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REPRESENTATIONS OF NON–COMMUTATIVE BANACH ALGEBRAS BY CONTINUOUS FUNCTIONS

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Abstract. A main result of the Gelfand theory states that any commutative semi-simple Banach algebra is isomorphic to an algebra of continuous complex-valued functions on a Hausdorff compact. In the present paper we extend this result to a class of non-commutative algebras. We introduce and compare several concepts of continuity of functions taking values in Banach algebras which differ from point to point and show that they, in a certain sense, coincide. Finally, we give applications to algebras of singular integral and convolution operators and to algebras of approximation methods.

§ 1. Introduction

A standard procedure for studying operator algebras and their images in the Calkin algebra is the *localization over central subalgebras* (which is also referred to as Douglas' local principle or Dauns-Hofmann theorem in the C^*-case and as Allan's local principle in its general setting):

Let $\mathfrak{A}$ be a unital Banach algebra and $\mathfrak{B}$ be a central subalgebra of $\mathfrak{A}$ containing the unit element. Given x belonging to the maximal ideal space $M(\mathfrak{B})$ of the commutative Banach algebra $\mathfrak{B}$ we denote by I_x the smallest closed two-sided ideal of $\mathfrak{A}$ containing x, by $\mathfrak{A}_x$ the quotient algebra $\mathfrak{A}/I_x$, and by Φ_x the canonical homomorphism from $\mathfrak{A}$ onto $\mathfrak{A}_x$. Then

(a) *if $a \in \mathfrak{A}$ then a is invertible in $\mathfrak{A}$ if and only if the cosets $\Phi_x(a)$ are invertible in $\mathfrak{A}_x$ for all $x \in M(\mathfrak{B})$.*

(b) *the mapping $M(\mathfrak{B}) \longrightarrow \mathbb{R}^+$, $x \mapsto \|\Phi_x(a)\|$ is upper semi-continuous (usc.) for all $a \in \mathfrak{A}$.*

(See [A], [BS], [D], [K], for proofs and applications.)

Given $a \in \mathfrak{A}$ we define a function $\Phi(a)$ on $M(\mathfrak{B})$ taking at $x \in M(\mathfrak{B})$ a value in $\mathfrak{A}_x$ by $\Phi(a)(x) = \Phi_x(a)$. Then the above localization principle can be restated as follows:

If $a \in \mathfrak{A}$ then a is invertible in $\mathfrak{A}$ if and only if $\Phi(a)$ is invertible.

This justifies to interprete $\Phi(\mathfrak{A})$ as a symbol algebra for $\mathfrak{A}$, $\Phi : \mathfrak{A} \longrightarrow \Phi(\mathfrak{A})$ as the symbol map, and $\Phi(a)$ as a symbol of a. If, in particular, $\bigcap_{x \in M(\mathfrak{B})} I_x = \{0\}$ then the symbol map is even an isomorphism. Thus, in this case, a complete description of the symbol algebra would yield a complete description of the original algebra, and this will just be the main object of our paper. More concretely, we will introduce and compare several concepts of continuity of functions taking values in certain Banach algebras which will allow to characterize the symbol algebra $\Phi(\mathfrak{A})$ as the algebra of *all continuous* functions on $M(\mathfrak{B})$. All concepts we shall propose are farreaching generalizations of the usual notion of continuity which preserve the most striking algebraic (to form a Banach algebra) and density (Stone-Weierstrass theorems) properties of continuous functions. Our first concept is a generalization of Simonenko's notion of upper semi-continuity ([SC]) which goes back to the authors ([RS 1,2]). The second one bases on continuous cross sections on vector bundles and is standard. Our third aproach is possibly new. Its main idea is to consider functions possessing one-sided limits at each point and then to compare these limits with the value of the function at the given point.

After this we shall illustrate our results by some well-known examples taken from operator theory (one- and multi dimensional singular integral operators, convolution operators) and from numerical analysis. For some background information about the concrete material needed in the concluding four sections we refer to [GK], [K], [MP], [SC] and [RS 1,2] for singular integral operators, to [BS] and [RS 1] for Fourier convolution operators, to [BS] and [HRS] for Toeplitz operators, and to [HS], [PS] and [HRS] for Banach algebra techniques in numerical analysis. In the cited papers and monographs one can also find remarks on the history of the topic and references to original works.

§ 2. Continuity via usc-subalgebras

Throughout this paper, let X be a Hausdorff compact and $(A_x)_{x\in X}$ be a family of Banach algebras with unit elements e_x, zero elements 0_x, and provided with norms $\|\|_x$, respectively. The set of all bounded functions on X which take at $x \in X$ a value in A_x will be denoted by $V(X, A_x)$. Provided with pointwise operations and with the norm

$$\|f\|_{V(X,A_x)} = \sup_{x\in X} \|f(x)\|_x,$$

$V(X, A_x)$ becomes a Banach algebra.

A subset E of $V(X, A_x)$ is called *maximal* if for all $x \in X$ and $a_x \in A_x$ there is a function $f \in E$ such that $f(x) = a_x$. A subset E of $V(X, A_x)$ is said to be *upper semi-continuous* (usc) if the function $X \longrightarrow \mathbb{R}^+$, $x \mapsto \|f(x)\|_x$ is usc for all $f \in E$.

Definition. *Let E be a maximal and usc (but not necessarily closed) subalgebra of $V(X, A_x)$. We say that the function $f \in V(X, A_x)$ is E-continuous, if for all $x_0 \in X$, for all $\varepsilon > 0$, and for all functions $g \in E$ with $g(x_0) = f(x_0)$ there is a neighborhood $U = U(x_0, f, g, \varepsilon)$ of x_0 such that $\|f(x) - g(x)\|_x < \varepsilon$ for all $x \in U$. The set of all E-continuous functions will be denoted by $C_{\text{usc}}(E)$.*

The following can be shown straightforwardly:

Proposition 1. (a) *$C_{\text{usc}}(E)$ is a closed subalgebra of $V(X, A_x)$.*

(b) *$E \subseteq C_{\text{usc}}(E)$.*

(c) *The algebra $C_{\text{usc}}(E)$ is maximal and usc.*

To justify our notations observe that if $A_x = A$ for all $x \in X$ and if E is the algebra of all constant functions from X into A then $C_{\text{usc}}(E)$ coincides with the algebra of all continuous functions from X into A (in the common sense).

For the next proposition which gives another description of the algebra $C_{\text{usc}}(E)$ we let $C(X) \subseteq V(X, A_x)$ stand for the set of all functions of the form $x \mapsto f(x)e_x$ where f runs through the continuous complex-valued functions on X. Obviously, $C(X)$ is a closed subalgebra of $V(X, A_x)$ which contains the unit element $x \mapsto e_x$ of $V(X, A_x)$.

Proposition 2. *Let E be a maximal, usc, and unital subalgebra of $V(X, A_x)$. Then*

(a) *$C(X) \subseteq C_{\text{usc}}(E)$.*

(b) *$C_{\text{usc}}(E)$ is the smallest closed subalgebra of $V(X, A_x)$ containing the algebras E and $C(X)$.*

Proof. (a) Let $f \in C(X)$, let g be the complex-valued continuous function determined by $f(x) = g(x)e_x$, and choose a function $h \in E$ such that $h(x_0) = f(x_0) = g(x_0)e_{x_0}$. Then

$$\|f(x) - h(x)\|_x \leqslant \|g(x)e_x - g(x_0)e_x\|_x + \|g(x_0)e_x - h(x)\|_x \\ \leqslant |g(x) - g(x_0)| + \|g(x_0)e_x - h(x)\|_x$$

which becomes as small as desired since the function $x \mapsto g(x)$ is continuous and since the function $x \mapsto g(x_0)e_x - h(x)$ is in E and, hence, the function $x \mapsto \|g(x_0)e_x - h(x)\|_x$ is usc.

(b) Let B denote the smallest closed subalgebra of $V(X, A_x)$ containing E and $C(X)$. Since $E \subseteq C_{\text{usc}}(E)$ and $C(X) \subseteq C_{\text{usc}}(E)$(cp. Prop. 1(b) and 2(a)) we get the inclusion $B \subseteq C_{\text{usc}}(E)$. For the reverse inclusion take a function $a \in c_{\text{usc}}(E)$. By the definition, for any point $x_0 \in X$ there is a function $a_{x_0} \in E$ with $a(x_0) = a_{x_0}(x_0)$ and, given $\varepsilon > 0$, $\|a(x) - a_{x_0}\|_x < \varepsilon$ for all x belonging to a neighborhood $U(x_0)$ of x_0. Choose a finite number of such neighborhoods $U(x_1), \dots, U(x_n)$ which cover X, and let $1 = g_1 + \dots + g_n$, $g_i \geqslant 0$, be a continuous partition of unity subordinate to this covering. Finally, set $a_\varepsilon : g_1 a_{x_1} + \dots + g_n a_{x_n}$. Then

$$\begin{aligned} \|a - a_\varepsilon\| &= \sup_{x\in X} \|a(x) - \sum_{i=1}^{n} g_i(x) a_{x_i}(x)\|_x \\ &= \sup_{x\in X} \| \sum_{i=1}^{n} g_i(x)(a(x) - a_{x_i}(x)\|_x \\ &= \sup_{x\in X} \| \sum_{i: x\in U(x_i)} g_i(x)(a(x) - a_{x_i}(x))\|_x \\ &\leqq \sup_{x\in X} \sum_{i: x\in U(x_i)} g_i(x) \cdot \varepsilon = \varepsilon, \end{aligned}$$

and since $a_\varepsilon \in B$ and B is closed we get $a \in B$ as desired. •

In particular this proposition shows that if $C(X)$ belongs to the closure $\operatorname{clos} E$ of E in $V(X, A_x)$ then $C_{\text{usc}}(E) = \operatorname{clos} E$. A sufficient condition for the inclusion $C(X) \subseteq \operatorname{clos} E$ is given by the following *Stone-Weierstrass theorem*. For the special case when $A_x = A_y$ for all x, y, this theorlem was established in [K, Th. 25.2]. The proof given here leans against Krupnik's one.

Theorem 1. *Assume that any algebra A_x contains a unital c^*-subalgebra B_x, and let B stand for the class of all functions $f \in \operatorname{clos} E$ for which $f(x) \in B_x$ for all $x \in X$, and for which the function $f^* : x \mapsto f^*(x)$ belongs to $\operatorname{clos} E$, too. If B separates the points of X then $C(X) \subseteq \operatorname{clos} E$ (and, thus, $C_{\text{usc}}(E) = \operatorname{clos} E$).*

Remember that an algebra B separates the points of X if for any two distinct points x, $y \in X$ there is a function $b \in B$ such that $b(x) = e_x$ and $b(y) = 0_y$.

Proof. We shall show that $C(X) \subseteq B$. At first we claim that if $x_1, x_2 \in X$ and $x_1 \neq x_2$ then there is a function $c \in B$ satisfying $c(x_1) = e_{x_1}$ and $c(x) = 0_x$ for all x in a certain neighborhood of x_2.

Since B separates the points of X there is a function b in B such that $b(x_1) = e_{x_1}$ and $b(x_2) = 0_{x_2}$. Without loss we can assume that b belongs to the conus B^+ of the positive elements of B (otherwise replace b by bb^*). Let $\operatorname{alg} b$ stand for the smallest C^*-subalgebra of B which contains b. Since the spectrum $\sigma(b)$ of b lies in the interval $[0, \|b\|]$ and since, by the Gelfand-Naimark theorem, the algebra $\operatorname{alg} b$ is isometrically isomorphic to the algebra $C_{\mathbb{C}}(\sigma(b))$ of all complex-valued continuous functions on $\sigma(b)$, to each function $f \in C_{\mathbb{C}}[0, \|b\|]$

it can be associated a function $f(b) \in \operatorname{alg} b$ whose image in $C_{\mathbb{C}}(\sigma(b))$ coincides with the restriction of f onto $\sigma(b)$.

Take a number a between 0 and 1 and choose a non-negative function $f \in C_{\mathbb{C}}[0, \|b\|]$ such that $f(s) = 0$ if $0 \leqslant s \leqslant a$ and $f(s) = 1$ if $1 \leqslant s \leqslant b$. Then the function $f(b)$ is just the function c we looked for. Indeed, since $b \in B$ we have $f(b) \in B$. Further, define an open neighborhood U of x_2 by $U = \{x \in X : \|b(x)\|_x < a\}$, and let $x \in U$. Then

$$\begin{aligned}\|c(x)\|_x^2 &= \|f(b)(x)\|_x^2 = \|f(b(x))\|_x^2 \\ &= \max_{\lambda \in \sigma(f(b(x)) f(b(x))^*)} \lambda = \max_{y \in \sigma(b(x))} |f(y)|^2 \\ &\leqslant \max_{y \in [0,a]} |f(y)|^2 = 0.\end{aligned}$$

Hence, $c(x) = 0_x$ for $x \in U$, and similarly one shows that $c(x_1) = e_{x_1}$ which proves our claim.

In the second step we shall show that, given a compact subset K of X and an $x_1 \in X \setminus K$, there is a function $d \in B^+$ with $d(x_1) = e_{x_1}$ and $d(x) = 0_x$ for all $x \in K$.

By our above claim, we can associate to any point $y \in K$ an open neighborhood $U_y \subseteq X \setminus \{x_1\}$ and a function $c_y \in B$ such that $c(x_1) = e_{x_1}$ and $c_y(x) = 0_x$ for all $x \in U_y$. Choose a finite covering $U_{y_1}, \ldots, U_{y_k}$ of K. Then it is easy to see that the function $d = c_{y_1} \ldots c_{y_k} c_{y_k}^* \ldots c_{y_1}^*$ has the desired properties.

For the third step, let K and L be disjoint and compact subsets of X. We will verify that there is a function $f \in B^+$ such that $f(x) = 0_x$ for all $x \in K$ and $f(x) \geqslant e_x$ for all $x \in L$. Indeed, if $y \in L$ then, by the second step, there is a function $d_y \in B^+$ such that $d_y(y) = e_y$ and $d_y(x) = 0_x$ for all $x \in L$. From the usc and from $d_y \in B^+$ one concludes that $d_y(x) \geqslant e_x/2$ for all x belonging to a certain neighborhood U_y of y. Now choose a finite covering $U_{y_1}, \ldots, U_{yl}$ of L and set $f = 2(d_{y_1} + \cdots + d_{yl})$ to get the wanted function.

Finally, we shall show that the function $x \mapsto g(x)e_x$ is in B whenever r is a continuous real-valued function on X. Since any complex-valued function is a complex linear combination of real-valued functions, this would prove our theorem.

Let $X = U_1 \cup \cdots \cup U_n$ be an open covering of X wuch that the variation of g on U_i does not exceed a previously given $\varepsilon > 0$, and choose another open covering $X = V_1 \cup \cdots \cup V_n$ with clos $V_i \subseteq U_i$. By the third step, one finds functions f_i satisfying $f_i(x) \geqslant e_x$ for all $x \in V_i$ and $f_i(x) = 0_x$ for all $x \in X \setminus U_i$. Put $f = f_1 + \cdots + f_n$. Because $f(x) \geqslant e_x$ for all $x \in X$, there is a uniquely determined element $\tilde{f} \in B^+$ for which $(\tilde{f}^2 f)(x) = (f \tilde{f}^2)(x) = e_x$. Now choose arbitrary points $u_i \in U_i$ and set $g_i = g(u_i)$ and $h_i = \tilde{f} f_i \tilde{f}$. Taking into account that $(\sum_i h_i)(x) = (\tilde{f} f \tilde{f})(x) = e_x$ we arrive at

$$g(x)e_x - \sum_i g_i h_i(x) = \sum_i (g(x) - g_i) h_i(x)$$

whence (remember that $h_i \in B^+$)

$$-\varepsilon \sum_i h_i(x) \leqslant \sum_i (g(x) - g_i) h_i(x) \leqslant \varepsilon \sum_i h_i(x)$$

or, equivalently,

$$-\varepsilon e_x \leqslant \sum_i (g(x) - g_i) h_i(x) \leqslant \varepsilon e_x,$$

which on its hand implies that $\| \sum_i (g(x) - g_i) h_i(x) \|_x < \varepsilon$ and, thus, $\|g - \sum_i g_i h_i\| < \varepsilon$. Since $h_i \in B$ we have $g \in B$, and we are done. •

Now we return to the localization principle from the introduction. The symbol map Φ yields a continuous homomorphism into the algebra $V(M(\mathfrak{B}), \mathfrak{A}_x)$, and the symbol algebra $\Phi(\mathfrak{A})$ is usc and maximal in $V(M(\mathfrak{B}), \mathfrak{A}_x)$. Further, the algebra $\Phi(\mathfrak{B})$ is contained in $C(M(\mathfrak{B}))$ and separates the points of $M(\mathfrak{B})$. Thus, Proposition 2 and Theorem 1 give the following characterization of the symbol algebra $\Phi(\mathfrak{A})$ (notice the analogy between this proposition and the corresponding assertion from the Gelfand theory of commutative Banach algebras which states that the algebra of the Gelfand transforms is contained in the algebra of all continuous complex-valued functions on the maximal ideal space):

Proposition 3. *The closure of $\Phi(\mathfrak{A})$ in $V(M(\mathfrak{B}), \mathfrak{A}_x)$ is contained in the algebra $C_{\text{usc}}(\Phi(\mathfrak{A}))$.*

Assume moreover, that the algebra $\mathfrak{A}$ is KMS with respect to $\mathfrak{B}$, i.e. $\|a\| = \sup_{x \in M(\mathfrak{B})} \|\Phi_x(a)\|$ for all $a \in \mathfrak{A}$ (compare [BKS] for details). Then the symbol map is an isometrical isomorphism, and the symbol algebra $\Phi(\mathfrak{A})$ is closed in $V(M(\mathfrak{B}), \mathfrak{A}_x)$. Summarizing this we obtain an analogue of the Gelfand-Naimark theorem for commutative C^*-algebras:

Theorem 2. *If the algebra $\mathfrak{A}$ is KMS with respect to $\mathfrak{B}$ then $\Phi(\mathfrak{A}) = C_{\text{usc}}(\Phi(\mathfrak{A}))$, and the algebras $\mathfrak{A}$ and $\Phi(\mathfrak{A})$ are isometrically isomorphic.*

§ 3. Continuity via topology

Let X and A_x as above, define $Z = \bigcup_{x \in X} (x, A_x)$, and let p stand for the canonical projection $Z \to X$, $(x, a) \mapsto x$. The triple (p, Z, X) is called a Banach algebra bundle if there is a topology T on Z having the following property: If V runs through the open sets in X, s runs through the class of all (in the common sense) continuous functions from V into Z, and ε runs through the positive real numbers then the sets $U(V, s, \varepsilon) := \{(x, a) \in Z : x \in V$ and $\|a - s(x)\|_x < \varepsilon\}$ form a basis of the topology T.

Let (p, Z, X) be a Banach algebra bundle. If $U \subseteq Z$ is an open set and if (x, U') is the restriction of U onto the fibre (x, A_x) then U' is open in the natural topology of A_x, and conversely, any open subset U' of A_x is of this form. This shows in particular, that the operations in A_x are continuous with respect to the topology T. Moreover, the projection p is always continuous. Indeed, let V be open in X. Then $p^{-1}(V) = \bigcup_{x \in V} (x, A_x) = \bigcup_{\varepsilon \in \mathbb{R}+} U(V, s, \varepsilon)$ if s is an arbitrarily taken continuous function from X into Z.

Let $C_{\text{top}}(T)$ denote the set of all functions $f \in V(X, A_x)$ such that the function $x \mapsto (x, f(x))$ from X into Z is continuous (i.e. a continuous cross section). Obviously, $C_{\text{top}}(T)$ is a closed subalgebra of $V(X, A_x)$.

Theorem 3. *Assume that $x \mapsto (x, e_x)$ is a continuous function. Then*
(a) $C(X) \subseteq C_{\text{top}}(T)$.
(b) *if E is a usc and maximal subalgebra of $V(X, A_x)$ and $E \subseteq C_{\text{top}}(T)$ then $C_{\text{top}}(T) C_{\text{usc}}(E)$.*

Proof. (a) Let g be a continuous complex-valued function on X, choose an open basis set $U(V, s, \varepsilon) \in T$ and put $W := \{x \in X : (x, g(x)e_x) \in U(V, s, \varepsilon)\}$. We claim that W is open in X.

By the definition of the U's we have $W = \{x \in V : \|g(x)e_x - s(x)\|_x < \varepsilon\}$. If W is empty then nothing is to prove. So let $x_0 \in W$, define ε_0 and δ by $\varepsilon_0 = \|g(x_0)e_{x_0} - s(x_0)\|_{x_0}$ and $\delta = (\varepsilon - \varepsilon_0)/3$, and consider the sets $V_1 = \{x \in X : |g(x) - g(x_0)| < \delta\}$ and $V_2 = \{x \in X : \|g(x_0)e_x - s(x)\|_x < \varepsilon_0 + \delta\}$. The set V_1 is open since g is continuous. In order to see that

V_2 is also open notice that the function $b : x \mapsto (x, g(x_0)e_x)$ is continuous by assumption and that $V_2 = b^{-1}(U(V, s, \varepsilon_0 + \delta))$. Hence, $V_1 \cap V_2$ is open in X, and the estimation

$$\begin{aligned}\|g(x)e_x - s(x)\|_x \leqslant \|g(x)e_x - g(x_0)e_x\|_x + \\ \|g(x_0)e_x - s(x)\|_x \\ \leqslant |g(x) - g(x_0)| + \|g(x_0)e_x - s(x)\|_x \\ \leqslant (\varepsilon - \varepsilon_0)/3 + \varepsilon_0 + (\varepsilon - \varepsilon_0)/3 < \varepsilon\end{aligned}$$

holding for all $x \in V_1 \cap V_2$ shows that $x_0 \in V_1 \cap V_2 \subseteq W$, i.e. x_0 is an interior point of W, and W must be open.

(b) Let $f \in C_{\text{top}}(T)$, $g \in E$, and $f(x_0) = g(x_0)$. Since both f and g are in $C_{\text{top}}(T)$ we conclude that the set $\{x \in X : \|f(x) - g(x)\|_x < \varepsilon\}$ is open in X for all $\varepsilon > 0$ in the very same way as we have verified in part (a) that V_2 is open. Thus, $C_{\text{top}}(T) \subseteq C_{\text{usc}}(E)$, and the reverse inclusion follows from $C(X) \subseteq C_{\text{top}}(T)$ (by part (a)), from $E \subseteq C_{\text{top}}(T)$ (by assumption), and from our characterization of $C_{\text{usc}}(E)$ as the smallest closed subalgebra of $V(X, A_x)$ containing E and $C(X)$ (by Prop. 2(b)). •

Naturally, the following question arises: Given a usc and maximal subalgebra E of $V(X, A_x)$, does there always exist a topology T on $Z = \bigcup_{x \in X} (x, A_x)$ such that (p, Z, X) is a Banach algebra bundle and that $E \subseteq C_{\text{top}}(T)$? The following proposition answers this question.

Proposition 4. (a) *If E is as above then there is a topology $T(E)$ on Z such that (p, Z, X) is a Banach algebra bundle and $E \subseteq C_{\text{top}}(T(E))$.*

(b) *Conversely, given a topology T on Z which makes (p, Z, X) to be a Banach algebra bundle, and if $C_{\text{top}}(T)$ is unital and maximal then there is a usc and maximal subalgebra E of $V(X, A_x)$ such that $C_{\text{top}}(T) = C_{\text{usc}}(E)$.*

Proof. (a) The proof is analogous to the C^*-algebra case where the generation of a topology on Z starting from an algebra is a standard procedure. We remark only that a prebasis for the wanted topology is formed by the sets

$$U(s, \varepsilon) = \{(x, a) \in Z : \|a - s(x)\|_x < \varepsilon\}$$

with s running through E and ε running through $\mathbb{R}^+$.

(b) If $f : X \to Z$ is continuous then the function $x \mapsto \|f(x)\|$ is necessarily usc. Thus, one can take $C_{\text{top}}(T)$ as the algebra E. •

§ 4. Continuity via connecting homomorphisms

The notion of continuity we shall now discuss is only applicable in some special situations. But if it is the case, it yields (in our opinion) a better visuality than other concepts of continuity.

As above, let X be a compact Hausdorff space and $x \in X$. An open and connected neighborhood U of x will be called *separating* if the following holds: If $U \setminus \{x\} = \bigcup_j \Gamma_j$ is the decomposition of $U \setminus \{x\}$ into its connected components Γ_j, and if $U' \subseteq U$ is another connected neighborhood of x then the sets $U' \cap \Gamma_j$ are connected again. Now assume we have the following situation:

(i) Each point $x \in X$ possesses a separating neighborhood $U(x)$.

(ii) If $U(x) \setminus \{x\} = \bigcup_j \Gamma_j(x)$ is the decomposition of $U(x) \setminus \{x\}$ into its connected components then $A_y = A_z$ for all y and z belonging to the same component $\Gamma_j(x)$.

Thus, since all algebras over $\Gamma_j(x)$ coincide, we have a natural notion of continuity on $\Gamma_j(x)$, and we only need conditions to "connect these continuities" at x.

To that end, let $A_j(x)$ denote one of the algebras A_y with $y \in \Gamma_j(x)$, and assume we are given a system $L = \{L_j(x), x \in X\}$ of continuous algebra homomorphisms $L_j(x) : A_x \longrightarrow A_j(x)$.

Definition. *A function $f \in V(X, A_x)$ is said to be L-continuous if the limit* $\lim\limits_{\substack{y \to x \\ y \in \Gamma_j(x)}} f(y)$ *exists for all $x \in X$, and if this limit equals $L_j(x)(f(x))$.*

Notice that the separating property of the neighborhoods $U(x)$ guaranties the independence of this definition from the concrete choice of the $U(x)$.

The set of all L-continuous functions on X will be denoted by $C_{\text{con}}(L)$. It is easy to check that $C_{\text{con}}(L)$ forms a closed subalgebra of $V(X, A_x)$.

Theorem 4. *Let the homomorphisms $L_j(x)$ be unital (i.e. $L_j(x)(e_x)$ is the unit element in $A_j(x)$). Then*

(a) $C(X) \subseteq C_{\text{con}}(L)$.

(b) *if E is a usc and maximal subalgebra of $V(X, A_x)$ and $E \subseteq C_{\text{con}}(L)$ then $C_{\text{con}}(L) = C_{\text{usc}}(E)$.*

Proof. (a) Let $e_j(x)$ denote the unit element of $A_j(x)$ and g be a continuous complex-valued function on X. Then, obviously, the limit $\lim\limits_{\substack{y \to x \\ y \in \Gamma_j(x)}} g(y)e_j(x)$ exists, and it equals $g(x)e_j(x)$. On the other hand, $L_j(x)(g(x)e_x) = g(x)e_j(x)$ by our assumption.

(b) Let $f \in C_{\text{con}}(L)$, $g \in E$, and $f(x_0) = g(x_0)$. Because $E \subseteq C_{\text{con}}(L)$, the limit $\lim\limits_{\substack{x \to x_0 \\ x \in \Gamma_j(x_0)}} (f(x) - g(x))$ exists, and it equals $L_j(x_0)(f(x_0) - g(x_0))$. Thus, given $\varepsilon > 0$ there is an open neighborhood U_j of x_0 such that

$$\|f(x) - g(x)\|_x \in \|L_j(x_0)(f(x_0) - g(x_0))\|_{x_0} + \varepsilon = \varepsilon$$

for all $x \in \Gamma_j(x_0) \cap U_j$. This yields the upper semi-continuity of the function $x \mapsto \|f(x) - g(x)\|_x$ on the component $\Gamma_j(x_0)$. Now put $U := \bigcup\limits_j (\Gamma_j(x_0) \cap U_j)$ to obtain $\|f(x) - g(x)\|_x < \varepsilon$ for all $x \in U$ and, hence, $C_{\text{con}}(L) \subseteq C_{\text{usc}}(E)$. The reverse inclusion follows from $E \subseteq C_{\text{con}}(L)$, $C(X) \subseteq C_{\text{con}}(L)$, and from Proposition 2(b). •

The following is the analogue of Proposition 4.

Proposition 5. (a) *If E is as above and if the limits* $\lim\limits_{\substack{y \to x \\ y \in \Gamma_j(x)}} f(y)$ *exist for all $f \in E$ and $x \in X$ then there is a set $L = L(E)$ of homomorphisms such that $E \subseteq C_{\text{con}}(L(E))$.*

(b) *Conversely, if L is a system of homomorphisms with $\|L_j(x)\| = 1$ for all x and j, and if $C_{\text{con}}(L)$ is unital and maximal, then there is a usc maximal subalgebra E of $V(X, A_x)$ such that $C_{\text{con}}(L) = C_{\text{usc}}(E)$.*

Proof. Given $a \in A_x$ choose a function $f \in E$ with $f(x) = a$, and set $L_j(x) := \lim\limits_{\substack{y \to x \\ y \in \Gamma_j(x)}} f(y)$. We show that this definition does not depend on the choice of f: Let $g \in E$ be another function with $g(x) = a$. Since $f - g \in E$, the mapping $y \mapsto \|f(y) - g(y)\|_y$ is usc at x, i.e. given $\varepsilon > 0$ there is a neighborhood U of x such that

$$\|f(y) - g(y)\|_y \leqslant \|f(x) - g(x)\|_x + \varepsilon = \varepsilon$$

for all $y \in U$. Thus, $\lim\limits_{\substack{y \to x \\ y \in \Gamma_j(x)}} f(y) = \lim\limits_{\substack{y \to x \\ y \in \Gamma_j(x)}} g(y)$. Moreover, if $f \in E$ then $\|f(y)\| \leqslant \|f(x)\| + \varepsilon$ for all y belonging to a certain neighborhood of x. This implies that $\|L_j(x)(f(x))\| \leqslant \|f(x)\|$, i.e. the so-defined mappings are continuous and of norm 1. Finally, it is an immediate consequence of the definition that the $L_j(x)$'s are algebra homomorphisms, and so the proof of (a) is complete.

(b) We shall show that the algebra $C_{\text{con}}(L)$ is usc. Then one can set $E := C_{\text{con}}(L)$ to prove the assertion. Let $f \in C_{\text{con}}(L)$. Since $\|L_j(x)f(x)\| \leqslant \|f(x)\|$ we have $\|\lim\limits_{\substack{y \to x \\ y \in \Gamma_j(x)}} f(y)\|_y \leqslant \|f(x)\|$, i.e. for all components $\Gamma_j(x)$ and $\varepsilon > 0$ there is an open neighborhood $U_j(x)$ of x such that $\|f(y)\|_y \leqslant \|f(x)\|_x + \varepsilon$ for all $y \in U_j(x) \cap \Gamma_j(x)$. Put $U(x) = \bigcap\limits_j U_j(x)$. Then, for all $y \in U(x)$, $\|f(y)\|_y \leqslant \|f(x)\|_x + \varepsilon$, i.e. the function $x \mapsto \|f(x)\|_x$ is usc.

§ 5. Algebras of singular integral operators on composed curves

Let Γ be a composed curve in the complex plane, i.e. Γ is the union of a finite number of pairwise compatible simple arcs. Remember that a simple arc is a bounded oriented curve which is homeomorphic to a closed interval and which satisfies the Lyapunov condition, i.e. there exists a unique tangent at each point $t \in \Gamma$, and if these tangents are endowed with an orientation being in accordance with the orientation of the curve Γ at t then the angle $\theta(t)$ which is formed by the so-oriented tangent at t and by the real axis (in its natural orientation) depends Hölder continuously on t. A pair (Γ_1, Γ_2) of simple arcs is called compatible if they have at most one point in common and if, in case $\Gamma_1 \cap \Gamma_2 \neq \varnothing$, the common point is end point both of Γ_1 and of Γ_2 and if the one-sided tangents of Γ_1 and Γ_2 at the common point do not coincide.

Given a finite subset Γ' of Γ and a sequence $\alpha = (\alpha_z)_{z \in \Gamma}$ of real numbers we define the Khvedelidze weight function ω on $\Gamma \setminus \Gamma'$ by $\omega(t) = \prod\limits_{z \in \Gamma'} |t - z|^{\alpha_z}$, and we let $L^p_\Gamma(\alpha)$ denote the weighted Lebesgue space on Γ consisting of all classes of functions f with

$$\|f\|_{L^p_\Gamma(\alpha)} := \Big(\int\limits_\Gamma |f(t)|^p \omega(t)^p |dt|\Big)^{1/p} < \infty.$$

Here and hereafter, assume that $0 < \alpha_z + 1/p < 1$ for all $z \in \Gamma'$. Under this condition, the singular integral operator S_Γ,

$$(S_\Gamma f)(t) := \frac{1}{\pi i} \int\limits_\Gamma \frac{f(s)}{s - t}\, ds, \quad t \in \Gamma,$$

is bounded on $L^p_\Gamma(\alpha)$.

A piecewise continuous function on Γ is a function possessing finite limits at each point t of Γ along each arc ending in t. Since any piecewise continuous function a on Γ is bounded, the operator aI of multiplication by a is bounded on $L^p_\Gamma(\alpha)$. Let $PC(\Gamma)$ stand for the algebra of all piecewise continuous functions on Γ provided with the supremum norm, and let $P\sum^p_\Gamma(\alpha)$ refer to the smallest closed subalgebra of $\mathfrak{L}(L^p_\Gamma(\alpha))$ which contains the operator S_Γ and all multiplication operators aI with $a \in PC(\Gamma)$.

The algebra $P\sum_\Gamma^p(\alpha)$ contains the ideal $K(L^p_\Gamma(\alpha))$ of all compact operators on $L^p_\Gamma(\alpha)$, and if the function f is continuous on Γ then the coset $fI + K(L^p_\Gamma(\alpha))$ belongs to the center of the quotient algebra $P\sum_\Gamma^p(\alpha)/K(L^p_\Gamma(\alpha))$ (see [GK] or [RS 1,2] for details). Thus one can localize this quotient algebra over a central subalgebra which is isometrically isomorphic to the algebra of all continuous functions on Γ, and the maximal ideal space of which is homeomorphic to Γ.

Given $z \in \Gamma$ denote the corresponding local algebra by $P_z\sum_\Gamma^p(\alpha)$. To give another description of this algebra, set $\alpha_z = 0$ for $z \in \Gamma \setminus \Gamma'$, and define the "local" curve Γ_z by

$$\Gamma_z = \bigcup_{j=1}^{k} e^{i(\theta_j(z)-\theta_1(z))}\mathbb{R}^+,$$

where we have assumed that, for any sufficiently small connected neighborhood U_z of z, the open set $(U_z \cap \Gamma)\setminus\{z\}$ consists of k connected components $\Gamma_1(z), \ldots, \Gamma_k(z)$, and where $\theta_j(z)$ denotes the angle between oriented tangent of $\Gamma_j(z)$ at z and the real axis.

The orientation of $e^{i(\theta_j(z)-\theta_1(z))}\mathbb{R}^+$ is chosen in accordance with the natural orientation on $\Gamma_j(z)$ (to z or away from z). Let $\mathfrak{B}^p_{\Gamma_z}(\alpha_z)$ denote the smallest closed subalgebra of $\mathfrak{L}(L^p_{\Gamma_z}(\alpha_z))$ containing the singular integral operator S_{Γ_z} and the operators of multiplication by the characteristic functions of the half axes $e^{i(\theta_j(z)-\theta_1(z))}\mathbb{R}^+$. In [RS 1,2] it was proved that the local algebras $P_z\sum_\Gamma^p(\alpha)$ are isomertically isomorphic to the algebras $\mathfrak{B}^p_{\Gamma_z}(\alpha_z)$.

Combining these isomorphisms with the symbol map Φ defined in the introduction we get a natural homomorphism ψ from $P\sum_\Gamma^p(\alpha)/K(L^p_\Gamma(\alpha))$ into the algebra $V(\Gamma, \mathfrak{B}^p_{\Gamma_z}(\alpha_z))$. Taking further into account that the algebra $P\sum_\Gamma^p(\alpha)/K(L^p_\Gamma(\alpha))$ is KMS with respect to its above constructed central subalgebra this yields via Theorem 2:

Proposition 6. *The algebras* $P\sum_\Gamma^p(\alpha)/K(L^p_\Gamma(\alpha))$ *and* $\psi(P\sum_\Gamma^p(\alpha)/K(L^p_\Gamma(\alpha)))$ *are isometrically isomorphic, and* $\psi(P\sum_\Gamma^p(\alpha)/K(L^p_\Gamma(\alpha))) = C_{\text{usc}}(\psi(P\sum_\Gamma^p(\alpha)/K(L^p_\Gamma(\alpha))))$.

We shall give now another description of this quotient algebra by means of a family of connecting homomorpisms. This is possible since, with the exception of a finite number of points, all algebras $\mathfrak{B}^p_{\Gamma_z}(\alpha_z)$ are equal to $\mathfrak{B}^p_{\mathbb{R}}(0)$.

To this end, we need the images of the generating cosets $S_\Gamma + K(L^p_\Gamma(\alpha))$ and $aI + K(L^p_\Gamma(\alpha))$ (with $a \in PC(\Gamma)$) of the quotient algebra $P\sum_\Gamma^p(\alpha)/K(L^p_\Gamma(\alpha))$ under the mapping ψ explicitely. For this, abbreviate the characteristic function of the half axis $e^{i\theta_j(z)}\mathbb{R}^+$ to $\chi_j(z)$, and write $a_j(z)$ for the limit $\lim\limits_{\substack{y\to z\\ y\in\Gamma_j(z)}} a(y)$. Then (see [Rs 1,2])

$$\psi(S_\Gamma + K(L^p_\Gamma(\alpha)))(z) = S_{\Gamma_z} \in \mathfrak{B}^p_{\Gamma_z}(\alpha_z) \tag{1}$$

and

$$(aI + K(L^p_\Gamma(\alpha)))(z) = \sum_j a_j(z)\chi_j(z) \in \mathfrak{B}^p_{\Gamma_z}(\alpha_z). \tag{2}$$

Taking into account that $\mathfrak{B}^p_{\Gamma_z}(\alpha_z) = \mathfrak{B}^p_{\mathbb{R}}(0)$ for almost all $z \in \Gamma$ and the definition of a piecewise continuous function it is now not too hard to derive that the limits

$$\lim_{\substack{y\to z\\ y\in\Gamma_j(z)}} \psi(S_\Gamma + K(L^p_\Gamma(\alpha)))(y) \quad \text{and} \quad \lim_{\substack{y\to z\\ y\in\Gamma_j(z)}} \psi(aI + K(L^p_\Gamma(\alpha)))(y)$$

exist for all $z \in \Gamma$, and that they are equal to $S_{\mathbb{R}} \in \mathfrak{B}^p_{\mathbb{R}}(0)$ and $a_j(z)I \in \mathfrak{B}^p_{\mathbb{R}}(0)$, respectively. Thus, the limits $\lim\limits_{\substack{y\to z\\ y\in\Gamma_j(z)}} \psi(A)(y)$ exist for all $z \in \Gamma$ and $A \in P\sum_\Gamma^p(\alpha)/K(L^p_\Gamma(\alpha))$, and we can define the connecting homomorphisms $L_j(z)$ in accordance with Proposition 5 as follows: Given $B \in \mathfrak{B}^p_{\Gamma_z}(\alpha_z)$ choose an operator $A \in P\sum^p_\Gamma(\alpha)$ such that $\psi(A + K(L^p_\Gamma(\alpha)))(z) = B$ and define

$$L_j(z)(B) := \lim_{\substack{y \to z \\ y \in \Gamma_j(z)}} (A + K(L^p_\Gamma(\alpha)))(y).$$

Let L denote the family of all so-declared homomorphisms $L_j(z)$. Then the Propositions 5 and 6 yield:

Theorem 5. *The algebra $P\sum^p_\Gamma(\alpha)/K(L^p_\Gamma(\alpha))$ is isometrically isomorphic to the algebra $C_{\mathrm{con}}(L)$.*

We conclude this section with an explicit representation of the homomorphisms $L_j(z)$ which will be given without proof. On defining the function ω_z on $\Gamma_z \setminus \{0\}$ by $\omega_z(y) = |y|^{\alpha_z}$, the mapping $\Phi_1 : A \mapsto \omega_z A \omega_z^{-1}$ becomes an isomertical isomorphism from $\mathfrak{B}^p_{\Gamma_z}(\alpha_z)$ onto $\mathfrak{B}^p_{\Gamma_z}(0)$ (see [RS 1,2] for this and the next steps).

For $A \in \mathfrak{B}^p_{\Gamma_z}(0)$ we define $\Phi_2(a) = \chi_j(z)A\chi_j(z)$. Further, let η_z be the isometry from $L^p_{e^{i\theta_j(z)}\mathbb{R}^+}(0)$ onto $L^p_{\mathbb{R}^+}(0)$ given by $(\eta_z f)(y) = f(e^{i\theta_j(z)}y)$, $y \in \mathbb{R}^+$, and set $\Phi_3 : A \mapsto \eta_z A \eta_z^{-1}$. The mapping Φ_3 is an isometrical isomorphism from $\chi_j(z)\mathfrak{B}^p_{\Gamma_z}(0)\chi_j(z)$ onto $\mathfrak{B}^p_{\mathbb{R}^+}(0)$. Finally, let Φ_4 be the embedding of $\mathfrak{B}^p_{\mathbb{R}^+}(0)$ into $\mathfrak{B}^p_{\mathbb{R}}(0)$ and define Φ_5 for $A \in \mathfrak{B}^p_{\mathbb{R}}(0)$ by

$$\Phi_5(A) := s - \lim_{t \to 0} Z_{t^{-1}} U_{-1} A U_1 Z_t$$

where $(U_s f)(y) = f(y - s)$ and $(Z_t f)(y) = f(y/t)$. Then $L_j(z) = \Phi_5\Phi_4\Phi_3\Phi_2\Phi_1$. Roughly speaking, the whole procedure makes nothing else than the homogenization of the operator A at an arbitrary inner point of the half axis $e^{i\theta_j(z)}\mathbb{R}^+$.

§ 6. Algebras of multiplication and Fourier convolution operators

Denote by PC_∞ the algebra of all pieciwise continuous functions of $\mathbb{R}$ which are continuous at infinity (i.e. $\lim_{t\to\infty} a(t) = \lim_{t\to-\infty} a(t)$ for all $a \in PC_\infty$) and by PC_p the algebra of all piecewise continuous Fourier multipliers on $L^p_{\mathbb{R}}$ possesing a finite total variation. Let $\mathfrak{A}$ stand for the smallest closed subalgebra of $\mathfrak{L}(L^p_{\mathbb{R}})$ containing all multiplication operators aI with $a \in PC_\infty$ and all Fourier convolution operators $W^0(b)$ with $b \in PC_p$. We will study the quotient algebra $\mathfrak{A}^\pi = \mathfrak{A}/K(L^p_{\mathbb{R}})$ which makes sense since the ideal $K(L^p_{\mathbb{R}})$ of all compact operators on $L^p_{\mathbb{R}}$ is contained in $\mathfrak{A}$.

If the function $c \in PC_\infty$ is continuous on $\mathbb{R}$ then the coset $\pi(c) := cI + K(L^p_{\mathbb{R}})$ belongs to the center of $\mathfrak{A}^\pi$. Thus, we can localize $\mathfrak{A}^\pi$ over a central subalgebra which is isomorphic to the algebra of all continuous functions in PC_∞, and which has the one-point compactification $\dot{\mathbb{R}}$ of the real axis as its maximal ideal space. Denote the local algebra being associated with $x \in \dot{\mathbb{R}}$ by $\mathfrak{A}^\pi_x$.

For $x \neq \infty$, the local algebras $\mathfrak{A}^\pi_x$ have been studied in [RS 1]. They are all isometrically isomorphic to the algebra $\mathfrak{B}^p_{\mathbb{R}}(0)$ introduced in the preceding section. For $x = \infty$ we give the following description of $\mathfrak{A}^\pi_x$:

Proposition 7. *The algebra $\mathfrak{A}^\pi_\infty$ is isometrically isomorphic to the closure of the algebra $W^0(PC_p)$ in $\mathfrak{L}(L^p_{\mathbb{R}})$.*

Proof. Denote the closure of the algebra $W^0(PC_p)$ in $\mathfrak{L}(L^p_{\mathbb{R}})$ by $\mathfrak{B}$ and the canonical homomorphisms from $\mathfrak{A}$ onto $\mathfrak{A}^\pi$ and $\mathfrak{A}^\pi_x$ by π and Φ^π_x, respectively.

At first we show that $\|B\| = \|\Phi^{\pi}_{\infty}(B)\|$ for all $B \in \mathfrak{B}$. Since the operators in $\mathfrak{B}$ are shift-invariant we get the equality $\|B\| = \|\pi(B)\|$ in a standard way, and from the definition it is clear that $\|\pi(B)\| \geqslant \|\Phi^{\pi}_{\infty}(B)\|$. It remains to show that $\|\pi(B)\| \leqslant \|\Phi^{\pi}_{\infty}(B)\|$.

As we know from the KMS-concept, $\|\pi(B)\| = \sup_{s\in\mathbb{R}}\|\Phi^{\pi}_{s}(B)\|$.

Taking into account the above mentioned result that all algebras $\mathfrak{A}^{\pi}_{s}$ are isometrically isomorphic, and evaluating these isomorphisms for $\Phi^{\pi}_{s}(B)$ one easily gets $\|\Phi^{\pi}_{s}(B)\| = \|\Phi^{\pi}_{t}(B)\|$ for all s, $t \in \mathbb{R}$ and $B \in \mathfrak{B}$. This in combination with the upper semi-continuity of the mapping $\dot{\mathbb{R}} \to \mathbb{R}^{+}, s \mapsto \|\Phi^{\pi}_{s}(B)\|$ shows that, in fact, $\|\pi(B)\| = \|\Phi^{\pi}_{\infty}(B)\|$. Consequently, the algebra $\Phi^{\pi}_{\infty}(\mathfrak{B})$ is closed in $\mathfrak{A}^{\pi}_{\infty}$.

Further, if A is a finite sum of products of multiplication and convolution operators, then $\Phi^{\pi}_{\infty}(A) \subseteq \Phi^{\pi}_{\infty}(\mathfrak{B})$ which implies that the algebra $\Phi^{\pi}_{\infty}(\mathfrak{B})$ is dense in $\mathfrak{A}^{\pi}_{\infty}$. Combining this we get $\Phi^{\pi}_{\infty}(\mathfrak{B}) = \mathfrak{A}^{\pi}_{\infty}$ as desired. •

On defining $\mathfrak{B}_x := \mathfrak{B}^{p}_{\mathbb{R}}(0)$ for $x \in \mathbb{R}$ and $\mathfrak{B}_{\infty} := \mathfrak{B}$ we get a natural homomorphism ψ from $\mathfrak{A}^{\pi}$ into $V(\dot{\mathbb{R}}, \mathfrak{B}_x)$, and the KMS-property of $\mathfrak{A}^{\pi}$ and Theorem 2 yield:

Proposition 8. *The algebras $\mathfrak{A}^{\pi}$ and $\psi(\mathfrak{A}^{\pi})$ are isometrically isomorphic, and $\psi(\mathfrak{A}^{\pi}) = C_{\text{usc}}(\psi(\mathfrak{A}^{\pi}))$.*

Now we describe the algebra $\psi(\mathfrak{A}^{\pi})$ via connecting homomorphisms once more.

If U is a finite, open and connected neighborhood of $x \in \mathbb{R}$ then we write $U \setminus \{x\} = \Gamma_{+}(x) \cup \Gamma_{-}(x)$ for the decomposition of $U \setminus \{x\}$ into its connected components and assume that $\Gamma_{-}(x) < \Gamma_{+}(x)$. For a connected neighborhood U of ∞ not containing the point 0 we write analogously $U \setminus \{\infty\} = \Gamma_{+}(\infty) \cup \Gamma_{-}(\infty)$ where $\Gamma_{\pm}(\infty) \leqslant \mathbb{R}^{\pm}$. Now define for $x \in \mathbb{R}$ and $A \in \mathfrak{B}^{p}_{\mathbb{R}}(0)$

$$L_{\pm}(x)(A) = s - \lim_{s\to\pm\infty} U_{-s}AU_{s} \tag{3}$$

and for $x = \infty$ and $B = W^{0}(b) \in \mathfrak{B}$

$$L_{\pm}(\infty)(B) = W^{0}(b(+\infty)\chi_{\mathbb{R}^{+}} + b(-\infty)\chi_{\mathbb{R}^{-}}),$$

and set $L = \{L_{\pm}(x), x \in \dot{\mathbb{R}}\}$.

Theorem 6. *The algebras $\mathfrak{A}^{\pi}$, $\psi(\mathfrak{A}^{\pi})$ and $C_{\text{con}}(L)$ are isometrically isomorphic.*

Proof. First notice that the strong limits in (3) exist for all operators $A \in \mathfrak{B}^{p}_{\mathbb{R}}(0)$. This is due to the elementary identities $U_{-s}S_{\mathbb{R}}U_{s} = S_{\mathbb{R}}$ and $U_{-s}\chi_{\mathbb{R}^{+}}U_{s} = \chi_{[-s,\infty)}$. Further it is obvious that all homomorphisms in L are unital. So, following Theorem 4, we have only to verify that $\psi(\mathfrak{A}^{\pi})$ is in $C_{\text{con}}(L)$ to get the assertion.

As generating elements of the algebra $\mathfrak{A}^{\pi}$ one can choose the cosets $\pi(a)$ with $a \in PC_{\infty}$ and $\pi(W^{0}(b))$ with $b \in PC_{p}$. Thus, it suffices to check whether $\psi(\pi(a))$ and $\psi(\pi(W^{0}(b)))$ belong to $C_{\text{con}}(L)$ for all a and b as above. Notice that

$$\psi(\pi(a))(x) = \begin{cases} a(x-0)\chi_{\mathbb{R}^{-}} + a(x+0)\chi_{\mathbb{R}^{+}}, & x \in \mathbb{R} \\ a(\infty)I, & x = \infty \end{cases}$$

and

$$\psi(\pi(W^{0}(b)))(x) = \begin{cases} W^{0}(b(+\infty)\chi_{\mathbb{R}^{+}} + b(-\infty)\chi_{\mathbb{R}^{-}}), & x \in \mathbb{R} \\ W^{0}(b), & x = \infty. \end{cases}$$

Hence,

$$\begin{aligned} \lim_{\substack{y\to x \\ y\gtrless x}} \psi(\pi(a))(y) &= a(x \pm 0)I \\ &= L_{\pm}(x)(a(x-0)\chi_{\mathbb{R}^{-}} + a(x+0)\chi_{\mathbb{R}^{+}}) \end{aligned}$$

and

$$\lim_{\substack{y \to x \\ y \gtrless x}} \psi(\pi(W^0(b)))(y) = W^0(b(+\infty)\chi_{\mathbb{R}^+} + b(-\infty)\chi_{\mathbb{R}^-})$$
$$= L_\pm(x)(W^0(b(+\infty)\chi_{\mathbb{R}^+} + b(-\infty)\chi_{\mathbb{R}^-}))$$

for all $x \in \mathbb{R}$, and similarly one gets the L-continuity at $x = \infty$(remember that $W^0(\chi_{\mathbb{R}^+}) = (I - S_{\mathbb{R}})/2$ for the inclusion $\psi(\pi(W^0(b)))(x) \in \mathfrak{B}^p_{\mathbb{R}}(0))$. •

§ 7. Algebras of multidimensional singular integral operators

For the sake of brevity we only consider singular integral operators on $\mathbb{R}^n$ having continuous coefficients. Let $\mathfrak{A}$ stand for the smallest closed subalgebra of $\mathfrak{L}(L^p_{\mathbb{R}^n})$ $(1 < p < \infty, n \in \mathbb{N})$ generated by all operators of the form

$$\sum_{m=0}^{j} \sum_{k=1}^{\varkappa_m} a_m^{(k)} A_m^{(k)} + T$$

where $A_0^{(1)} = 1, T \in K(L^p_{\mathbb{R}^n})$, $a_m^{(k)} \in C(\dot{\mathbb{R}}^n)$ with $\dot{\mathbb{R}}^n$ being defined as the one-point compactification of $\mathbb{R}^n$, and

$$(A_m^{(k)} u)(x) = \int\limits_{\mathbb{R}^n} \frac{Y_m^{(l)}(\frac{x-y}{|x-y|})u(y)}{|x-y|^n}\, dy$$

with $Y_m^{(l)}$ denoting the l-th spherical function of order m defined on the sphere S^{n-1}. Recall that $\{Y_m^{(l)}\}(l = 1, \ldots, \varkappa_m; m = 0, 1, \ldots)$ actually is an orthogonal normed system in the space $L^2(S^{n-1})$, and that the upper index l numbers the linearly independent spherical functions of one and the same order m.

Here we list some properties of these operators which are well-known and can be found in [MP], for instance.

(i) All operators $A_m^{(k)}$ commute with all translation operators U_s and with all operators Z_t, $t > 0$.

(ii) If $a \in C(\dot{\mathbb{R}}^n)$ then $aA_m^{(k)} - A_m^{(k)} aI \in K(L^p_{\mathbb{R}^n})$.

(iii) The algebra $\mathfrak{A}^\pi := \mathfrak{A}/K(L^p_{\mathbb{R}^n})$ is commutative, and it contains a copy of the C^*-algebra $C(\dot{\mathbb{R}}^n)$.

(iv) If

$$(Fu)(x) = \int\limits_{\mathbb{R}^n} e^{2\pi i(x,y)} u(y)\, dy$$

denotes the Fourier transform then, for u being smooth and having a compact support,

$$A_m^{(k)} u = F^{-1} \hat{k}\, F u$$

where

$$\hat{k}(z) = \pi^{n/2} i^m \frac{\Gamma(m/2)}{\Gamma((m+n)/2)} Y_m^{(l)}(\frac{z}{|z|})$$

is the Fourier transform of the function

$$k(x) = \frac{Y^{(l)}(\frac{x}{|x|})}{|x|^n}$$

in the sense of distributions.

(v) The smallest closed subalgebra $\mathfrak{B}$ of $\mathfrak{L}(L^p_{\mathbb{R}^n})$ containing all operators $A^{(k)}_m$ is commutative, inverse closed, semi-simple, and its maximal ideal space is homeomorphic to the sphere S^{n-1}.

Applying the results of § 2 one gets

Theorem 7. (a) *The algebra $\mathfrak{A}^\pi$ is KMS with respect to $C(\dot{\mathbb{R}}^n)$ and inverse closed in $\mathfrak{L}(L^p_{\mathbb{R}^n})/K(L^p_{\mathbb{R}^n})$.*

(b) *For each $x \in \dot{\mathbb{R}}^n$, the local algebra $(\mathfrak{A}^\pi)_x$ is isomertically isomorphic to the algebra $\mathfrak{B}$.*

(c) *The algebra $\mathfrak{A}^\pi$ is isometrically isomorphic to the algebra of all continuous functions (in the common sense) defined on $\dot{\mathbb{R}}^n$ and taking values in $\mathfrak{B}$.*

Sketch of the proof. (a) See [BKS] for a similar proof. (b) Use the homogenization techniques developed in [RS 1,2] and shortly explained after Theorem 5. (c) Apply Theorem 2 and Proposition 2(b) with $\Phi(\mathfrak{A}^\pi) = E$, and have in mind the discussion after Proposition 1.

§ 8. Algebras of approximation methods

Given $d \geqslant 0$ and $n \geqslant 1$ let $S^{d,n}$ denote the space of smoothest polynomial splines of degree d over the partition $\{(i + d/2)/n\}_{i\in\mathbb{Z}}$ of $\mathbb{R}$ which belong to $L^p_{\mathbb{R}}$, and write l^p for the Banach space of all sequences $(x_k)_{k\in\mathbb{Z}}$ for which

$$\|(x_k)\| := (\sum_k |x_k|^p)^{1/p} < \infty.$$

A fundamental result on spline spaces is the following estimation which goes back to de Boor: There is a constant $C > 0$ such that

$$\frac{1}{C}\|\sum_k f_k\phi_k^{d,n}\|_{L^p_{\mathbb{R}}} \leqslant n^{-1/p}\|(f_k)\|_{l^p} \leqslant c\|\sum_k f_k\phi_k^{d,n}\|_{L^p_{\mathbb{R}}}$$

for all functions $f = \sum_k f_k\phi_k^{d,n} \in S^{d,n}$. Herein, the spline functions $\phi_k^{d,n}$ stand for the elements of a special basis of $S^{d,n}$.

Thus, the mapping $E^{d,n} : l^p \to S^{d,n}$, $(f_k) \mapsto \sum_k f_k\phi_k^{d,n}$ is correctly defined and continuous, and the mapping

$$W^{d,n} : \mathcal{L}(S^{d,n}) \to \mathcal{L}(l^p), \qquad A \mapsto (E^{d,n})^{-1}AE^{d,n},$$

is a continuous algebra homomorphism between $\mathcal{L}(S^{d,n})$ and $\mathcal{L}(l^p)$.

Let A be a bounded linear operator on $L^p_{\mathbb{R}}$. By a spline approximation method for solving the equation $Au = f$ we mean the following: Choose a parameter $d \geqslant 0$ and consider a sequence of approximating equations $A^{(n)}u^{(n)} = R^{(n)}f$ where $A^{(n)} : S^{d,n} \to S^{d,n}$, where the solution $u^{(n)}$ is seeked in the spline space $S^{d,n}$, and where the operators $R^{(n)}$ are certain projections onto $S^{d,n}$.

Let $\mathcal{F}$ be the family of all approximation methods, i.e. of all sequences $(A^{(n)})$ with $A^{(n)} \in \mathcal{L}(S^{d,n})$ for which $\sup_n\|A^{(n)}\| < \infty$. On defining elementwise operations and a norm by $\|(A^{(n)})\| = \sup_n\|A^{(n)}\|$, $\mathcal{F}$ becomes a Banach algebra.

A subalgebra of $\mathcal{F}$ which is on the one hand sufficiently large to contain a lot of important numerical methods for solving singular integral equations with piecewise continuous coefficients such as Galerkin, collocation, qualocation, and quadrature methods, and which is, on the other hand, small enough to give some hope for a successful treatment, can be constructed as follows: Let $\mathfrak{A}$ be the smallest closed subalgebra of $\mathcal{F}$ which contains all sequences $(A^{(n)}) \in \mathcal{F}$ having the property that the operators $W^{d,n}(A^{(n)})$ do not depend on n and belong to the algebra $\mathcal{T}_p$ (see below), and which contains the shifted sequences $(U_{-[\frac{s}{n}]}A^{(n)}U_{[\frac{s}{n}]})$ for all $s \in \mathbb{R}$ whenever $(A^{(n)}) \in \mathfrak{A}$. Here, as usual, $[\cdot]$ denotes the entier function, and $\mathcal{T}_p$ is the smallest closed subalgebra of $\mathcal{L}(l^p)$ which includes all operators $T^0(a)$ of multiplication by a piecewise continuous function a on the unit circle which has a finite total variation, and which contains the natural projection $P : (x_k) \to (\dots, 0, 0, x_0, x_1, \dots)$.

The algebra $\mathfrak{A}$ itself is of a rather complicated structure (similarly as the algebras $P\sum_\Gamma^p(\alpha)$ or $\mathfrak{A}$ considered above). But there is a canonical procedure to construct an ideal $\mathcal{J}$ of $\mathfrak{A}$ (which could be compared with the ideal of all compact operators in operator theory) such that the quotient algebra $\mathfrak{A}/\mathcal{J}$ gets a center being rich enough for localization. Since the explicit form of this ideal is not essential in what follows, we recommend the reader to [HS], [HRS], or [PS] for details.

Let $\psi^{\mathcal{J}}$ stand for the canonical homomorphism from $\mathfrak{A}$ onto $\mathfrak{A}/\mathcal{J}$, write $L^{(n)}$ for the Galerkin projection operator from $L^p_{\mathbb{R}}$ onto $S^{d,n}$, and let f denote a function which is continuous on $\dot{\mathbb{R}}$. Then the coset $\psi^{\mathcal{J}}((L^{(n)}f|_{S^{d,n}})_{n\geqslant 1})$ belongs to the center of $\mathfrak{A}/\mathcal{J}$, and this offers the possibility of localizing the algebra $\mathfrak{A}/\mathcal{J}$ over the one-point compactification $\dot{\mathbb{R}}$ of $\mathbb{R}$. More over, it turns out that the algebra is KMS with respect to its so-declared subalgebra.

The local algebras $\mathfrak{A}^{\mathcal{J}}_x (x \in \dot{\mathbb{R}})$ associated with this localization were determined in [HRS] as follows: If $x \in \mathbb{R}$ then $\mathfrak{A}^{\mathcal{J}}_x$ is isometrically isomorphic to $\mathcal{T}_p$, and $\mathfrak{A}^{\mathcal{J}}_\infty$ is isometrically isomorphic to the quotient $\mathcal{T}^\pi_p := \mathcal{T}_p/K(l^p)$.

$$\text{Setting } A_x := \begin{cases} \mathcal{T}_p \text{ if } x \in \mathbb{R} \\ \mathcal{T}^\pi_p \text{ if } x = \infty \end{cases} \text{ and writing } \psi \text{ for}$$

the natural homomorphism from $\mathfrak{A}/\mathcal{J}$ into $V(\dot{\mathbb{R}}, A_x)$ we arrive at

Proposition 9. *The algebras $\mathfrak{A}/\mathcal{J}$ and $\psi(\mathfrak{A}/\mathcal{J})$ are isomertically isomorphic, and $\psi(\mathfrak{A}/\mathcal{J}) = C_{\text{usc}}(\psi(\mathfrak{A}/\mathcal{J}))$.*

We conclude this paper by reformulating this proposition in terms of continuity via connecting homomorphisms. Denote by V_n the shift operator on l^p, i.e. the operator $V_n(x_k) = (x_{k-n})$, and define for $A \in \mathcal{T}_p$ and $x \in \mathbb{R}$:

$$L_\pm(x)(a) = s - \lim_{s\to\mp\infty} V_{-s}AV_s$$

and, for $A \in \mathcal{T}^\pi_p$ with $A \in A' + K(l^p)$,

$$L_\pm(\infty)(a) = s - \lim_{s\to\pm\infty} V_{-s}A'V_s.$$

It is not too hard to check that these definitions make sense and that $L_\pm(\infty)$ is correctly defined. (For the latter assertion notice that $V_{-s}KVs \to 0$ strongly as $s \to \pm\infty$ for any compact operator K).

Put $L = \{L_\pm(x), x \in \dot{\mathbb{R}}\}$. In an analogous manner as for Theorem 6 one derives:

Theorem 8. *The algebras $\mathfrak{A}/\mathcal{J}$, $\psi(\mathfrak{A}/\mathcal{J})$, and $C_{\text{con}}(L)$ are isomertically isomorphic.*

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