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On subgroups of the unitary group over a semilocal ring

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In [1], [2], [5] there is a description of the subgroups of the general linear group $GL(n, \Lambda)$ over a semilocal ring Λ, containing the group $D(n, \Lambda)$ of diagonal matrices. The analogous question has also been solved for split groups of the other classical types (for $G\ Sp_{2l}$ see [6], an account of the results for GO_n is due to appear in the "Bulletin of the Polish Academy of Sciences"). In this note we solve the analogous question for the subgroups of the 4-dimensional unitary group corresponding to a form of maximal Witt index.

Let Λ be an associative ring with an identity and an involution ρ (antiautomorphism of order 2). Let $\rho x = \bar{x}$; $R = \{x \in \Lambda \mid x = \bar{x}\}$ be the invariant ring; let Λ^* and R^* be the multiplicative groups of the rings Λ and R, respectively. In the main results of this note we assume that Λ is a semilocal ring for which 1) $2 \in R^*$, 2) every residue field of R contains at least 7 elements, 3) there exists a $\theta \in \Lambda^*$ such that $\bar{\theta} = -\theta$.

We denote by f an $(l \times l)$-matrix that has 1's on the second diagonal and 0's everywhere else, and by F the block matrix of order $2l$ of the form

$$F = \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix}.$$

The general unitary group $\Gamma = GU(2l, \Lambda)$ consists of those matrices x of order $2l$ over Λ for which $xF\bar{x}^t = \lambda F$ for some $\lambda \in R^*$, where x^t is the transpose of x. By $\Delta = \Delta(2l, \Lambda)$ we denote the subgroup of diagonal matrices contained in Γ.

We recall that the array $\sigma = (\sigma_{ij})$, $1 \leqslant i, j \leqslant n$ of two-sided ideals σ_{ij} of Λ is called a D-net of ideals in Λ of order n if $\sigma_{ir}\sigma_{rj} \subseteq \sigma_{ij}$ and $\sigma_{ii} = \Lambda$ for all i, j, and r (see [1], [2]). To every net there corresponds a subgroup $G(\sigma)$ in the general linear group $G = GL(n, \Lambda)$ consisting of those matrices $a = (a_{ij}) \in G$ for which $a_{ij}, a'_{ij} \equiv \delta_{ij} (\mathrm{mod}\ \sigma_{ij})$ for all i, j, where $a^{-1} = (a'_{ij})$ is the matrix inverse to a.

For $r = 1, \ldots, n$ we put $r^* = n + 1 - r$ and we call a net $\sigma = (\sigma_{ij})$ of order n over a ring Λ with an involution unitary if $\sigma_{i^*j} = \bar{\sigma}_{ji}$ for all i and j. When $n = 2^l$, a unitary net determines a unitary net subgroup $\Gamma(\sigma) = G(\sigma) \cap \Gamma$ of $\Gamma = GU(2l, \Lambda)$. By $N_\Gamma(\sigma)$ we denote the normalizer of $\Gamma(\sigma)$ in Γ.

As is customary, we denote by e the identity matrix and by e_{ij} the matrix unit in which there is a 1 in the place (i, j) and 0's elsewhere. Unitary transvections are defined by $T_{ij}(\alpha) = e + \alpha e_{ij} - \bar{\alpha} e_{j^*i^*}$, $\alpha \in \Lambda$, for $i \neq j, j^*$ and $T_{ii^*}(\alpha) = e + \alpha e_{ii^*}$, $\alpha \in R$. For a unitary net σ we put $\hat{\sigma}_{ij} = \sigma_{ij}$ if $i \neq j^*$ and $\hat{\sigma}_{ii^*} = \sigma_{ii^*} \cap R$. Let $E_\Gamma(\sigma)$ be the subgroup generated by the unitary transvections $T_{ij}(\alpha)$, $\alpha \in \hat{\sigma}_{ij}$, $i \neq j$. For a subgroup H of Γ normalized by Δ we put $\hat{\sigma}_{ij} = \{\alpha \in \Lambda;\ T_{ij}(\alpha) \in H\}$. Under insignificant additional assumptions (which hold in particular, if Λ is a semilocal ring satisfying 1)-3) above) all $\hat{\sigma}_{ij}$, $i \neq j^*$, are ideals of Λ, and the $\hat{\sigma}_{ii^*}$ are ideals in R, and if we put $\sigma_{ij} = \Lambda\hat{\sigma}_{ij}$ for $i \neq j$ and $\sigma_{ii} = \Lambda$, then the set of ideals $\sigma = (\sigma_{ij})$ forms a unitary net, which we call the net associated with H (see [4]). Here $E_\Gamma(\sigma) \leqslant H$ and σ is the largest net with this property.

***Theorem* 1.** *Let Λ be a semilocal ring with an involution satisfying* 1)-3). *Then for any subgroup H of $\Gamma = GU(2l, \Lambda)$ containing the group $\Delta = \Delta(2l, \Lambda)$*

$$\Gamma(\sigma) \leqslant H \leqslant N_\Gamma(\sigma),$$

where σ is the net associated with H.

We recall that a subgroup Δ of Γ is called pronormal in Γ if the subgroups Δ and $x\Delta x^{-1}$ for any $x \in \Gamma$ are already conjugate in the subgroup $\langle \Delta, x\Delta x^{-1} \rangle$ generated by them.

***Theorem* 2.** *Let Λ be a matrix local ring with an involution satisfying* 1)-3). *Then the subgroup $\Delta = \Delta(2l, \Lambda)$ is pronormal in $\Gamma = GU(2l, \Lambda)$.*

In other words, this theorem asserts that if H and F are two subgroups of Γ containing Δ and if $xHx^{-1} = F$ for some $x \in \Gamma$, then $x = \pi y$, where $y \in H$ and $\pi \in N_\Gamma(\Delta)$.

The proofs of Theorems 1 and 2 are based largely on the same arguments as in [1], [2], [4]-[6], but are technically somewhat more complicated.

Remarks. 1°. For the case of a finite field Theorem 1 is proved by completely different methods in [8]. 2°. The analogue of Theorem 1 also holds when 3) fails, but instead of nets one considers sets of ideals in which the condition $\sigma_{ij}\sigma_{ji^*} \subseteq \sigma_{ii^*}$ is replaced by the weaker one: Tr $\sigma_{ij}\sigma_{ji^*} \subseteq \sigma_{ii^*}$, where $\mathrm{Tr}\, I = \{a + \bar{a},\ a \in I\}$. 3°. The analogue of Theorem 1 holds for subgroups normalized by Δ, but here the inclusions are replaced by the following: $E_\Gamma(\sigma) \leqslant H \leqslant N_\Gamma(\sigma)$. 4°. Theorem 2 enables us, in particular, to compute the factor groups $N_\Gamma(\sigma)/\Gamma(\sigma)$ arising in Theorem 1. They are all factor groups of the Weyl group $N_\Gamma(\Delta)/\Delta$. 5°. Outwardly, the problem solved in the present note coincides with that solved in [3], [7]. But actually, this is not so, even when in every dimension there is only one unitary group. The point is that the torus considered in these papers is anisotropic, whereas in this note it is as close to the splitting one as possible.

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