of φ is equal to $(\mathrm{SL}(U') \times \mathrm{SL}(W)) \cap \mathrm{Stab}(V'_\bullet)$ and is connected by the induction hypothesis.

2) There exist $e_i = v_j \in U$ such that the vector $v_{j'}$ is not equal to $e_i + e_{i'}$ for any i', j'. Then K contains the one-dimensional torus $T = \mathrm{diag}(\lambda, 1, \dots, 1, \lambda^{-1}, 1, \dots, 1)$, where λ^{-1} is in the ith place. By multiplying by $t \in \mathbb{C}^\times$ the inverse images of all elements of $(\mathrm{SL}(U') \times \mathrm{SL}(W)) \cap \mathrm{Stab}(V'_\bullet)$ we obtain the inverse images of all elements of $(\mathrm{GL}(U') \times \mathrm{SL}(W)) \cap \mathrm{Stab}(V'_\bullet)$. As in the preceding case, the kernel is unipotent and therefore connected, the image is connected by Proposition 2.

3) There exist i, j, i', j' such that $e_i = v_j \in W$, $v_{j'} = e_i + e_{i'}$. We can assume that $\dim(U) > 1$, since otherwise $H = \mathrm{SL}(W)$ and the induction assertion consists in the fact that a Borel subgroup is connected. Let $e_s + e_k = v_l$. Then the group K contains the one-dimensional torus

$$T = \{\mathrm{diag}(\lambda, 1, \dots, 1, \lambda^{-1}, 1, \dots, 1, \lambda^{-1}, 1, \dots, 1, \lambda, 1, \dots, 1)\},$$

where λ is in the 1st and ith places, λ^{-1} is in the sth and kth places. Similarly to the preceding case, the image is equal to $(\mathrm{GL}(U') \times \mathrm{SL}(W)) \cap \mathrm{Stab}(V'_\bullet)$. If there are no s, k, l such that $e_k + e_s = v_l$, then we can take $s = 2$, $k > m$, $k \neq i$, and the torus T will be contained in the subgroup K. We have the same kernel and image as in the preceding case.

Now let $v_1 = e_1 + e_{m+1}$, $e_1 \in U$, $e_{m+1} \in W$, $U' = U/\langle e_1 \rangle$, $W' = W/\langle e_{m+1} \rangle$, $\varphi\colon K \to \mathrm{GL}(U' \oplus W')$, $V'_\bullet = V_\bullet/\langle e_1, e_{m+1} \rangle$. We can assume that $\dim(U) > 1$ and $\dim(W) > 1$; otherwise the assertion that is being proved is equivalent to the assertion that a Borel subgroup of SL is connected. Then either there exists $v_k = e_i + e_j$, $i > 1$, $j > m+1$, $e_i \in U$, $e_j \in W$, or there exist $v_k = e_i \in U$ and $v_l = e_j \in W$, $i > 1$, $j > m+1$. In both cases, the group contains the one-dimensional torus

$$T = \{\mathrm{diag}(\lambda, 1, \dots, 1, \lambda^{-1}, 1, \dots, 1, \lambda, 1, \dots, 1, \lambda^{-1})\},$$

where λ is in the 1st and $(m+1)$st places, and λ^{-1} in the ith and jth places. As in the preceding case, the image is equal to

$$\bigl\{(A, B) \in \mathrm{GL}(U') \times \mathrm{GL}(W') : \det(A) = \det(B)\bigr\} \cap \mathrm{Stab}(V'_\bullet).$$

The elements of the kernel of the map φ have the form

$$\left(\begin{array}{cc|cc} 1 & * & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & E & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1 & * \\ & & 0 & E \end{array}\right),$$

so that the kernel is connected. Thus, the kernel and the image of φ are connected by Proposition 3, so that the subgroup K is also connected.

§ 4. Case $\mathbf{S(GL(m) \times GL(n)) \subset SL(m+n)}$, $\boldsymbol{m \neq n}$

Proposition 5. *The subgroup* $\mathrm{S}(\mathrm{GL}(m) \times \mathrm{GL}(n)) \subset \mathrm{SL}(m+n)$ *is parabolically connected.*

Proof. We conduct the proof in similar fashion by induction on

$$(m, n) = (\dim(U), \dim(W))$$

using the following notation:

$$G = \mathrm{S}(\mathrm{GL}(U) \times \mathrm{GL}(W)), \qquad K = H \cap \mathrm{Stab}(V_\bullet),$$

where $V_\bullet$ is a complete flag, $V = U \oplus W$. The basis of induction for $m = 0$ or $n = 0$ consists in the fact that a Borel subgroup of the special linear group is connected. In the space V we choose bases $\{e_1, \dots, e_{m+n}\}$ and $\{v_1, \dots, v_{m+n}\}$ by Lemma 1.

Let $v_1 = e_1 \in U$ (the case $v_1 = e_{m+1} \in W$ is considered in similar fashion), $U' = U/\langle e_1 \rangle$, $\varphi\colon K \to \mathrm{GL}(U' \oplus W)$, $V'_\bullet = V_\bullet/\langle e_1 \rangle$. The kernel $\ker(\varphi)$ consists of matrices of the form

$$\left(\begin{array}{cc|c} 1 & * & \multirow{2}{*}{0} \\ 0 & E & \\ \hline \multicolumn{2}{c|}{0} & E \end{array}\right),$$

it is connected, and the image is equal to $K' = (\mathrm{GL}(U') \times \mathrm{GL}(W)) \cap \mathrm{Stab}(V'_\bullet)$, since for any $g' \in K'$ we can choose in the inverse image the matrix element a_{11} to be such that the determinant of the element of the inverse image becomes equal to 1.

Let $v_1 = e_1 + e_{m+1}$, $U' = U/\langle e_1 \rangle$, $W' = W\langle e_{m+1} \rangle$, $\varphi\colon K \to \mathrm{GL}(U' \oplus W')$, and $V'_\bullet = V_\bullet/\langle e_1, e_{m+1} \rangle$. The kernel $\ker(\varphi)$ consists of matrices of the form

$$\left(\begin{array}{cc|cc} \lambda & * & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & E & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \lambda & * \\ & & 0 & E \end{array}\right), \qquad \lambda = \pm 1.$$

Since $m \neq n$, we can assume that $m + n > 2$, and one of the following two cases holds.

1) There exist i and j such that $e_i = v_j \in U$. Then

$$\ker(\varphi) \subset \mathrm{diag}(\lambda, 1, \dots, 1, \lambda^{-2}, 1, \dots, 1, \lambda, 1, \dots) = T,$$

where λ is in the 1st and mth places, and λ^{-2} is in the ith place.

2) There exist i and j such that $e_i = v_j \in W$. Then

$$\ker(\varphi) \subset \mathrm{diag}(\lambda, 1, \dots, 1, \lambda, 1, \dots, 1, \lambda^{-2}, 1, \dots) = T,$$

where λ is in the 1st and mth places and λ^{-2} is in the ith place.

The kernel $\ker(\varphi)$ is contained in the connected component of the identity element of the group K, since it is contained in the torus T. The image of φ is equal to $(\mathrm{GL}(U') \times \mathrm{GL}(W')) \cap \mathrm{Stab}(V'_\bullet)$ and is connected by Proposition 1. Thus, the subgroup K is connected.

§ 5. Cases $\mathbf{Sp}(2n) \subset \mathbf{SL}(2n)$, $\mathbf{Sp}(2n) \subset \mathbf{SL}(2n+1)$, and $\mathbf{Sp}(2n) \times \mathbf{T}^1 \subset \mathbf{SL}(2n+1)$

Proposition 6. *The subgroup* $\mathrm{Sp}(2n) \subset \mathrm{SL}(2n+1)$ *is parabolically connected.*

Proof. Let $V = U \oplus W$, $\dim(U) = 2n$, $\dim(W) = 1$, let ω be a skew-symmetric form on V such that the restriction $\omega|_U$ is nondegenerate, let $W = \ker(\omega)$, let $V_\bullet$ be a complete flag in the space V, let $K = \mathrm{Sp}(2n) \cap \mathrm{Stab}(V_\bullet)$. In the space V we choose a basis $\{e_1, \dots, e_{2n+1}\}$ by Lemma 3. We can explicitly write out equations on the matrix elements with respect to this basis. Let $A = (a_{ij}) \in \mathrm{SL}(V)$. The condition $A \in K$ is equivalent to the conditions $A|_U \in \mathrm{SL}(U)$, $A^t \Omega A = \Omega$, where $\Omega = (\omega(e_i, e_j))$, and $Av_i \in V_i$, $i = 1, \dots, 2n+1$. Since $(a_{ij}) \in \mathrm{Sp}(2n) \subset \mathrm{SL}(2n+1)$, in this basis we have $a_{2n+1,2n+1} = 1$ and $a_{i,2n+1} = a_{2n+1,i} = 0$, $i = 1, \dots, 2n$. Therefore we consider matrices of size $2n \times 2n$.

We introduce some notation. Let $I = \{1, \dots, 2n\}$,

$$S = \{i : \exists\, j \quad v_j = e_i + e_{2n+1}\}. \tag{5.1}$$

Let $I^\flat, I^\sharp \subset I$ be subsets such that if $\omega(e_i, e_j) = 1$, then $i \in I^\flat$, $j \in I^\sharp$. The conditions that the basis $\{e_i\}_{i \in I}$ is hyperbolic and the restriction of the form ω to the hyperplane U is nondegenerate imply that $I^\flat \sqcup I^\sharp = I$. We also denote $i = j^\flat$, $i^\sharp = j$ for convenience of writing formulae. The index $\bar{i}$ is equal to $i^\flat$ or $i^\sharp$ depending on which of the formulae is defined. Here the symbol $i^\sharp$ makes sense only for $i \in I^\flat$ (similarly for $j^\flat$). We now write down explicitly equations on the matrix elements defining the group K:

$$\begin{cases} a_{i,2n+1} = 0, \quad i = 1, \dots, 2n; & (5.2)\\ a_{2n+1,i} = 0, \quad i = 1, \dots, 2n; & (5.3)\\ a_{2n+1,2n+1} = 1; & (5.4)\\ a_{i,j} = 0, \quad i > j; & (5.5)\\ \displaystyle\sum_{i \in I^\flat} a_{i,l} a_{i^\sharp,m} - \sum_{i \in I^\sharp} a_{i,l} a_{i^\flat,m} = \omega(e_l, e_m), \quad l, m = 1, \dots, 2n; & (5.6)\\ \displaystyle\sum_{i \in S} a_{i,k} = \begin{cases} 1, & k \in S; \\ 0, & k \notin S. \end{cases} & (5.7) \end{cases}$$

Let us explain how this system of equations was obtained. Equations (5.2)–(5.4) are equivalent to the condition $A \in \mathrm{SL}(W)$.

The invariance of the flag $Av_i \in V_i$ implies that

$$Av_i \in \langle e_1, \dots, e_i, e_{2n+1} \rangle.$$

Next, $Ae_{2n+1} = e_{2n+1}$, $v_i = e_i$ or $v_i = e_i + e_{2n+1}$ for $i = 1, \dots, 2n$, whence $Ae_i \in \langle e_1, \dots, e_i, e_{2n+1} \rangle$. This fact and (5.3) imply (5.5).

Equations (5.6) are equivalent to the fact that $A^t \Omega A = \Omega$.

Let $k \in S$. Then

$$Av_k = A(e_k + e_{2n+1}) = Ae_k + e_{2n+1} \in \bigl\langle \{e_i\}_{i \notin S, i \leqslant k}, \{e_i + e_{2n+1}\}_{i \in S, i \leqslant k} \bigr\rangle.$$

Here,

$$Ae_k = \sum_{i=1}^{k} a_{i,k} e_i, \qquad Ae_{2n+1} = e_{2n+1}.$$

Hence,

$$e_{2n+1} = \sum_{S\ni i\leqslant k} a_{i,k} e_{2n+1}, \quad \text{that is,} \sum_{S\ni i\leqslant k} a_{i,k} = 1 \quad \Longleftrightarrow \quad \sum_{i\in S} a_{i,k} = 1.$$

For $k \notin S$ we conclude in similar fashion that $\sum_{i\in S} a_{i,k} = 0$.

From the obtained system of equations we now derive the conditions $A \in \mathrm{SL}(W)$, $A^t\Omega A = \Omega$, and $Av_i \in V_i$ for $i = 1, \ldots, 2n+1$. The first two follow from that system in obvious fashion. We now derive the third condition.

Let $k \in S$. Then

$$\begin{aligned} Av_k = Ae_k + e_{2n+1} &= \sum_{i=1}^{k} a_{i,k} e_i + e_{2n+1} \\ &= \sum_{i=1}^{k} a_{i,k} e_i + \sum_{S\ni i\leqslant k} a_{i,k} e_{2n+1} = \sum_{i=1}^{k} a_{i,k} v_i \in V_i. \end{aligned}$$

The case $k \notin S$ is considered in similar fashion.

We now prove the assertion by induction. The subgroup $\mathrm{SL}(2) \times \{1\} \subset \mathrm{SL}(3)$ is parabolically connected by Proposition 4. Suppose that we have proved that the subgroup $\mathrm{Sp}(2n-2) \subset \mathrm{SL}(2n-1)$ is parabolically connected. We now prove that then the subgroup $\mathrm{Sp}(2n) \subset \mathrm{SL}(2n+1)$ is also parabolically connected. Consider the map $\varphi\colon K \to \mathrm{GL}(V')$, $V' = \langle e'_1, \ldots, e'_{2n-2}\rangle$, obtained by removing from the matrix the rows and columns with numbers 1 and $1^\sharp$. We claim that this is a homomorphism. Since the group K preserves the form ω, we have $a_{k1^\sharp} = \pm\omega(e_1, ge_k) = \pm\lambda^{-1}\omega(ge_1, ge_k) = 0$ for any $k \neq 1^\sharp$ for any $g \in K$. We claim that the deleted elements in the two matrices (a_{ij}) and (b_{ij}) do not affect the non-deleted elements in their product (c_{ij}) for (a_{ij}), $(b_{ij}) \in K$. We have $c_{ij} = \sum_k a_{ik}b_{kj}$. Let $\{i, j\} \cap \{1, 1^\sharp\} = \varnothing$. Then the deleted elements participate only in two products: $a_{i1}b_{1j}$ and $a_{i1^\sharp}b_{1^\sharp j}$. But $b_{1j} = 0$, since $1 \neq j$ due to the invariance of $\langle e_1\rangle$, and $a_{i1^\sharp} = 0$, since $i \neq 1^\sharp$. Thus, φ is indeed a homomorphism. Its kernel is a semidirect product of a one-dimensional torus by a unipotent group and is therefore connected. Consider its image $K' = \varphi(K) \subset \mathrm{GL}(2n-2)$. Let $\xi\colon \{2, \ldots, 1^\sharp - 1, 1^\sharp + 1, \ldots, 2n\}$ be a monotonic bijective map, let $V'_\bullet$ be the flag consisting of the spaces $V'_{\xi(k)} = \langle e_{\xi(i_1)}, \ldots, e_{\xi(i_k)}\rangle$, where V_k is generated by e_1, e_{i_1}, $\ldots$, e_{i_k} and also, possibly, $e_{1^\sharp}$, $k \neq 1, 1^\sharp$. The group K' preserves the flag $V'_\bullet$. Let ω' be the nondegenerate skew-symmetric form on the space V' the matrix of which is obtained from the matrix of ω by removing the 1st and $1^\sharp$st rows and columns. Clearly, the group K' stabilizes ω'. The homomorphism $\varphi\colon K \to \mathrm{Stab}(V'_\bullet) \cap \mathrm{Stab}(\omega')$ is surjective, since for finding the inverse image of an element of K' we need to add the deleted rows and columns to the matrix. In order to obtain an element of K, we can set the added diagonal elements to be equal to 1, and the non-diagonal to be equal to 0. The image K' is connected by the induction hypothesis.

Proposition 7. *The subgroup* $\mathrm{Sp}(2n) \subset \mathrm{SL}(2n)$ *is parabolically connected.*

Proof. We reduce this assertion to the parabolic connectedness of $\mathrm{Sp}(2n) \subset \mathrm{SL}(2n+1)$. Indeed, let $\mathrm{Sp}(2n)$ act on a hyperplane U of V. Then the connectedness of the intersection of $\mathrm{Sp}(2n)$ with the stabilizer of any flag $V_\bullet$ implies the connectedness of the intersection with the stabilizer of any flag $U_\bullet \cup \{V\}$, that is, the connectedness of the intersection with any Borel subgroup of $\mathrm{SL}(V)$ implies the connectedness of the intersection with any Borel subgroup of $\mathrm{SL}(U)$.

Proposition 8. *The subgroup* $\mathrm{Sp}(2n) \cdot \mathrm{T}^1 \subset \mathrm{SL}(2n+1)$ *is parabolically connected.*

Proof. Let $V = U \oplus W$, $\dim(W) = 1$, $\mathrm{Sp}(2n) \subset \mathrm{SL}(U)$, and let B be some Borel subgroup of $\mathrm{SL}(V)$. We define the homomorphism

$$\varphi\colon \mathrm{Sp}(2n) \cdot T^1 \to \mathbb{C}^\times, \qquad \varphi(A) = A|_W.$$

We choose in V a basis $\{e_1, \dots, e_{2n+1}\}$ by Lemma 3. In the notation of the proof of Proposition 6, for every pair $(i, \bar{\imath})$, $i \in S$, we set $t_i = \lambda^{-n}$, $t_{\bar{\imath}} = \lambda^{n+1}$, $t_{2n+1} = \lambda^{-n}$, where S is defined by formula (5.1). In the case $i, \bar{\imath} \notin S$, we define t_i and $t_{\bar{\imath}}$ in the same way as if any one of the indices i or $\bar{\imath}$ belongs to S. Then

$$\mathrm{diag}(t_1, \dots, t_{2n+1}) \in K = (\mathrm{Sp}(2n) \cdot \mathrm{T}^1) \cap B.$$

Hence the image satisfies

$$\varphi(\mathrm{Sp}(2n) \cdot \mathrm{T}^1 \cap B) \simeq \mathbb{C}^\times.$$

The kernel is equal to $(\mathrm{Sp}(2n) \cdot Z_{2n}) \cap B$ and is disconnected, since the group $Z_{2n} = \{\lambda E : \lambda^{2n} = 1\}$ is finite. But $Z_{2n} \subset \mathrm{T}^{1\prime}$, the one-dimensional torus is connected and intersects all the connected components of the kernel of φ since $\mathrm{Sp}(2n) \cap B$ is connected, so that $\ker(\varphi) \subset K^\circ$, the image is connected, so that the inverse image is also connected.

§ 6. Absence of parabolic connectedness

We consider the special orthogonal group $\mathrm{SO}(n)$, $n \geqslant 2$, preserving the standard quadratic form $x_1^2 + \dots + x_n^2$. Then its intersection with the group of upper-triangular matrices is finite and therefore disconnected. This shows that $\mathrm{SO}(n) \subset \mathrm{SL}(n)$ is not parabolically connected.

Proposition 9. *The subgroup* $\mathrm{S}(\mathrm{GL}(n) \times \mathrm{GL}(n)) \subset \mathrm{SL}(2n)$ *is not parabolically connected.*

Proof. Let $\{e_1, \dots, e_n\}$ be a basis in U, $\{e_{n+1}, \dots, e_{2n}\}$ a basis in W, let

$$V_i = V_{i-1} \oplus \langle e_i + e_{n+i} \rangle, \qquad i = 1, \dots, n,$$

where we assume that V_0 is equal to zero, and let

$$V_{n+i} = V_{n+i-1} \oplus \langle e_i \rangle, \qquad i = 1, \dots, n.$$

Consider the group $K = \mathrm{S}(\mathrm{GL}(U)\times\mathrm{GL}(W))\cap\mathrm{Stab}(V_\bullet)$. Choose an element $g\in K$ and consider the matrix (a_{ij}) of this element in the basis $\{e_1,\dots,e_{2n}\}$. Since $g(e_i+e_{n+i})\in V_i$, the equality $a_{ii}+a_{i,n+i}=a_{n+i,i}+a_{n+i,n+i}$ holds. But $a_{i,n+i}=a_{n+i,i}=0$, since the subspaces U and W are invariant, so that $a_{ii}=a_{n+i,n+i}=\lambda_i$ for $i=1,\dots,n$. It is easy to observe that the diagonal blocks in the matrix are upper-triangular. Therefore, $\lambda_1^2\cdots\lambda_n^2=\det(g)=1$, that is, $\lambda_1\cdots\lambda_n=\pm1$. The group is decomposed into two disjoint closed non-empty subsets, that is, it is disconnected.

Thus, Theorem 1 is completely proved.

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