A. A. Osinovskaya, I. D. Suprunenko, Inductive systems of representations with small highest weights for natural embeddings of symplectic groups,
*Тр. Ин-та матем.*, 2014, том 22, номер 2, 109–118

*Национальная академия наук Беларуси*
*Труды Института математики. 2014. Том 22. № 2. С. 109–118*

*УДК 512.554.32*

# INDUCTIVE SYSTEMS OF REPRESENTATIONS WITH SMALL HIGHEST WEIGHTS FOR NATURAL EMBEDDINGS OF SYMPLECTIC GROUPS

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*Received 10.09.2014*

For natural embeddings of symplectic groups, inductive systems of irreducible representations where the maximum of the highest weight value on the maximal root is equal to $2$ are studied. For such embeddings of algebraic groups of type $C_n$ in characteristic $3$, the inductive system of representations generated by irreducible representations with highest weight $2\omega_n$ is determined. It is proved that any inductive system of representations of such groups consisting of representations with the value of the highest weight on the maximal root at most $2$ and containing representations with such value equal to $2$ contains the subsystem generated by the standard representations or the subsystem generated by the representations with highest weight $\omega_n$. For algebraic groups of type $C_n$ in characteristic $3$, the restrictions of certain irreducible modules to subsystem subgroups of type $C_{n-1}$ are described.

For natural embeddings of symplectic groups, inductive systems of irreducible representations where the maximum of the highest weight value on the maximal root is equal to $2$ are studied. For such embeddings of algebraic groups of type $C_n$ in characteristic $3$, the inductive system of representations generated by irreducible representations with highest weight $2\omega_n$ is determined. It is proved that any inductive system of representations of such groups consisting of representations with the value of the highest weight on the maximal root at most $2$ and containing representations with such value equal to $2$ contains the subsystem generated by the standard representations or the subsystem generated by the representations with highest weight $\omega_n$. The proof of the main result is based on an explicit description of the restrictions of certain irreducible modules for algebraic groups of type $C_n$ in characteristic $3$ to subsystem subgroups of type $C_{n-1}$.

In what follows $K$ is an algebraically closed field of an odd characteristic $p$, $\mathbb{Z}$ is the set of all integers, $\mathbb{N}$ is the set of positive integers, $\mathbb{C}$ is the field of complex numbers, $G_n = C_n(K)$, $\omega_i^n$ and $\alpha_i^n$ with $1 \leq i \leq n$ are the fundamental weights and the simple roots of $G_n$, $\langle \mu, \alpha \rangle$ is the value of a weight $\mu$ on a root $\alpha$, $\varphi(\omega)$ is the irreducible representation with highest weight $\omega$, $M(\omega)$ is the module affording $\varphi(\omega)$, $\varphi \downarrow H$ ($M \downarrow H$) is the restriction of a representation $\varphi$ (a module $M$) to a subgroup $H$, $M^\mu$ is the weight component of weight $\mu$ in a module $M$, $\operatorname{Irr}\varphi$ is the set of composition factors of a representation $\varphi$ (multiplicities are not taken into account).

Let

$$\Gamma_1 \subset \Gamma_2 \subset \ldots \subset \Gamma_n \subset \ldots \qquad (1)$$

be a series of fixed embeddings of algebraic groups $\Gamma_n$ over $K$ and $\Phi_n$ be a nonempty finite set of irreducible rational representations of $\Gamma_n$. A collection $\Phi = \{\Phi_n | n \in \mathbb{N}\}$ is called *an inductive system* of representations for the series (1) if

$$\bigcup_{\varphi \in \Phi_{n+1}} \mathrm{Irr}(\varphi \downarrow \Gamma_n) = \Phi_n$$

for all $n \in \mathbb{N}$. The notion of an inductive system of representations was introduced by A. Zalesskii in [12]. Such systems can be regarded as an asymptotic version of the branching rules for the embeddings (1). Observe that in positive characteristic one cannot expect to find explicit analogues of the classical branching rules in characteristic 0 which have quite a lot of applications, so their asymptotic versions can be useful. Moreover, inductive systems can be applied to the study of ideals in group algebras of locally finite groups.

So far we know little about the structure of inductive systems. Minimal and minimal nontrivial inductive systems of modular representations for natural embeddings of special linear groups were classified by A. Baranov and I. Suprunenko [2]. In this case only the system that consists of the trivial representations is minimal, all inductive systems contain it; all minimal nontrivial inductive systems consist of nontrivial irreducible representations of the minimal dimension and trivial ones. For other classical groups the question on the minimal inductive systems is substantially more difficult and is far from solution. In [1] for the natural embeddings of the classical groups, inductive systems of representations with totally bounded weight multiplicities are classified and an analogue of the Steinberg tensor product theorem for arbitrary indecomposable inductive systems for such embeddings is proved.

Let $l \in \mathbb{N}$ or $l = 0$ and $\varphi_n$ be an irreducible representation of $\Gamma_n$ for $n > l$. We say that an inductive system of representations $\Phi = \{\Phi_n | n \in \mathbb{N}\}$ is generated by a collection $\{\varphi_n | n > l\}$ if $\varphi_n \in \Phi_n$ for $n > l$ and for any representation $\rho \in \Phi_n$ there exists $m > \max\{n, l\}$ such that $\rho \in \mathrm{Irr}(\varphi_m \downarrow \Gamma_n)$. In this situation we shall use the notation $\Phi = \langle \varphi_n | n > l \rangle$. It follows from [1, Lemma 4.3] that the system $\Phi$ is correctly determined.

It is well known that for $n > k$ the root subgroups in $G_n$ associated with the roots

$$\pm\alpha^n_{n-k+1}, \ldots, \pm\alpha^n_n$$

generate a subgroup isomorphic to $G_k$. Identifying $G_k$ with such subgroup, we get a series of natural embeddings

$$G_1 \subset G_2 \subset \ldots \subset G_n \subset \ldots. \tag{2}$$

In what follows we consider only inductive systems for the series (2).

**Theorem 1.** *Let* $p = 3$. *Then*

$$\langle \varphi(2\omega^n_n) | n > 0 \rangle_n = \{0, \varphi(\omega^n_1), \varphi(\omega^n_{n-2} + \omega^n_n), \varphi(2\omega^n_k),\ 1 \leq k \leq n,\ \varphi(\omega^n_j + \omega^n_{j+1}),\ 1 \leq j \leq n-1\}.$$

**Corollary 1.** *Let* $p = 3$. *Then* $\langle \varphi(2\omega^n_{n-1}) | n > 1 \rangle = \langle \varphi(\omega^n_{n-1} + \omega^n_n) | n > 1 \rangle$.

Let $\delta(\Phi)$ be the maximum of the values of the highest weights of representations from an inductive system $\Phi$ on the maximal roots of the corresponding groups. It follows from [1, Lemma 4.1] that the parameter $\delta(\Phi)$ is defined correctly. For $n \in \mathbb{N}$ set

$$\mathcal{R}^u_n = \{\omega^n_i | n + 1 - u \leq i \leq n\}.$$

By [3, Theorem 1.3], $\mathcal{R}^{p-1}$ is a minimal inductive system.

The following lemma holds for all odd $p$.

**Lemma 1.** *Let $p > 2$ and $\Phi$ be an inductive system of representations for the series* (2) *with $\delta(\Phi) = 2$. Then $\Phi$ contains the subsystem generated by the standard representations, or the subsystem $\langle\varphi(2\omega_n^n)|n > 0\rangle$, or $\mathcal{R}^{p-1}$.*

**Corollary 2.** *Let $p = 3$ and $\Phi$ be an inductive system of representations for the series (2) with $\delta(\Phi) = 2$. Then $\Phi$ contains the subsystem generated by the standard representations or $\mathcal{R}^{p-1}$.*

For $\omega \in \{2\omega_j^n,\ 1 \leq j \leq n,\ \omega_k^n + \omega_{k+1}^n,\ 1 \leq k \leq n-1,\ \omega_{n-2}^n + \omega_n^n\}$ the restrictions $\varphi(\omega) \downarrow G_{n-1}$ are described (Lemma 5).

This research is supported by the Belarusian Republican Foundation for Fundamental Research, Project F12-060.

In what follows for the root $\pm\alpha_j^n$, $t \in K$ and $d \in \mathbb{N}$ denote by $X_{\pm j}$, $x_{\pm j}(t)$, $\mathcal{X}_{\pm j}$, and $X_{\pm j,d}$ the root element in the Lie algebra of $G_n$, the root element in $G_n$, the root subgroup in $G_n$, and the element in the hyperalgebra of $G_n$ associated with $\pm\alpha_j^n$, $t$, and $d$, respectively; $W$ is the Weyl group of $G_n$. It will always be clear from the context what group is considered. Denote by $G_n(\beta_1, \dots, \beta_k)$ the subgroup in $G_n$ generated by all root subgroups for roots $\pm\beta_1, \dots, \pm\beta_k$. Set $G_n(i_1, \dots, i_k) = G_n(\alpha_{i_1}, \dots, \alpha_{i_k})$. Throughout the text if $H \subset G$ are semisimple algebraic groups with fixed maximal tori $T_1 \subset H$ and $T \subset G$ and $T_1 \subset T$, then $\lambda \downarrow H$ is the restriction of a weight $\lambda$ from $T$ to $T_1$. For a $G_n$-module $M$ denote by $v^+$ a highest weight vector, $\omega(v)$ is the weight of a weight vector $v$, $\omega_H(v) = \omega(v) \downarrow H$, $\Lambda(M)$ is the set of weights of $M$. Below $\omega(M)$ ($\omega(\varphi)$) is the highest weight of a module $M$ (a representation $\varphi$). It is well known that

$$x_{\pm j}(t)(v) = v + \sum_{d=1}^{\infty} X_{\pm j,d} t^d(v)$$

and that $X_{\pm j,d} = X_{\pm j}^d/d!$ for $d < p$ (see, for instance, [4, Proposition 5.13]). We shall use this facts frequently without special comments.

We need some preliminary lemmas to prove the results stated above.

**Lemma 2** [4, Lemma 5.14], [8, 1.5], and [10, 2.1].

(*i*) *For the operators $X_{i,d}$ the following equality holds:*

$$X_{-i}X_{i,d} = X_{i,d}X_{-i} - H_i X_{i,d-1} + (d-1)X_{i,d-1}. \tag{3}$$

*Here $H_i = [X_i, X_{-i}]$.*

(*ii*) *Let $M$ be a $G_n$-module, $\mu$ be a weight of $G_n$, $v \in M^\mu \setminus \{0\}$, $X_{i,b}v = 0$ for $b > 0$, and $\langle\mu, \alpha_i\rangle = c \geq 0$. Then $X_i X_{-i,b}v = (c - b + 1)X_{-i,b-1}v$ and $X_{-i,c}v \neq 0$. In particular, if $0 < c < p$, one has $X_{-i,d}v \neq 0$ for $0 < d \leq c$.*

**Theorem 2** (Jantzen [7], Smith [9]). *Let $H \subset G_n$ be the subgroup generated by $\mathcal{X}_{\pm i_1}, \dots, \mathcal{X}_{\pm i_j}$ and $M$ be an irreducible $G_n$-module. Then $KHv^+ \subset M$ is an irreducible $H$-module with highest weight $\omega_H(v^+)$ and a direct summand of the $H$-module $M$.*

**Lemma 3** [11, Lemma 2.46]. *Let $G$ be a simple algebraic group over $K$ of type $A_r$, $B_r$, or $C_r$, $\omega_i$ be the fundamental weights of $G$, and $\alpha_i$ be its simple roots ($1 \leq i \leq r$). Suppose that $V$ is an indecomposable $G$-module with highest weight $\sum_{k=1}^r m_k\omega_k$ and $v \in V$ is a nonzero highest weight vector. Let $1 \leq l, j \leq r$. Set $b_k = -\langle\alpha_{k+1}, \alpha_k\rangle$ and $c_k = -\langle\alpha_{k-1}, \alpha_k\rangle$. For an integer $d$ with $0 < d \leq m_j$ define the vector $v(l, j, d)$ as follows. Put $d_j = d$. If $l < j$, set $d_k = m_k + d_{k+1}b_k$ for $l \leq k < j$. If $l > j$, put $d_k = m_k + d_{k-1}c_k$ for $l \geq k > j$. Now set*

$$v(l, j, d) = X_{-l,d_l} \dots X_{-k,d_k} \dots X_{-j,d}v.$$

*For $l = j$ put $v(l, j, d) = X_{-l,d}v$. Assume that $0 < m_j < p$ or $d = m_j$. Then $v(l, j, d) \neq 0$ and $X_{t,b}v(l, j, d) = 0$ for positive $t \neq l$ and $b > 0$. Hence $\mathcal{X}_t$ fixes $v(l, j, d)$.*

Until the end of the paper $p = 3$, except Lemmas 1 and 6.

**Lemma 4.**

1) *Let* $\omega = 2\omega_2^2$ *and* $M = M(\omega)$. *Then there exists a nonzero vector* $v \in M$ *with* $\omega(v) = \omega - 2\alpha_1^2 - 2\alpha_2^2$ *that is fixed by* $\mathcal{X}_2$.

2) *Let* $\omega = \omega_1^3 + \omega_3^3$ *and* $M = M(\omega)$. *Then there exists a nonzero vector* $w \in M$ *with* $\omega(v) = \omega - 2\alpha_1^3 - 2\alpha_2^3 - \alpha_3^3$ *that is fixed by* $\mathcal{X}_i$ *for* $i = 2$ *and* 3.

**Proof.** Set $\lambda = \omega - 2\alpha_1^2 - 2\alpha_2^2$ in Case 1 and $\lambda = \omega - 2\alpha_1^3 - 2\alpha_2^3 - \alpha_3^3$ in Case 2; put $\lambda' = \lambda + \alpha_2^n$. We shall show that in both cases $\dim M^\lambda > \dim M^{\lambda'}$. This implies that there exists a nonzero vector $v \in M^\lambda$ such that $X_2 v = 0$. Hence $\mathcal{X}_2$ fixes $v$.

*Case 1.* Put

$$v_1 = X_{-2}(X_{-1})^2 X_{-2} v^+,$$

$$v_2 = X_{-1}X_{-2}X_{-1}X_{-2}v^+.$$

We claim that $v_1$ and $v_2$ are linearly independent. Suppose that $w = av_1 + bv_2 = 0$ with $a, b \in K$. By Formula (3), $X_2 w = (2a + b)(X_{-1})^2 X_{-2} v^+$ and $(X_1)^2 w = (4a + 12b)(X_{-2})^2 v^+$. Several applications of Lemma 2(ii) imply that $(X_{-1})^2 X_{-2} v^+$ and $(X_{-2})^2 v^+ \neq 0$. Hence $2a + b = 4a + 12b = 0$ which forces $a = b = 0$. Therefore $v_1$ and $v_2$ are linearly independent and $\dim M^\lambda \geq 2$. Observe that the weight $\lambda'$ lies in the same $W$-orbit with the weight $\omega - \alpha_2^2$ and hence $\dim M^{\lambda'} = 1$. This completes the proof in Case 1.

*Case 2.* We claim that $\lambda + \alpha_3^3 \notin \Lambda(M)$. Indeed, $\lambda + \alpha_3^3$ lies in the same $W$-orbit with the weight $\mu = \omega - 2\alpha_1^3$. Since $\langle \mu, \alpha_1^3 \rangle = -3$, it is clear that $\mu$ and $\lambda + \alpha_3^3 \notin \Lambda(M)$. Hence $\mathcal{X}_3$ fixes all vectors of weight $\lambda$. Since $\lambda'$ is in the same $W$-orbit with $\omega - \alpha_2^3 - \alpha_3^3$, it is clear that $\dim M^{\lambda'} = 1$. Set $\nu = \omega - \alpha_1^3 - \alpha_2^3 - \alpha_3^3$. Since $\lambda$ and $\nu$ are in the same $W$-orbit, we have $\dim M^\lambda = \dim M^\nu$. Put

$$w_1 = X_{-3}X_{-2}X_{-1}v^+,$$

$$w_2 = X_{-1}X_{-2}X_{-3}v^+.$$

Suppose that $w = aw_1 + bw_2 = 0$ with $a, b \in K$. Then $X_2 w = X_3 w = 0$. Several applications of Lemma 2(ii) yield that $X_{-2}X_{-1}v^+$ and $X_{-1}X_{-3}v^+ \neq 0$. Since $X_2 w = (a + 2b)X_{-1}X_{-3}v^+$ and $X_3 w = 2aX_{-2}X_{-1}v^+$ by Formula (3), we conclude that $a = b = 0$. Hence $w_1$ and $w_2$ are linearly independent and $\dim M^\lambda = \dim M^\nu > 2$. This completes the proof.

Put $\omega_0^n = \omega_{-k}^n = 0$ for $k \in \mathbb{N}$.

**Lemma 5.** *Let* $n > 1$. *Then*

1) $\varphi(2\omega_n^n) \downarrow G_{n-1} \cong 3\varphi(2\omega_{n-1}^{n-1}) \oplus 2\varphi(\omega_{n-2}^{n-1} + \omega_{n-1}^{n-1}) \oplus \varphi(2\omega_{n-2}^{n-1})$;

2) $\varphi(2\omega_{n-1}^n) \downarrow G_{n-1} \cong \varphi(2\omega_{n-1}^{n-1}) \oplus 2\varphi(\omega_{n-2}^{n-1} + \omega_{n-1}^{n-1}) \oplus 3\varphi(2\omega_{n-2}^{n-1}) \oplus 2\varphi(\omega_{n-3}^{n-1} + \omega_{n-2}^{n-1}) \oplus \varphi(\omega_{n-3}^{n-1} + \omega_{n-1}^{n-1}) \oplus \varphi(2\omega_{n-3}^{n-1})$ *for* $n > 2$;
$\varphi(2\omega_1^2) \downarrow G_1 \cong 6\varphi(0) \oplus 2\varphi(\omega_1^1) \oplus \varphi(2\omega_1^1)$;

3) $\varphi(2\omega_k^n) \downarrow G_{n-1} \cong \varphi(2\omega_k^{n-1}) \oplus 2\varphi(\omega_{k-1}^{n-1} + \omega_k^{n-1}) \oplus 3\varphi(2\omega_{k-1}^{n-1}) \oplus 2\varphi(\omega_{k-2}^{n-1} + \omega_{k-1}^{n-1}) \oplus \varphi(2\omega_{k-2}^{n-1})$ *for* $1 \leq k \leq n-2$;

4) $\varphi(\omega_{n-1}^n + \omega_n^n) \downarrow G_{n-1} \cong 2\varphi(2\omega_{n-1}^{n-1}) \oplus 4\varphi(\omega_{n-2}^{n-1} + \omega_{n-1}^{n-1}) \oplus 2\varphi(2\omega_{n-2}^{n-1}) \oplus \varphi(\omega_{n-3}^{n-1} + \omega_{n-2}^{n-1}) \oplus 2\varphi(\omega_{n-3}^{n-1} + \omega_{n-1}^{n-1})$ *for* $n > 2$;
$\varphi(\omega_1^2 + \omega_2^2) \downarrow G_1 \cong 3\varphi(0) \oplus 4\varphi(\omega_1^1) \oplus 2\varphi(2\omega_1^1)$;

5) $\varphi(\omega_k^n+\omega_{k+1}^n)\downarrow G_{n-1}\cong\varphi(\omega_k^{n-1}+\omega_{k+1}^{n-1})\oplus 2\varphi(2\omega_k^{n-1})\oplus 3\varphi(\omega_{k-1}^{n-1}+\omega_k^{n-1})\oplus 2\varphi(2\omega_{k-1}^{n-1})\oplus$ $\oplus\,\varphi(\omega_{k-2}^{n-1}+\omega_{k-1}^{n-1})$ *for* $1\le k\le n-2$;

6) $\varphi(\omega_{n-2}^n+\omega_n^n)\downarrow G_{n-1}\cong\varphi(2\omega_{n-2}^{n-1})\oplus 2\varphi(\omega_{n-2}^{n-1}+\omega_{n-1}^{n-1})\oplus 3\varphi(\omega_{n-3}^{n-1}+\omega_{n-1}^{n-1})$ *for* $n>2$; $\varphi(\omega_2^2)\downarrow G_1=2\varphi(\omega_1^1)\oplus\varphi(0)$.

*Here and below the expression* $\varphi\downarrow G_{n-1}\cong a_1\varphi_1\oplus\ldots\oplus a_k\varphi_k$ *means that such restriction is a direct sum of* $a_1$ *copies of* $\varphi_1$, $\ldots$, *and* $a_k$ *copies of* $\varphi_k$.

**Proof.** Set $\Gamma_n=SL(2n,K)$. Denote by $\theta(\mu)$ the irreducible representation of $\Gamma_n$ with highest weight $\mu$ and by $\lambda_i^n$, $1\le i\le 2n-1$, the fundamental weights of $\Gamma_n$. We assume that $\lambda_0^n=\lambda_{-k}^n=0$ for $k\in\mathbb{N}$ and $\lambda_2^1=0$. The group $G_n=Sp(2n,K)$ can be embedded into $\Gamma_n$ in a natural way. There exist maximal tori $T_1$ and $T$ in $G_n$ and $\Gamma_n$, respectively, such that $T_1=T\cap G_n$. One can assume that $\lambda_i^n\downarrow T_1=\omega_i^n$ for $1\le i\le n$ and $\lambda_i^n\downarrow T_1=\omega_{2n-i}$ for $n<i\le 2n-1$. By [14, Proposition 2.2], for $n>1$

$$\theta(2\lambda_n^n)\downarrow G_n\cong\varphi(2\omega_n^n)\oplus\varphi(\omega_{n-2}^n+\omega_n^n),\tag{4}$$

$$\theta((2-j)\lambda_k^n+j\lambda_{k+1}^n)\downarrow G_n\cong\varphi((2-j)\omega_k^n+j\omega_{k+1}^n)\tag{5}$$

for $j\in\{0,1\}$, $k\in\{0,\ldots,n-1\}$, and

$$\theta((2-j)\lambda_k^n+j\lambda_{k+1}^n)\downarrow G_n\cong\varphi((2-j)\omega_{2n-k}^n+j\omega_{2n-k-1}^n)\tag{6}$$

for $j\in\{0,1\}$, $k\in\{n,\ldots,2n-1\}$, $(2-j)\lambda_k^n+j\lambda_{k+1}^n\ne 2\lambda_n^n$. It is clear that

$$\theta(2\lambda_1^1)\downarrow G_1\cong\varphi(2\omega_1^1),\quad \theta(\lambda_1^1)\downarrow G_1\cong\varphi(\omega_1^1)$$

since $G_1=\Gamma_1$.

It follows from [14, Corollary 1.5] that

$$\theta(2\lambda_n^n)\downarrow\Gamma_{n-1}\cong\theta(2\lambda_{n-2}^{n-1})\oplus 2\theta(\lambda_{n-2}^{n-1}+\lambda_{n-1}^{n-1})\oplus 3\theta(2\lambda_{n-1}^{n-1})\oplus 2\theta(\lambda_{n-1}^{n-1}+\lambda_n^{n-1})\oplus\theta(2\lambda_n^{n-1}).$$

Combining the formulas above, one can deduce that

$$(\varphi(2\omega_n^n)\oplus\varphi(\omega_{n-2}^n+\omega_n^n))\downarrow G_{n-1}\cong 2\varphi(2\omega_{n-2}^{n-1})\oplus 4\varphi(\omega_{n-2}^{n-1}+\omega_{n-1}^{n-1})\oplus 3\varphi(2\omega_{n-1}^{n-1})\oplus 3\varphi(\omega_{n-3}^{n-1}+\omega_{n-1}^{n-1})\tag{7}$$

for $n>2$ and

$$(\varphi(2\omega_2^2)\oplus\varphi(\omega_2^2))\downarrow G_1\cong 2\varphi(0)\oplus 4\varphi(\omega_1^1)\oplus 3\varphi(2\omega_1^1).\tag{8}$$

It follows from results of [13] (Formula (1) and the proof of the main theorem) that

$$\varphi(\omega_2^2)\downarrow G_1\cong 2\varphi(\omega_1^1)\oplus\varphi(0).$$

Therefore (8) yields that

$$\varphi(2\omega_2^2)\downarrow G_1\cong\varphi(0)\oplus 2\varphi(\omega_1^1)\oplus 3\varphi(2\omega_1^1).$$

Set $U_1=M(2\omega_n^n)\downarrow G_{n-1}$ and $U_2=M(\omega_{n-2}^n+\omega_n^n)\downarrow G_{n-1}$. To prove Items 1 and 6 of the lemma for $n>2$, we shall construct explicitly certain composition factors of $U_1$ and $U_2$. By (7), $U_1$ and $U_2$ are completely reducible. Below in this proof $M=M(2\omega_n^n)$ or $M(\omega_{n-2}^n+\omega_n^n)$, $\omega$ is the highest weight of $M$. For a fixed nonnegative integer $j$ denote by $\Lambda_j$ the subset in $\Lambda(M)$ consisting of the weights of the form $\omega-j\alpha_1^n-\sum_{i=2}^n x_i\alpha_i^n$ and by $M_j$ the subspace in $M$ generated by all $M^\mu$ with $\mu\in\Lambda_j$. It is clear that $M_j$ is a direct summand of the $G_{n-1}$-module $M$ (naturally,

$M_j = 0$ for large enough $j$ ). Hence if $L \in \operatorname{Irr} M_j$ for at least $s$ different $j$, then the multiplicity of $L$ as a composition factor of $M \downarrow G_{n-1}$ is at least $s$. Set $H = G_n(1, \dots, n-2)$.

First let $\omega = 2\omega_n^n$. By Theorem 2, $M(2\omega_{n-1}^{n-1})$ is a direct summand of $U_1$. Put

$$v_1 = X_{-1}X_{-2}\dots X_{-n}v^+,$$

$$v_2 = X_{-1,2}X_{-2,2}\dots X_{-(n-1),2}X_{-n}v^+,$$

$$v_3 = X_{-1,3}X_{-2,3}\dots X_{-(n-1),3}X_{-n,2}v^+,$$

$$v_4 = X_{-1,4}X_{-2,4}\dots X_{-(n-1),4}X_{-n,2}v^+.$$

We claim that the vectors $v_i \neq 0$ and are fixed by $\mathcal{X}_l$ for $2 \leq l \leq n$. For $v_2$ and $v_4$ this follows from Lemma 3. Set $v_1' = X_{-(n-1)}X_{-n}v^+$ and $v_3' = X_{-(n-1),3}X_{-n,2}v^+$. By Lemma 2(ii), $v_1' \neq 0$. Using Formula (3) and Lemma 2(ii), we conclude that $X_nX_{n-1}v_3' = 2X_{-(n-1),2}X_{-n}v^+ \neq 0$. Hence $v_3' \neq 0$. Consider $v_1$ and $v_3$ as vectors in the $H$-modules generated by $v_1'$ and $v_3'$, respectively, and conclude that $v_1 = v_1'(1, n-2, 1)$ and $v_3 = v_3'(1, n-2, 3)$. Then Lemma 3 implies that $v_1$ and $v_3 \neq 0$ and are fixed by $\mathcal{X}_j$ for $2 \leq j \leq n-2$. We claim that $\mathcal{X}_{n-1}$ and $\mathcal{X}_n$ fix $v_1$ and $v_3$ as well. It suffices to show that $X_{n-1}v_1 = X_nv_1 = X_{n-1,s}v_3 = X_nv_3 = 0$ with $1 \leq s \leq 3$. It is clear that $\omega(v_1) + \alpha_n^n \notin \Lambda(M)$ and hence $X_nv_1 = 0$. Set

$$u_1 = X_{-(n-2)}X_{-(n-1)}X_{-n}v^+,$$

$$u_2 = X_{-(n-1),3}X_{-n,2}v^+,$$

$$u_3 = X_{-(n-2),3}u_2.$$

Taking into account the commutation relations for the operators $X_{\pm i,k}$, we reduce the question to proving the following equalities: $X_{n-1}u_1 = X_nu_2 = X_{n-1,s}u_3 = 0$. Set $\mu = \omega(u_1) + \alpha_{n-1}^n$, $\lambda = \omega(u_2) + \alpha_n^n$, and $\nu(s) = \omega(u_3) + s\alpha_{n-1}^n$. Observe that $\langle \mu, \alpha_{n-2}^n \rangle = -2$, $\langle \lambda, \alpha_{n-1}^n \rangle = -4$, and $\langle \nu(s), \alpha_{n-2}^n \rangle < -3$. Since $\Lambda(M)$ is invariant under the action of $W$, this implies that $\mu$, $\lambda$, and $\nu(s) \notin \Lambda(M)$ and yields the desired equalities. Hence $\mathcal{X}_j$ fixes $v_i$ for $1 \leq i \leq 4$ and $j > 1$.

Therefore $v_i$ generates an indecomposable $G_{n-1}$-module $V_i$ with highest weight $\delta_i = \omega_{G_{n-1}}(v_i)$. We have $\delta_1 = \delta_3 = \omega_{n-2}^{n-1} + \omega_{n-1}^{n-1}$ and $\delta_2 = \delta_4 = 2\omega_{n-1}^{n-1}$.

By Lemma 4(1), there exists a nonzero vector $v \in M$ of weight $\omega - 2\alpha_{n-1}^n - 2\alpha_n^n$ that is fixed by $\mathcal{X}_n$. Put $u = X_{-1,2}X_{-2,2}\dots X_{-(n-2),2}v$. Applying Lemma 3 to the $H$-module generated by $v$, we conclude that $u \neq 0$ and is fixed by $\mathcal{X}_j$ for $2 \leq j \leq n-2$. Observe that the weight $\omega(u) + \alpha_{n-1}^n$ lies in the same $W$-orbit with the weight $\tau = \omega(v) - 2\alpha_{n-2}^n + \alpha_{n-1}^n \notin \Lambda(M)$ (the latter is clear since $\langle \tau, \alpha_{n-2}^n \rangle = -3$ ). So $\mathcal{X}_{n-1}$ fixes $u$ as well. The group $\mathcal{X}_n$ commutes with $X_{-j,d}$ for $j < n$ and so it fixes $u$ as it fixes $v$. Therefore $u$ generates an indecomposable $G_{n-1}$-module with highest weight $\omega_{G_{n-1}}(u) = 2\omega_{n-2}^{n-1}$. It is clear that $V_i \subset M_i$ for $1 \leq i \leq 4$. Now we can conclude that $U_1$ has at least 3 composition factors isomorphic to $M(2\omega_{n-1}^{n-1})$, at least 2 factors isomorphic to $M(\omega_{n-2}^{n-1} + \omega_{n-1}^{n-1})$, and at least 1 factor isomorphic to $M(2\omega_{n-2}^{n-1})$.

Now let $\omega = \omega_{n-2}^n + \omega_n^n$. The arguments are close to those for $\omega = 2\omega_n^n$. By Theorem 2, $M(\omega_{n-3}^{n-1} + \omega_{n-1}^{n-1})$ is a direct summand of $U_2$ (for $n = 3$ it has the form $M(\omega_2^2)$ ). Set

$$w_1 = X_{-1}X_{-2}\dots X_{-(n-2)}v^+,$$

$$w_2 = X_{-1,2}X_{-2,2}\dots X_{-(n-2),2}X_{-(n-1)}X_{-n}v^+,$$

$$w_3 = X_{-1,3}X_{-2,3}\dots X_{-(n-2),3}X_{-(n-1),2}X_{-n}v^+,$$

$$w_4 = X_{-1,4}X_{-2,4}\dots X_{-(n-3),4}X_{-(n-2),3}X_{-(n-1),3}X_{-n,2}X_{-(n-1)}X_{-(n-2)}v^+$$

for $n > 3$, and

$$w_4 = X_{-(n-2),3}X_{-(n-1),3}X_{-n,2}X_{-(n-1)}X_{-(n-2)}v^+$$

for $n = 3$.

We claim that $w_i \neq 0$ and are fixed by the groups $\mathcal{X}_l$ for $2 \leq l \leq n$. For $i = 1$ and 3 this follows from Lemma 3. The proof for $w_2$ is similar to that for $v_1$ above. We conclude that $X_{-(n-1)}X_{-n}v^+ \neq 0$, then apply Lemma 3 to the relevant $H$-module and show that $w_2 \neq 0$ and is fixed by $\mathcal{X}_l$ for $2 \leq l \leq n-2$, and finally show that $\omega(w_2) + \alpha^n_{n-1}$ and $\omega(w_2) + \alpha^n_n \notin \Lambda(M)$ and hence $\mathcal{X}_{n-1}$ and $\mathcal{X}_n$ fix $w_2$ as well.

Now put $S = G_n(1, \dots, n-1)$ and $m = X_{-n,2}X_{-(n-1)}X_{-(n-2)}v^+$. By Lemma 3, $m \neq 0$ and is fixed by $\mathcal{X}_l$ for $l \neq n$. Next, we regard $w_4$ as a vector in the $S$-module generated by $m$ and observe that $w_4 = m(1, n-1, 3)$. Then Lemma 3 implies that $w_4 \neq 0$ and is fixed by $\mathcal{X}_l$ for $2 \leq l \leq n-1$. Observe that the weight $\omega(w_4) + \alpha^n_n$ lies in the same $W$-orbit with the weight $\gamma = \omega - \alpha^n_{n-2} - 4\alpha^n_{n-1} - \alpha^n_n$. Since $\langle \gamma, \alpha^n_{n-1} \rangle = -5 < -4$, we deduce that $\gamma \notin \Lambda(M)$. Hence $X_n w_4 = 0$ and so $\mathcal{X}_n$ fixes $w_4$.

Hence the vectors $w_i$ generate indecomposable $G_{n-1}$-modules with highest weights $\mu_i = \omega_{G_{n-1}}(w_i)$. We have $\mu_1 = \mu_3 = \omega^{n-1}_{n-2} + \omega^{n-1}_{n-1}$, $\mu_2 = 2\omega^{n-1}_{n-2}$, $\mu_4 = \omega^{n-1}_{n-3} + \omega^{n-1}_{n-1}$ for $n > 3$, and $\mu_4 = \omega^{n-1}_{n-1}$ for $n = 3$.

By Lemma 4 (2), there exists a nonzero vector $w$ of weight $\omega - 2\alpha^n_{n-2} - 2\alpha^n_{n-1} - \alpha^n_n$ that is fixed by the groups $\mathcal{X}_{n-1}$ and $\mathcal{X}_n$. Put $t = X_{-1,2}X_{-2,2}\dots X_{-(n-3),2}w$ for $n > 3$ and $t = w$ otherwise. We claim that $t \neq 0$ and is fixed by $\mathcal{X}_l$ for $2 \leq l \leq n$. For $n = 3$ this is trivial. Let $n > 3$. Here the arguments are similar to those for the vector $u$ considered earlier in the case where $\omega = 2\omega^n_n$. First we apply Lemma 3 to the $G_n(1, \dots, n-3)$-module generated by $w$ and show that $t \neq 0$ and is fixed by the groups $\mathcal{X}_l$ for $2 \leq l \leq n-3$. The weight $\omega(t) + \alpha^n_{n-2}$ lies in the same $W$-orbit with $\chi = \omega - 2\alpha^n_{n-3} - \alpha^n_{n-2} - 2\alpha^n_{n-1} - \alpha^n_n$. As $\langle \chi, \alpha^n_{n-3} \rangle = -3 < -2$, the weight $\chi \notin \Lambda(M)$. Hence $X_{n-2}t = 0$ and $\mathcal{X}_{n-2}$ fixes $t$. Arguing as for the vector $u$ and for the group $\mathcal{X}_n$ earlier, we conclude that $\mathcal{X}_{n-1}$ and $\mathcal{X}_n$ fix $t$ as well. Therefore $t$ generates an indecomposable $G_{n-1}$-module with highest weight $\omega_{G_{n-1}}(t) = \omega^{n-1}_{n-3} + \omega^{n-1}_{n-1}$.

Since the vectors $w_i \in M_i$ for $1 \leq i \leq 4$ and $t \in M_2$, we can conclude that $U_2$ has at least 3 composition factors isomorphic to $M(\omega^{n-1}_{n-3} + \omega^{n-1}_{n-1})$, at least 2 composition factors isomorphic to $M(\omega^{n-1}_{n-2} + \omega^{n-1}_{n-1})$, and at least 1 composition factor isomorphic to $M(\omega^{n-1}_{n-2})$.

Now Formula (7) implies that

$$M(2\omega^n_n) \downarrow G_{n-1} \cong 3M(2\omega^{n-1}_{n-1}) \oplus 2M(\omega^{n-1}_{n-2} + \omega^{n-1}_{n-1}) \oplus M(2\omega^{n-1}_{n-2});$$

and

$$M(\omega^n_{n-2} + \omega^n_n) \downarrow G_{n-1} \cong M(2\omega^{n-1}_{n-2}) \oplus 2M(\omega^{n-1}_{n-2} + \omega^{n-1}_{n-1}) \oplus 3M(\omega^{n-1}_{n-3} + \omega^{n-1}_{n-1}).$$

This proves Items (1) and (6) of the lemma.

For $k < n$ it follows from [14, Corollary 1.5] that

$$\theta(2\lambda^n_k) \downarrow \Gamma_{n-1} \cong \theta(2\lambda^{n-1}_{k-2}) \oplus 2\theta(\lambda^{n-1}_{k-2} + \lambda^{n-1}_{k-1}) \oplus 3\theta(2\lambda^{n-1}_{k-1}) \oplus 2\theta(\lambda^{n-1}_{k-1} + \lambda^{n-1}_k) \oplus \theta(2\lambda^{n-1}_k)$$

and

$$\theta(\lambda^n_k + \lambda^n_{k+1}) \downarrow \Gamma_{n-1} \cong$$

$$\cong \theta(\lambda^{n-1}_{k-2} + \lambda^{n-1}_{k-1}) \oplus 2\theta(2\lambda^{n-1}_{k-1}) \oplus 3\theta(\lambda^{n-1}_{k-1} + \lambda^{n-1}_k) \oplus 2\theta(2\lambda^{n-1}_k) \oplus \theta(\lambda^{n-1}_k + \lambda^{n-1}_{k+1}).$$

Using Formulas (4)–(6), we prove Items (2)–(5) of the lemma.

We have proved that the restrictions of the modules mentioned in Lemma 5 to $G_{n-1}$ are completely reducible. For $\varphi(\omega^n_1)$, $\varphi(2\omega^n_k)$ with $1 \leq k \leq n$, and $\varphi(\omega^n_j + \omega^n_{j+1})$ with $1 \leq j \leq n-1$

this was proved earlier in [14]. The complete reducibility of the restriction $\varphi(\omega^n_{n-2}+\omega^n_n)\downarrow G_{n-1}$ is a new result.

**Proof of Theorem 1.** By Theorem 2, $\varphi(\omega^n_{n-i}+\omega^n_{n-j})\in \mathrm{Irr}(\varphi(\omega^{n+1}_{n+1-i}+\omega^{n+1}_{n+1-j})\downarrow G_n)$ for $j\le i\le n$. Hence the inductive system $\langle\varphi(\omega^n_{n-i}+\omega^n_{n-j})|n\ge i\rangle$ is correctly defined for fixed $i$ and $j$ with $j\le i$.

Put

$$\mathcal{F}_n=\{0,\varphi(\omega^n_1),\varphi(\omega^n_{n-2}+\omega^n_n),\varphi(2\omega^n_k),\ 1\le k\le n,\ \varphi(\omega^n_j+\omega^n_{j+1}),\ 1\le j\le n-1\}.$$

First observe that $\mathcal{F}=\{\mathcal{F}_n|n\in\mathbb{N}\}$ is an inductive system by Lemma 5 and Theorem 2.

For $m>n$ set $\mathrm{Irr}^m_n=\mathrm{Irr}(\varphi(2\omega^m_m)\downarrow G_n)$. Let $n=k+l>k$. We claim that $\varphi(2\omega^n_k)\in\mathrm{Irr}^{n+l}_n$. Use induction on $l$. For $l=1$ the claim follows directly from Lemma 5(1). Now assume that $l>1$ and the claim holds for $l-1$. Then $\varphi(2\omega^{n+1}_{k+1})\in\mathrm{Irr}^{n+l}_{n+1}$. Items (2) and (3) of Lemma 5 yield that $\varphi(2\omega^n_n)\in\mathrm{Irr}(\varphi(2\omega^{n+1}_{k+1})\downarrow G_n)$ and complete the proof of the claim.

Similar arguments imply that $\varphi(\omega^n_k+\omega^n_{k+1})\in\mathrm{Irr}^{n+l}_n$ for $n=k+l$ and $l>0$. Here we apply Lemma 5(1) for $l=1$, then use induction on $l$ and apply Items (4) and (5) of that lemma.

We have shown above that $\varphi(2\omega^{n+1}_1)\in\mathrm{Irr}^{2n}_{n+1}$. Then Items (2) and (3) of Lemma 5 imply that $\varphi(\omega^n_1)\in\mathrm{Irr}^{2n}_n$.

Recall that $\varphi(\omega^n_{n-2}+\omega^n_n)\in\mathrm{Irr}^{n+1}_n$ for $n>2$ by Lemma 5(6). It is well known that

$$\varphi(0)\in\mathrm{Irr}(\varphi(\omega^{n+1}_1)\downarrow G_n).$$

This completes that proof since $\langle\varphi(2\omega^n_n)|n>0\rangle\subset\mathcal{F}$.

**Proof of Corollary 1.** This follows from Theorem 1 and the fact that $\varphi(2\omega^{n-1}_{n-1})$ is a composition factor of the restrictions $\varphi(2\omega^n_{n-1})\downarrow G_{n-1}$ and $\varphi(\omega^n_{n-1}+\omega^n_n)\downarrow G_{n-1}$ by Lemma 5, Items (2) and (3).

The symbols $\varepsilon^n_j$, $1\le j\le n$, denote the weights of the standard $G_n$-modules defined in [5, Ch. VIII, § 13].

**Proof of Lemma 1.** It is well known that for $l\ge j\ge n$ the subgroup

$$\Gamma=G_l(\alpha^l_{j-n+1},\dots,\alpha^l_{j-1},2\varepsilon^l_j)$$

is conjugate to $G_n$. Hence $\mathrm{Irr}(\varphi\downarrow\Gamma)=\mathrm{Irr}(\varphi\downarrow G_n)$ for every representation $\varphi$ of $G_l$.

First suppose that there exists an integer $k$ such that for any $m>k$ the set $\Phi_m$ contains a representation $\varphi_m=\varphi(\omega^m_{m-j}+\omega^m_{m-i})$ with $i\le j\le k$. Fix such $k$. For $n>0$ and $l\ge n+k$ put

$$H_n=G_l(\alpha^l_{l-k-n+1},\dots,\alpha^l_{l-k-1},2\varepsilon^l_{l-k}).$$

Observe that $\omega(\varphi_l)\downarrow H_n=2\omega^n_n$. Hence $\varphi(2\omega^n_n)\in\mathrm{Irr}(\varphi_l\downarrow H_n)\subset\Phi_n$ for any $n>0$.

Now assume that there is no such $k$.

Suppose that for every $n$ there exists $m$ such that $\Phi_m$ contains a representation

$$\psi_m=\varphi(\omega^m_i+\omega^m_j)$$

with $i\le j$ and $m-j\ge n$. In this case $\varphi(\omega^{m-j+1}_1)$ or $\varphi(2\omega^{m-j+1}_1)\in\mathrm{Irr}(\psi_m\downarrow G_{m-j+1})$ by Theorem 2. Lemma 5, Items (2) and (3), yields that $\varphi(\omega^{m-j}_1)\in\Phi_{m-j}$ and hence $\varphi(\omega^n_1)\in\Phi_n$.

Finally suppose that for some integer $l$ the difference $m-j<l$ for all $m$ and all

$$\varphi(\omega^m_i+\omega^m_j)\in\Phi_m.$$

Our previous assumptions imply that for every $n$ there exist $t$ and a representation

$$\chi=\varphi(\omega^t_i+\omega^t_j)\in\Phi_t$$

with $j-i\geq n$. Put

$$H_n = G_t(\alpha^t_{j-n+1},\dots,\alpha^t_{j-1},2\varepsilon^t_j).$$

Then $\omega(\chi)\downarrow H_n=\omega^n_n$ and hence $\varphi(\omega^n_n)\in \mathrm{Irr}(\chi\downarrow H_n)\subset\Phi_n$. The lemma is proved.

**Proof of Corollary 2.** The assertion follows from Theorem 1 and Lemma 1.

It is clear that the inductive system generated by the standard representations contains the trivial subsystem.

The following lemma implies that the inductive system $\langle\varphi(2\omega^n_n)|n>0\rangle$ in characteristic 3 strongly differs from the similar system in characteristic 0.

**Lemma 6.** *Let* $n>4$, $G^{\mathbb{C}}_n=C_n(\mathbb{C})$, *and* $M(\omega)^{\mathbb{C}}$ *be the irreducible* $G^{\mathbb{C}}_n$*-module with highest weight* $\omega$. *Then*

1) $M(2\omega^n_n)^{\mathbb{C}}\downarrow G^{\mathbb{C}}_{n-1}\cong 3M(2\omega^{n-1}_{n-1})^{\mathbb{C}}\oplus 2M(\omega^{n-1}_{n-2}+\omega^{n-1}_{n-1})^{\mathbb{C}}\oplus M(2\omega^{n-1}_{n-2})^{\mathbb{C}}$;

2) $M(2\omega^n_{n-1})^{\mathbb{C}}\downarrow G^{\mathbb{C}}_{n-1}\cong M(2\omega^{n-1}_{n-1})^{\mathbb{C}}\oplus 2M(\omega^{n-1}_{n-2}+\omega^{n-1}_{n-1})^{\mathbb{C}}\oplus 3M(2\omega^{n-1}_{n-2})^{\mathbb{C}}\oplus 2M(\omega^{n-1}_{n-3}+\omega^{n-1}_{n-2})^{\mathbb{C}}\oplus$ $\oplus M(\omega^{n-1}_{n-3}+\omega^{n-1}_{n-1})^{\mathbb{C}}\oplus M(2\omega^{n-1}_{n-3})^{\mathbb{C}}$;

3) $M(2\omega^n_k)^{\mathbb{C}}\downarrow G^{\mathbb{C}}_{n-1}\cong M(2\omega^{n-1}_k)^{\mathbb{C}}\oplus 2M(\omega^{n-1}_{k-1}+\omega^{n-1}_k)^{\mathbb{C}}\oplus 3M(2\omega^{n-1}_{k-1})^{\mathbb{C}}\oplus 2M(\omega^{n-1}_{k-2}+\omega^{n-1}_{k-1})^{\mathbb{C}}\oplus$ $\oplus M(2\omega^{n-1}_{k-2})^{\mathbb{C}}$ *for* $1\leq k\leq n-2$;

4) $M(\omega^n_{n-1}+\omega^n_n)^{\mathbb{C}}\downarrow G^{\mathbb{C}}_{n-1}\cong 2M(2\omega^{n-1}_{n-1})^{\mathbb{C}}\oplus 4M(\omega^{n-1}_{n-2}+\omega^{n-1}_{n-1})^{\mathbb{C}}\oplus 2M(2\omega^{n-1}_{n-2})^{\mathbb{C}}\oplus M(\omega^{n-1}_{n-3}+$ $+\omega^{n-1}_{n-2})^{\mathbb{C}}\oplus 2M(\omega^{n-1}_{n-3}+\omega^{n-1}_{n-1})^{\mathbb{C}}$;

5) $M(\omega^n_k+\omega^n_{k+1})^{\mathbb{C}}\downarrow G^{\mathbb{C}}_{n-1}\cong M(\omega^{n-1}_k+\omega^{n-1}_{k+1})^{\mathbb{C}}\oplus 2M(2\omega^{n-1}_k)^{\mathbb{C}}\oplus 4M(\omega^{n-1}_{k-1}+\omega^{n-1}_k)^{\mathbb{C}}\oplus$ $\oplus 2M(2\omega^{n-1}_{k-1})^{\mathbb{C}}\oplus M(\omega^{n-1}_{k-2}+\omega^{n-1}_{k-1})^{\mathbb{C}}\oplus 2M(\omega^{n-1}_{k-1}+\omega^{n-1}_{k+1})^{\mathbb{C}}\oplus M(\omega^{n-1}_{k-2}+\omega^{n-1}_{k+1})^{\mathbb{C}}\oplus 2M(\omega^{n-1}_{k-2}+$ $+\omega^{n-1}_k)^{\mathbb{C}}$ *for* $1\leq k\leq n-2$;

6) $M(\omega^n_{n-2}+\omega^n_n)^{\mathbb{C}}\downarrow G^{\mathbb{C}}_{n-1}\cong M(2\omega^{n-1}_{n-2})^{\mathbb{C}}\oplus 2M(\omega^{n-1}_{n-2}+\omega^{n-1}_{n-1})^{\mathbb{C}}\oplus 4M(\omega^{n-1}_{n-3}+\omega^{n-1}_{n-1})^{\mathbb{C}}\oplus$ $\oplus 2M(\omega^{n-1}_{n-4}+\omega^{n-1}_{n-1})^{\mathbb{C}}\oplus 2M(\omega^{n-1}_{n-3}+\omega^{n-1}_{n-2})^{\mathbb{C}}\oplus M(\omega^{n-1}_{n-4}+\omega^{n-1}_{n-2})^{\mathbb{C}}$.

**Proof.** Let $\omega$ be the same as in the left hand side of the formulas in Items 1-6 of the lemma. Apply the classical branching rules [6, 8.1.5]. To state them, we need some notation. When we write weights of $G_n(\mathbb{C})$ in terms of $\varepsilon^n_i$, $(b_1,\dots,b_n)$ means $\sum_{i=1}^n b_i\varepsilon^n_i$. The set of all weights of $G^{\mathbb{C}}_n$ is equal to

$$\{(b_1,\dots,b_n)|b_1\geq b_2\geq\dots\geq b_n\geq 0,\ b_i\in\mathbb{Z},\ 1\leq i\leq n\}$$

and $(b_1,\dots,b_n)>(c_1,\dots,c_{n-1})$ means that all $b_j\in\mathbb{Z}$ and $b_1\geq c_1\geq b_2\geq b_{n-1}\geq c_{n-1}\geq b_n$. It is well known that $\omega^n_i=\varepsilon^n_1+\dots+\varepsilon^n_i$ for $i=1,\dots,n$. Hence for $1\leq i\leq j\leq n$ the weight $\omega=\omega^n_i+\omega^n_j=(2,\dots,2,1,\dots,1,0,\dots,0)$ where the sequence in the right hand side contains $i$ twos and $j-i$ ones. Denote this sequence by $b=b(\omega)$.

The branching rules imply that

$$M(\omega)^{\mathbb{C}}\downarrow G^{\mathbb{C}}_{n-1}\cong\oplus_{d<c}\oplus_{c<(b,0)}M(d)^{\mathbb{C}}.$$

Writing down this formula for all relevant $\omega$, we obtain the assertion of the lemma.

Note that by [13, Theorem, Item C] for the natural embeddings of groups of type $C_n$ in characteristic 5

$$\langle\varphi(2\omega^n_n)|n>0\rangle_n=\{\varphi(2\omega^n_n),\varphi(\omega^n_{n-1}+\omega^n_n)\}.$$

Therefore, such inductive system contains no trivial or standard representations for groups of rank greater than 1 and does not contain $\mathcal{R}^{p-1}$.

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**A. A. Osinovskaya, I. D. Suprunenko**


**Inductive systems of representations with small highest weights for natural embeddings of symplectic groups**

### Summary


For natural embeddings of symplectic groups, inductive systems of irreducible representations where the maximum of the highest weight value on the maximal root is equal to $2$ are studied. For such embeddings of algebraic groups of type $C_n$ in characteristic $3$, the inductive system of representations generated by irreducible representations with highest weight $2\omega_n$ is determined. It is proved that any inductive system of representations of such groups consisting of representations with the value of the highest weight on the maximal root at most $2$ and containing representations with such value equal to $2$ contains the subsystem generated by the standard representations or the subsystem generated by the representations with highest weight $\omega_n$. For algebraic groups of type $C_n$ in characteristic $3$, the restrictions of certain irreducible modules to subsystem subgroups of type $C_{n-1}$ are described.