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BEREZIN TRANSFORM AND THE LAPLACE-BELTRAMI OPERATOR

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Abstract. Let Ω be a domain in $\mathbf{C}$, $K(x,\overline{y})$ its Bergman kernel, Δ the Laplace-Beltrami operator on Ω, and B the Berezin transform on Ω, i.e., the integral operator with the kernel $|K(x,\overline{y})|^2/K(y,\overline{y})$. For domains that are complete in the Riemannian metric $K(x,\overline{x})^{1/2}\,|dx|$, it is shown that B is a function of Δ if and only if B commutes with Δ if and only if the above metric has constant curvature if and only if Ω is simply connected. This supplements the results of Berezin [5] and of Unterberger and Upmeier [19] for the unit disc. We also briefly treat the case of weighted Bergman spaces, and indicate a relationship with quantization on Ω.

1. Introduction. Let Ω be a domain in $\mathbf{C}$ and $A^2(\Omega)$ the Bergman space of analytic functions on Ω square-integrable with respect to the two-dimensional Lebesgue measure dx. If Ω, regarded as a Riemann surface, belongs to the class O_G (i.e., has no Green's function), the space $A^2(\Omega)$ contains only the function identically equal to zero. This happens if and only if Ω has zero analytic capacity with respect to some (hence every) point in it; for domains in $\mathbf{C}$, this is equivalent to saying that the complement $\mathbf{C}\setminus\Omega$ has zero logarithmic capacity. For the proofs of all these facts, see [18 , Sections II.3.C, X.2.B, VI.5.C, IV.3.B, VII.6.D, VII.6.E, VII.5.G and X.1.B]. In the rest of this paper, we shall assume that $\Omega\notin O_G$ (this is, in particular, satisfied by all bounded domains $\Omega\subset\mathbf{C}$). The space $A^2(\Omega)$ is then nontrivial; moreover [7 , Chapter 1], [13 , Section VIII.3], it is known that the evaluation functional at every point $x\in\Omega$ is continuous, i.e., for each $x\in\Omega$ there exists $C_x>0$ such that

$$|f(x)|^2\leqslant C_x\int_\Omega |f(y)|^2\,dy\qquad \forall f\in A^2(\Omega).$$

It follows that $A^2(\Omega)$ is a closed subspace in $L^2(\Omega)$ and is a *reproducing kernel Hilbert space*; that is, for each $x\in\Omega$, there is $K_x\in A^2(\Omega)$ such that

$$f(x)=\langle f,K_x\rangle\qquad \forall f\in A^2(\Omega).\tag{1}$$

Key words and phrases. Berezin transform, Laplace-Beltrami operator, Bergman kernel, curvature, quantization.

The function

$$K(x,\overline{y}) = \langle K_y, K_x\rangle = K_y(x) \tag{2}$$

is holomorphic in $\Omega \times \overline{\Omega}$ and is called[1] the reproducing kernel of $A^2(\Omega)$. It can be shown (see, e.g., [7]) that the best possible constant C_x above is equal to $K(x,\overline{x})$, and [18 , II.3.C]

$$K(x,\overline{x}) > 0 \qquad \forall x \in \Omega. \tag{3}$$

The *Berezin transform* B is an integral operator on Ω given by the formula

$$Bf(y) = \int_\Omega f(x) \cdot |k_y(x)|^2\, dx = \langle fk_y, k_y\rangle = \frac{\langle fK_y, K_y\rangle}{\langle K_y, K_y\rangle}, \tag{4}$$

where

$$k_y := \frac{K_y}{\|K_y\|}$$

is the normalized reproducing kernel. Alternatively, one can write

$$Bf(y) = \int_\Omega f(x) \cdot \frac{K(x,\overline{y})K(y,\overline{x})}{K(x,\overline{x})K(y,\overline{y})}\, d\iota(x), \tag{5}$$

where

$$d\iota(x) := K(x,\overline{x})\, dx \tag{6}$$

is the "invariant" measure on Ω. The Berezin transform plays an important role in the theory of Toeplitz and Hankel operators on $A^2(\Omega)$, and also in quantization procedures on Ω; see [6], [3], [14], [16], and [8], for instance.

Equipped with the Riemannian metric

$$ds^2 = K(z,\overline{z})\, |dz|^2, \tag{7}$$

Ω becomes a Riemannian manifold. Being of real dimension two, it is automatically a Kähler manifold, i.e., there exists a (unique) Hermitian connection on Ω that preserves the complex structure and is compatible with the metric (7). The holomorphic (or, in this case, Gaussian) *curvature* of this connection at a point $z \in \Omega$ is given by[2]

$$\kappa(z) = \frac{1}{K(z,\overline{z})} \frac{\partial^2}{\partial z \partial \overline{z}} \ln K(z,\overline{z}) = \frac{K_{12}(z,\overline{z})K(z,\overline{z}) - K_1(z,\overline{z})K_2(z,\overline{z})}{K(z,\overline{z})^3}, \tag{8}$$

[1] Sometimes $K(x,y)$ is called the reproducing kernel.
[2] Up to an inessential constant factor of $-2/\pi$.

where K_1 stands for the partial derivative of $K(z,\overline{z})$ with respect to the first argument, i.e.,

$$K_1(z,\overline{z}) := \frac{\partial}{\partial x}K(x,\overline{y})\Big|_{x=z,y=z} = \frac{\partial}{\partial z}K(z,\overline{z}), \tag{8a}$$

and similarly

$$K_2(z,\overline{z}) := \frac{\partial}{\partial \overline{y}}K(x,\overline{y})\Big|_{x=z,y=z} = \frac{\partial}{\partial \overline{z}}K(z,\overline{z}), \tag{8b}$$

etc. For more detailed information about these topics, see [10 , Chapter 2, §§ 12 – 13], [13], [9 , Chapter 8], or [18].

For a (real) Riemannian manifold with metric $ds^2 = g_{jk}\,dx^j dx^k$, the *Laplace-Beltrami operator* is defined by the formula

$$\Delta f := \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\big(\sqrt{g}\,g^{jk}\frac{\partial f}{\partial x^k}\big),$$

where $g = \det\|g_{ij}\|$ and (g^{jk}) is the inverse matrix of (g_{jk}) [13 , Section X.2.1], [4 , Appendix 3]. In our case ($g_{11} = g_{22} = g = K(z,\overline{z})$, $g_{12} = g_{21} = 0$), this reduces (modulo an immaterial constant factor) to

$$\Delta f = \frac{1}{K(z,\overline{z})}\frac{\partial^2 f}{\partial z\partial\overline{z}}. \tag{9}$$

It was shown by Berezin [5] and, in a more general context, by Unterberger and Upmeier [19] that for $\Omega = \mathbf{D}$ (the unit disc) the Berezin transform can be expressed as a function of the Laplace-Beltrami operator:

$$B = F(\Delta), \tag{10}$$

where

$$F : -s(1-s) \longmapsto \frac{\pi s(1-s)}{\sin \pi s}, \qquad \text{or} \qquad F(x) = \prod_{k=1}^{\infty}\left(1 - \frac{x}{k(k+1)}\right)^{-1}. \tag{11}$$

More precisely: Δ is a selfadjoint operator[3] on $L^2(\mathbf{D}, d\iota)$, where $d\iota$ is the measure given by (6); thus, the usual functional calculus (or the spectral theorem) for selfadjoint operators [17 , Theorem 13.24] can be used to construct $F(\Delta)$, and the latter coincides with the (bounded and selfadjoint) operator B on $L^2(\mathbf{D}, d\iota)$. See also [12].

Our aim in this paper is to investigate whether this situation persists in other domains $\Omega \subset \mathbf{C}$. Our main results are the following two theorems.

[3]Hermitian symmetricity is obvious from (9); that Δ is actually selfadjoint follows, e.g., from formula (II.1.18) in [2].

Theorem A. *The operators B and Δ commute only if the Riemannian metric* (7) *has constant curvature.*

Thus, a necessary condition the for existence of a relation of the form (10) is that

$$\kappa(z) \equiv \text{const}$$

throughout Ω. The second main result shows that, in some sense, this condition is also sufficient. Recall that a Riemannian manifold is called *complete* if it is complete as a metric space (in the metric induced by (7)). An equivalent formulation: every geodesic has infinite length in both directions, i.e., if we parametrize it by the natural parameter (= its arc-length, measured by (7)), the parameter will range through the entire interval $(-\infty, +\infty)$. Still another formulation: each point on the boundary of Ω is at an infinite distance (in the metric (7)) from any point in the interior of Ω.

Theorem B. *If Ω, equipped with the metric* (7), *is a complete Riemannian manifold of constant curvature, then Ω is simply connected and* (10) *is valid, with the function F given by* (11).

The proof of Theorem A is contained in §2; the proof of Theorem B occupies §3. In the last section (§4), we establish an analog of Theorem A for weighted Bergman spaces $A^2(\Omega, w)$, and point out a relationship between our results and the recent work of Cahen, Gutt and Rawnsley [8] on quantization of Kähler manifolds.

2. The necessary condition. Let $f \in \mathcal{D}(\Omega)$, i.e., f is an infinitely differentiable function on Ω with compact support. Then both $B\Delta f$ and $\Delta B f$ are well defined, and we have by (5)

$$\Delta B f(y) = \int_\Omega f(x)\,.\,\Delta_y \left(\frac{K(x,\overline{y})K(y,\overline{x})}{K(x,\overline{x})K(y,\overline{y})} \right) d\iota(x). \tag{12}$$

Since the usual (i.e., Euclidean) Laplacian $4\frac{\partial^2}{\partial x \partial \overline{x}}$ is symmetric with respect to Lebesgue (area) measure (i.e., as an operator on $L^2(\mathbf{C})$), it readily follows from (6) and (9) that Δ is symmetric with respect to the measure $d\iota$ (i.e., as an operator on $L^2(\Omega, d\iota)$). Consequently,

$$\begin{aligned} B\Delta f(y) &= \int_\Omega \Delta f(x) \cdot \frac{K(x,\overline{y})K(y,\overline{x})}{K(x,\overline{x})K(y,\overline{y})}\, d\iota(x) \\ &= \int_\Omega \frac{\partial^2}{\partial x \partial \overline{x}} f(x) \cdot \frac{K(x,\overline{y})K(y,\overline{x})}{K(x,\overline{x})K(y,\overline{y})}\, dx \\ &= \int_\Omega f(x) \cdot \frac{\partial^2}{\partial x \partial \overline{x}} \left(\frac{K(x,\overline{y})K(y,\overline{x})}{K(x,\overline{x})K(y,\overline{y})} \right) dx \\ &= \int_\Omega f(x) \,.\, \Delta_x \left(\frac{K(x,\overline{y})K(y,\overline{x})}{K(x,\overline{x})K(y,\overline{y})} \right) d\iota(x). \end{aligned} \tag{13}$$

Comparing (12) and (13), we see that $B\Delta f = \Delta B f$ for all $f \in \mathcal{D}(\Omega)$ if and only if

$$\Delta_x \frac{K(x,\overline{y})K(y,\overline{x})}{K(x,\overline{x})K(y,\overline{y})} = \Delta_y \frac{K(x,\overline{y})K(y,\overline{x})}{K(x,\overline{x})K(y,\overline{y})}.$$

Multiplying by $K(x,\overline{x})K(y,\overline{y}) > 0$ on both sides, we get

$$\frac{\partial^2}{\partial x \partial \overline{x}} \frac{K(x,\overline{y})K(y,\overline{x})}{K(x,\overline{x})} = \frac{\partial^2}{\partial y \partial \overline{y}} \frac{K(x,\overline{y})K(y,\overline{x})}{K(y,\overline{y})}. \tag{14}$$

Denote, for clarity, the left-hand side by $F(x,y)$. Then (14) can be rewritten as

$$F(x,y) = F(y,x). \tag{15}$$

Let also $\frac{\partial}{\partial x} F(x,y) =: G_1(x,y)$ and $\frac{\partial}{\partial y} F(x,y) =: G_2(x,y)$. If (15) is valid, then

$$G_1(y,x) = \frac{\partial}{\partial y} F(y,x) = \frac{\partial}{\partial y} F(x,y) = G_2(x,y),$$

whence, in particular,

$$G_1(x,x) = G_2(x,x) \qquad \forall x \in \Omega. \tag{16}$$

Thus, (16) is a necessary condition for (14), i.e., for $B\Delta = \Delta B$ on $\mathcal{D}(\Omega)$. We shall show that (16) is fulfilled only if the curvature $\kappa(z)$, given by (8), is a constant function.

We have, in the notation explained in (8a), (8b), etc.,

$$\frac{\partial}{\partial \overline{x}} \frac{K(x,\overline{y})K(y,\overline{x})}{K(x,\overline{x})} = \frac{K(x,\overline{y})K_2(y,\overline{x})}{K(x,\overline{x})} - \frac{K(x,\overline{y})K(y,\overline{x})K_2(x,\overline{x})}{K(x,\overline{x})^2},$$

whence

$$\begin{aligned} F(x,y) &= \frac{\partial^2}{\partial x \partial \overline{x}} \frac{K(x,\overline{y})K(y,\overline{x})}{K(x,\overline{x})} \\ &= \frac{K_1(x,\overline{y})K_2(y,\overline{x})}{K(x,\overline{x})} - \frac{K(x,\overline{y})K_2(y,\overline{x})K_1(x,\overline{x})}{K(x,\overline{x})^2} \\ &\quad - \frac{K_1(x,\overline{y})K(y,\overline{x})K_2(x,\overline{x})}{K(x,\overline{x})^2} - \frac{K(x,\overline{y})K(y,\overline{x})K_{12}(x,\overline{x})}{K(x,\overline{x})^2} \\ &\quad + \frac{K(x,\overline{y})K(y,\overline{x})K_2(x,\overline{x}) \,.\, 2K_1(x,\overline{x})}{K(x,\overline{x})^3}. \end{aligned} \tag{16a}$$

Thus,

$$\begin{aligned} G_2(x,y) &= \frac{\partial}{\partial y} F(x,y) \\ &= \frac{K_1(x,\overline{y})K_{12}(y,\overline{x})}{K(x,\overline{x})} - \frac{K(x,\overline{y})K_{12}(y,\overline{x})K_1(x,\overline{x})}{K(x,\overline{x})^2} \\ &\quad - \frac{K_1(x,\overline{y})K_1(y,\overline{x})K_2(x,\overline{x})}{K(x,\overline{x})^2} - \frac{K(x,\overline{y})K_1(y,\overline{x})K_{12}(x,\overline{x})}{K(x,\overline{x})^2} \\ &\quad + \frac{K(x,\overline{y})K_1(y,\overline{x})K_2(x,\overline{x}) \,.\, 2K_1(x,\overline{x})}{K(x,\overline{x})^3}, \end{aligned}$$

and

$$\begin{aligned} G_2(x,x) &= \frac{K_1K_{12}}{K} - \frac{KK_{12}K_1}{K^2} - \frac{K_1^2K_2}{K^2} - \frac{KK_1K_{12}}{K^2} + \frac{2KK_1^2K_2}{K^3} \\ &= \frac{K_1^2K_2 - KK_1K_{12}}{K^2} = -KK_1\kappa, \end{aligned} \tag{17}$$

where, for the sake of brevity, we write simply K, K_1, K_{12}, etc., instead of $K(x,\overline{x})$, $K_1(x,\overline{x})$, $K_{12}(x,\overline{x})$, etc.

The second calculation we need is somewhat more patulous:

$$
\begin{aligned}
G_1(x,x) &= \frac{\partial}{\partial x}F(x,y)\Big|_{y=x} \\
&= \left(\frac{K_{11}K_2}{K} - \frac{K_1K_2K_1}{K^2}\right) - \left(\frac{K_1K_2K_1}{K^2} + \frac{KK_2K_{11}}{K^2} - \frac{KK_2K_1\,.\,2KK_1}{K^4}\right) \\
&\quad - \left(\frac{K_{11}KK_2}{K^2} + \frac{K_1KK_{12}}{K^2} - \frac{K_1KK_2\,.\,2KK_1}{K^4}\right) \\
&\quad - \left(\frac{K_1KK_{12}}{K^2} + \frac{KKK_{112}}{K^2} - \frac{KKK_{12}\,.\,2KK_1}{K^4}\right) \\
&\quad + 2\left(\frac{K_1KK_2K_1}{K^3} + \frac{KKK_{12}K_1}{K^3} + \frac{KKK_2K_{11}}{K^3} - \frac{KKK_2K_1\,.\,3K^2K_1}{K^6}\right) \\
&= \frac{K_{11}K_2}{K} - \frac{K_1^2K_2}{K^2} - \frac{K_1^2K_2}{K^2} - \frac{K_2K_{11}}{K} + 2\frac{K_1^2K_2}{K^2} \\
&\quad - \frac{K_2K_{11}}{K} - \frac{K_1K_{12}}{K} + 2\frac{K_1^2K_2}{K^2} - \frac{K_1K_{12}}{K} - K_{112} \\
&\quad + \frac{2K_1K_{12}}{K} + \frac{2K_1^2K_2}{K^2} + \frac{2K_1K_{12}}{K} + \frac{2K_{11}K_2}{K} - \frac{6K_1^2K_2}{K^2} \\
&= -K_{112} + \frac{2K_1K_{12}}{K} + \frac{K_{11}K_2}{K} - \frac{2K_1^2K_2}{K^2} \\
&= -K_{112} + \frac{K_{11}K_2}{K} + 2K_1K\left(-\frac{K_1K_2}{K^3} + \frac{K_{12}K}{K^3}\right) \\
&= 2KK_1\kappa - \frac{1}{K}\frac{\partial}{\partial x}(K^3\kappa),
\end{aligned} \tag{18}
$$

where we have used the same notation K, K_1, K_2, etc., as in formula (17). Combining (17) and (18), we can rewrite condition (16) as follows:

$$3KK_1\kappa - \frac{1}{K}\frac{\partial}{\partial x}(K^3\kappa) = 0,$$

or, after multiplying both sides by $K > 0$,

$$\left(\frac{\partial}{\partial x}K^3\right).\kappa - \frac{\partial}{\partial x}(K^3\kappa) = 0,$$

that is,

$$K^3\frac{\partial\kappa}{\partial x} = 0,$$

whence (remember that $K > 0$ owing to (3))

$$\frac{\partial \kappa}{\partial x} = 0 \qquad \text{on } \Omega. \tag{19}$$

Since κ is a real-valued function, we have

$$\frac{\partial \kappa}{\partial \overline{x}} = \overline{\frac{\partial \overline{\kappa}}{\partial x}} = \overline{\frac{\partial \kappa}{\partial x}} = 0$$

as well. Thus,

$$\boxed{\kappa(x) \equiv \text{const}}$$

throughout Ω, which completes the proof of Theorem A.

3. Sufficiency. Conversely, suppose that our domain $\Omega \subset \mathbf{C}$, equipped with the metric (7), is a complete Riemannian manifold of constant curvature κ. By the Killing-Hopf theorem (see, e.g., [20 , Theorem 2.4.10]), such a Riemannian manifold is conformally equivalent to one of the following quotient spaces:

(a) $(\kappa > 0)$ $\quad \mathbf{S}^2/\Gamma$, where $\Gamma \subset O(3)$,
(b) $(\kappa = 0)$ $\quad \mathbf{R}^2/\Gamma$, where $\Gamma \subset E(2)$,
(c) $(\kappa < 0)$ $\quad \mathbf{D}/\Gamma$, where $\Gamma \subset SU(1,1)$,

for some subgroup Γ acting freely and properly discontinuously on the corresponding space. $\mathbf{S}^2$ is the Gauss sphere, $\mathbf{R}^2 \simeq \mathbf{C}$ the plane, and $\mathbf{D}$ the unit disc; $O(3)$ is the orthogonal group, $E(2)$ the group of affine transformations on $\mathbf{R}^2$, and $SU(1,1)$ the group of all conformal automorphisms of $\mathbf{D}$ (Möbius transformations).

Consider two domains $\Omega_1, \Omega_2 \subset \mathbf{C}$, and let $\phi : \Omega_1 \to \Omega_2$ be a biholomorphic equivalence, i.e., a conformal mapping of Ω_1 onto Ω_2. The (real) Jacobian corresponding to the change of variables $z_2 = \phi(z_1)$ is equal to $|\phi'(z_1)|^2$. It follows that the mapping

$$f(z_2) \mapsto f(\phi(z_1))\phi'(z_1)$$

is a norm-preserving bijection of $A^2(\Omega_2)$ onto $A^2(\Omega_1)$. This implies the well-known formula for the change of variables in the Bergman kernel:

$$K_{\Omega_1}(x_1, \overline{y}_1) = K_{\Omega_2}(\phi(x_1), \overline{\phi(y_1)}) \cdot \phi'(x_1)\overline{\phi'(y_1)}. \tag{20}$$

Substituting this into formula (5), we see that ($x_2 = \phi(x_1)$, $y_2 = \phi(y_1)$)

$$\begin{aligned} B_{\Omega_1}(f \circ \phi)(y_1) &= \int_{\Omega_1} f(\phi(x_1)) \cdot \frac{K_{\Omega_1}(x_1,\overline{y}_1)K_{\Omega_1}(y_1,\overline{x}_1)}{K_{\Omega_1}(x_1,\overline{x}_1)K_{\Omega_1}(y_1,\overline{y}_1)} \cdot K_{\Omega_1}(x_1,\overline{x}_1)\,dx_1 \\ &= \int_{\Omega_1} f(\phi(x_1)) \cdot \frac{K_{\Omega_2}(\phi(x_1),\overline{\phi(y_1)})K_{\Omega_2}(\phi(y_1),\overline{\phi(x_1)})}{K_{\Omega_2}(\phi(x_1),\overline{\phi(x_1)})K_{\Omega_2}(\phi(y_1),\overline{\phi(y_1)})} \cdot \\ &\qquad \cdot K_{\Omega_2}(\phi(x_1),\overline{\phi(x_1)})\,|\phi'(x_1)|^2\,dx_1 \\ &= \int_{\Omega_2} f(x_2)\,\frac{K_{\Omega_2}(x_2,\overline{y}_2)K_{\Omega_2}(y_2,\overline{x}_2)}{K_{\Omega_2}(x_2,\overline{x}_2)K_{\Omega_2}(y_2,\overline{y}_2)} \cdot K_{\Omega_2}(x_2,\overline{x}_2)\,dx_2 \\ &= B_{\Omega_2}f(y_2), \end{aligned}$$

or

$$B_{\Omega_1}(f \circ \phi) = (B_{\Omega_2}f) \circ \phi.$$

In other words, the pullback of B_{Ω_1} is B_{Ω_2}, i.e., the Berezin transform is invariant under biholomorphic mappings. Similarly, in accordance with the chain rule,

$$\begin{aligned} \Delta_{\Omega_1}(f \circ \phi) &= \frac{1}{K_{\Omega_1}(y_1,\overline{y}_1)} \cdot \frac{\partial^2}{\partial y_1 \partial \overline{y}_1} f \circ \phi = \\ &= \frac{1}{K_{\Omega_2}(y_2,\overline{y}_2)\,|\phi'(y_2)|^2} \cdot |\phi'(y_2)|^2 \frac{\partial^2}{\partial y_2 \partial \overline{y}_2} f = \Delta_{\Omega_2} f(y_2), \end{aligned}$$

i.e., the operator Δ is also invariant. Of course, combining the last formula with (20), wee see that the curvature is invariant as well. Thus, any interrelations of the phenomena in question (such as B's commuting with Δ, B being a function of Δ, or κ being constant) will be preserved under biholomorphic equivalence. It follows that, for the purpose at hand, we can assume not only that Ω is conformally equivalent to one of the quotient spaces (a) through (c), but that Ω actually *coincides* with one of these spaces.

The first case (a) can never occur, since S^2, whence also S^2/Γ, is compact, while Ω is not. In the second case, further elaboration [20 , Section 2.5] shows that R^2/Γ is (biholomorphically equivalent to) one of the following spaces

(1) the entire complex plane;
(2) the cylinder, or the punctured plane;
(3) the torus;
(4) Klein's bottle;
(5) the Möbius strip.

(4) and (5) cannot occur, since these manifolds are not orientable (whereas Ω is). The torus is compact, unlike Ω, so (3) cannot occur either. This leaves but (1) and (2), which, however, do not satisfy our condition that the space $A^2(\Omega)$ be nontrivial (they belong to O_G). We conclude that our domain Ω is isometric to $\mathbf{D}/\Gamma$, where Γ is a subgroup of $SU(1,1)$ that acts freely and properly discontinuously on $\mathbf{D}$.

Mutatis mutandis, this means that Ω is a domain of hyperbolic type and the metric (7) coincides with the Poincaré metric (see [15] or [9]) on Ω. Recall that the upper half-plane $\mathbf{U}$ is conformally equivalent to $\mathbf{D}$, and the Poincaré metric on $\mathbf{U}$ is given by

$$ds^2 = \frac{1}{(\operatorname{Im}\xi)^2}\,|d\xi|^2 = -\frac{4}{(\xi-\overline{\xi})^2}\,|d\xi|^2, \qquad \xi \in \mathbf{U}.$$

If $\phi : \mathbf{U} \to \Omega$ is the covering map from $\mathbf{U} \simeq \mathbf{D}$ onto $\Omega \simeq \mathbf{D}/\Gamma$, the Poincaré metric on Ω is given by

$$ds^2 = \frac{1}{|\phi'(\xi)|^2\,(\operatorname{Im}\xi)^2}\,|dz|^2, \qquad \text{where } \xi \in \mathbf{U}, z = \phi(\xi) \in \Omega. \tag{21}$$

Naturally, this expression is independent of the choice of $\xi \in \phi^{-1}(z)$. Indeed, if ω is any conformal automorphism of $\mathbf{U}$, then

$$\omega(\xi) = \frac{a\xi+b}{c\xi+d}, \qquad \text{with } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{R}),$$

and

$$2\operatorname{Im}\omega(\xi) = \frac{a\xi+b}{c\xi+d} - \frac{a\overline{\xi}+b}{c\overline{\xi}+d} = \frac{(ad-bc)(\xi-\overline{\xi})}{|c\xi+d|^2} = \frac{2\operatorname{Im}\xi}{|c\xi+d|^2}.$$

On the other hand, $\omega \in \Gamma$ implies $\phi \circ \omega = \phi$, whence

$$\phi'(\xi) = (\phi\circ\omega)'(\xi) = \phi'(\omega(\xi))\omega'(\xi) = \phi'(\omega(\xi)) \cdot \frac{1}{(c\xi+d)^2}, \tag{22}$$

so that, indeed,

$$|\phi'(\omega(\xi))|\operatorname{Im}\omega(\xi) = |\phi'(\xi)|\operatorname{Im}\xi \qquad \forall \omega \in \Gamma.$$

Now the requirement that the Poincaré metric (21) coincide with the one given by (7) means that[4]

$$K(z,\overline{z}) = |\phi'(\xi)|^{-2}(\xi-\overline{\xi})^{-2}, \qquad \xi \in \mathbf{U}, \quad z = \phi(\xi) \in \Omega.$$

[4]Modulo an inessential constant factor.

Since a holomorphic function of two complex variables is uniquely determined by its restriction to the anti-diagonal $\{(\xi,\zeta): \zeta=\overline{\xi}\}$ (see, for instance, [11 , Proposition 1]), we must actually have

$$K(x,\overline{y}) = \frac{1}{\phi'(\xi)\overline{\phi'(\zeta)}(\xi-\overline{\zeta})^2}, \qquad \xi,\zeta\in\mathbf{U}, \quad x=\phi(\xi) \text{ and } y=\phi(\zeta)\in\Omega.$$

It follows that the function

$$F(\xi,\zeta) := \phi'(\xi)\overline{\phi'(\zeta)}(\xi-\overline{\zeta})^2, \qquad \xi,\zeta\in\mathbf{U},$$

must satisfy the invariance condition

$$F(\xi,\zeta) = F(\omega(\xi),\omega_1(\zeta)) \qquad \forall\xi,\zeta\in\mathbf{U},\ \forall\omega,\omega_1\in\Gamma.$$

By (22), this can be rewritten as

$$(\omega(\xi)-\overline{\omega_1(\zeta)})^2 = (\xi-\overline{\zeta})^2\omega'(\xi)\overline{\omega_1'(\zeta)}.$$

In particular, for $\omega_1 =$id,

$$\left(\frac{\omega(\xi)-\overline{\zeta}}{\xi-\overline{\zeta}}\right)^2 = \omega'(\xi) \qquad \forall\xi,\zeta\in\mathbf{U},\ \omega\in\Gamma. \tag{23}$$

Letting $\zeta\to\infty\in\partial\mathbf{U}$, we get, by continuity,

$$\omega'(\xi)=1 \qquad \forall\xi\in\mathbf{U},\ \omega\in\Gamma,$$

and letting $\zeta\to 0\in\partial\mathbf{U}$ we get

$$\frac{\omega(\xi)^2}{\xi^2} = \omega'(\xi) \qquad \forall\xi\in\mathbf{U},\ \omega\in\Gamma.$$

Combining the last two formulas, we see that (23) can only be fulfilled when $\Gamma=\{\text{id}\}$, i.e., when $\phi:\mathbf{U}\to\Omega$ is biholomorphic and Ω is conformally equivalent to $\mathbf{U}$, or, which is the same, to $\mathbf{D}$. As we have already observed, the reproducing kernel K, the Laplace-Beltrami operator[5] Δ, and the Berezin transform B behave nicely under

[5] In this case also known as the *invariant Laplacian.*

conformal equivalence, i.e., the pullback via ϕ of each of these objects on Ω will be the corresponding object on $\mathbf{D}$. Since relation (10), i.e.,

$$B = F(\Delta),$$

is satisfied on $\mathbf{D}$ with the function F given by (11), it follows that this relation persists on Ω as well. This completes the proof of Theorem B.

Remark. It is not clear to what extent the completeness requirement is restrictive in Theorem B. A typical example of a domain Ω not complete under the metric (7) can be obtained by taking a complete domain Ω_1, and deleting from it a sufficiently "thin" piece (e.g., a single interior point). Then the original domain Ω_1 can be recovered from Ω by attaching the so-called *regular boundary* of Ω. See [1] for more on this topic. The author does not know whether all incomplete domains in $\mathbf{C}$ arise in this way.

4. Weights. Now, suppose that $w > 0$ is a positive measurable function on Ω, and consider the weighted Bergman space $A^2(\Omega, w)$ of all holomorphic functions on Ω square-integrable against the measure $w(x)\,dx$. We shall assume that Ω and w are such that

for each $x \in \Omega$, there exists $f \in A^2(\Omega, w)$ such that $f(x) \neq 0$; (24)

for each compact set $K \subset \Omega$, there exists $C_K < +\infty$ such that

$$|f(x)| \leqslant C_K \cdot \int_\Omega |f(y)|^2\, w(y)\,dy \qquad \forall f \in A^2(\Omega, w),\ x \in K. \tag{25}$$

It follows from (24) that, in particular, $A^2(\Omega, w) \neq \{0\}$; (25) says that the point evaluations are bounded, and uniformly so on the compact subsets of Ω. It is easy to see that (24) and (25) are satisfied, for instance, if w is Lebesgue integrable over Ω and bounded away from zero on compact subsets of Ω. Indeed, then (24) is fulfilled since the constant functions belong to $A^2(\Omega, w)$, and (25) is a simple consequence of (the area version of) the mean value theorem for analytic functions and the Schwarz inequality.

Under condition (25), slight modifications in the standard proof of the existence of the Bergman kernel function (see [13 , Section VIII.3], or [7]) show that $A^2(\Omega, w)$ is a closed subspace of $L^2(\Omega, w(x)\,dx)$ — hence, a Hilbert space — which admits a reproducing kernel: for each $x \in \Omega$, there is $K_x \in A^2(\Omega, w)$ such that

$$f(x) = \langle f, K_x \rangle \qquad \forall f \in A^2(\Omega, w).$$

The reproducing kernel

$$K(x,\overline{y}) := \langle K_y, K_x \rangle = K_y(x)$$

is a holomorphic function on $\Omega \times \overline{\Omega}$, and it follows from (24) that

$$K(x,\overline{x}) > 0 \qquad \forall x \in \Omega. \tag{26}$$

Parallelling our developments for the unweighted case, we define the *Berezin transform* B by the formula

$$Bf(y) := \int_\Omega f(x)\, \frac{K(x,\overline{y})K(y,\overline{x})}{K(y,\overline{y})}\, w(x)\, dx.$$

It is easy to see that the integral exists, for instance, for any $f \in L^\infty(\Omega)$. B is formally selfadjoint with respect to the measure

$$K(x,\overline{x})\, w(x)\, dx. \tag{27}$$

Thus, it is natural to consider the differential operator

$$\Delta := \frac{1}{K(x,\overline{x})w(x)}\, \frac{\partial^2}{\partial x \partial \overline{x}},$$

which is also formally selfadjoint with respect to (27), and ask whether B is not a function of Δ. We have the following analog of Theorem A.

Theorem C. *A necessary condition for B and Δ to commute is given by*

$$\Delta \log K(x,\overline{x}) \equiv \textit{const}$$

throughout Ω.

Proof. The proof is entirely similar to that of Theorem A, so we shall proceed a little more quickly than in §2. For $f \in \mathcal{D}(\Omega)$, due expressions for ΔBf and $B\Delta f$ can be obtained as in (12) and (13), with dx replaced by $w(x)\, dx$. Thus, a necessary condition for $\Delta B = B\Delta$ is given by

$$\Delta_x \frac{K(x,\overline{y})K(y,\overline{x})}{K(x,\overline{x})K(y,\overline{y})} = \Delta_y \frac{K(x,\overline{y})K(y,\overline{x})}{K(x,\overline{x})K(y,\overline{y})},$$

or, upon multiplying both sides by $K(x,\overline{x})K(y,\overline{y}) > 0$,

$$\frac{1}{w(x)}\frac{\partial^2}{\partial x\partial\overline{x}}\frac{K(x,\overline{y})K(y,\overline{x})}{K(x,\overline{x})} = \frac{1}{w(y)}\frac{\partial^2}{\partial y\partial\overline{y}}\frac{K(x,\overline{y})K(y,\overline{x})}{K(y,\overline{y})}.$$

Denoting the left-hand side by $F_w(x,y)$, we rewrite this as follows:

$$F_w(x,y) = F_w(y,x). \tag{28}$$

Putting $G_{w1}(x,y) := \frac{\partial}{\partial x}F_w(x,y)$ and $G_{w2}(x,y) := \frac{\partial}{\partial y}F_w(x,y)$, we get from (28)

$$G_{w1}(y,x) = G_{w2}(x,y),$$

whence, in particular,

$$G_{w1}(x,x) = G_{w2}(x,x) \qquad \forall x \in \Omega. \tag{29}$$

Now,

$$G_{w2}(x,y) = \frac{\partial}{\partial y}\left(\frac{1}{w(x)}F(x,y)\right) = \frac{1}{w(x)}G_2(x,y),$$

$$G_{w1}(x,y) = \frac{\partial}{\partial x}\left(\frac{1}{w(x)}F(x,y)\right) = \frac{1}{w(x)}G_1(x,y) - \frac{1}{w(x)^2}\frac{\partial w}{\partial x}F(x,y),$$

where F, G_1 and G_2 have the same meaning as in §2. Substituting in (29), we get

$$\frac{1}{w(x)}\left[G_1(x,x) - G_2(x,x)\right] = \frac{1}{w(x)^2}\frac{\partial w}{\partial x}F(x,x). \tag{30}$$

Using formula (16a), we have

$$\begin{aligned} F(x,x) &= \frac{K_1K_2}{K} - \frac{KK_1K_2}{K^2} - \frac{K_1KK_2}{K^2} - \frac{KKK_{12}}{K^2} + \frac{2KKK_2K_1}{K^3} \\ &= \frac{K_1K_2}{K} - \frac{K_1K_2}{K} - \frac{K_1K_2}{K} - K_{12} + 2\frac{K_1K_2}{K} = \frac{K_1K_2}{K} - K_{12} \\ &= -K^2\kappa, \end{aligned} \tag{31}$$

where K_1, K_2, etc., have the same meaning as in (17) and (18), and

$$\kappa = \frac{K_{12}K - K_1K_2}{K^3} = \frac{1}{K}\frac{\partial^2}{\partial x\partial\overline{x}}\log K,$$

as in (8). Substituting (17), (18), and (31) in (30), we obtain

$$\frac{1}{w}\cdot\left[3KK_1\kappa-\frac{1}{K}\frac{\partial}{\partial x}(K^3\kappa)\right]=-K^2\kappa\cdot\frac{1}{w^2}\frac{\partial w}{\partial x},$$

or

$$\frac{1}{Kw}\cdot\left(-K^3\frac{\partial\kappa}{\partial x}\right)=-K^2\kappa\cdot\frac{1}{w^2}\frac{\partial w}{\partial x}.$$

Division of both sides by $-K^2\kappa/w$ yields

$$\frac{1}{\kappa}\frac{\partial\kappa}{\partial x}=\frac{1}{w}\frac{\partial w}{\partial x},$$

that is,

$$\frac{\partial}{\partial x}\left(\frac{\kappa}{w}\right)=0.$$

Since κ/w is real valued, this again implies that

$$\frac{\partial}{\partial\overline{x}}\left(\frac{\kappa}{w}\right)=0\qquad\text{as well,}$$

so, finally,

$$\boxed{\frac{\kappa}{w}\equiv\text{const}}\tag{32}$$

throughout Ω, which completes the proof.

Example 1. Take $\Omega=\mathbf{D}$, the unit disc, and $w(x)=(1-|x|^2)^\alpha$, $\alpha>0$. Then (see, e.g., [3, §4])

$$K(x,\overline{y})=\frac{\alpha+1}{\pi}(1-x\overline{y})^{-\alpha-2},$$

so

$$\frac{\partial^2}{\partial x\partial\overline{x}}\log K(x,\overline{x})=\frac{\alpha+2}{(1-x\overline{x})^2},$$

$$\kappa=\pi\frac{\alpha+2}{\alpha+1}(1-x\overline{x})^\alpha=\pi\frac{\alpha+2}{\alpha+1}\cdot w,$$

and we see that the condition of Theorem C is satisfied.

Example 2. Take $\Omega = \mathbf{C}$, the entire complex plane, and $w(x) = e^{-\alpha|x|^2}$. The space $A^2(\Omega, w)$ is then the Fock (or Segal-Bargmann) space on $\mathbf{C}$. It can be shown that (see [6])

$$K(x,\overline{y}) = \frac{1}{2\pi} e^{\alpha x \overline{y}},$$

$$\frac{\partial^2}{\partial x \partial \overline{x}} \log K(x,\overline{x}) = \alpha,$$

so

$$\kappa = 2\pi\, e^{-\alpha x \overline{x}} \cdot \alpha = 2\pi\alpha \cdot w,$$

and, again, the condition of Theorem C is fulfilled.

We finish this section by mentioning a relationship between our results and the recent work of Cahen, Gutt, and Rawnsley on geometric quantization on Kähler manifolds. Let us briefly recall the relevant notions from [8]. Consider a Kähler manifold M, with Kähler form ω, and let L be a Hermitian complex line bundle over M with a connection ∇ that leaves the Hermitian structure h of L invariant. The latter condition means that

$$X(h(s,s')) = h(\nabla_X s, s') + h(s, \nabla_X s')$$

for all smooth sections s, s' of L and all vector fields X on M. Let R be the curvature form of (L, ∇). Then L is called a *prequantization bundle* if

$$R = -2\pi i\, \omega. \tag{33}$$

If we introduce a positive Kähler polarization on M (see [8] for details), L acquires a natural structure of a holomorphic line bundle, and we let $\mathcal{H}$ be the space of holomorphic sections s of L whose norm is square-integrable against the Liouville measure $\omega^n/n!$:

$$\|s\|^2_{\mathcal{H}} := \int_M h(s,s)\, \frac{\omega^n}{n!} < +\infty.$$

For s, $s' \in \mathcal{H}$, we denote by $\langle s, s' \rangle$ the scalar product obtained by polarizing $\|s\|_{\mathcal{H}}$. For $x \in M$, choose some $q \in L_x \setminus \{0\}$ and let s be an element of $\mathcal{H}$. Since the fiber L_x of L is one-dimensional, evaluation of s at x gives a multiple $l_q(s)$ of q, and, since s is holomorphic, $l_q(s)$ is a continuous linear functional of s. Thus, there exists $e_q \in \mathcal{H}$ such that

$$l_q(s) = \langle s, e_q \rangle.$$

It is easy to see that the expression

$$h_x(q,q)\cdot\|e_q\|^2_{\mathcal{H}}$$

is independent of the choice of q and of the local coordinate system at x, and, consequently, defines a global function θ on M :

$$\theta(x) := h_x(q,q)\cdot\|e_q\|^2_{\mathcal{H}}, \qquad x\in M,\ q\in L_x\setminus\{0\}.$$

Under the conditions (33) and

$$\theta\equiv \text{ const} \qquad \text{throughout } M, \tag{34}$$

the authors of [8] obtained a number of interesting results about both geometric quantization and Berezin quantization on compact Kähler manifolds M. We are going to relate conditions (33) and (34) to our condition (32).

Thus, we take a domain $\Omega\subset\mathbf{C}$ for M. Since M is of complex dimension one, it is automatically a Kähler manifold (any 2-form on M is a Kähler form). Further, since $M\subset\mathbf{C}$, we can afford the luxury of working with a single global coordinate chart, and identify the bundle L with $M\times\mathbf{C}$. The Hermitian structure is then given by

$$h_x(s,s') = e(x)\cdot s(x)\overline{s'(x)},$$

where the sections s, s' are identified with holomorphic functions on M, and the metric coefficient $e(x)>0$ is a smooth function of $x\in M$. The corresponding curvature form is then given by

$$R = -2\pi\frac{\partial^2}{\partial z\partial\overline{z}}\log e(z)\cdot dz\wedge d\overline{z},$$

and condition (33) reads simply as follows:

$$\omega = -i\frac{\partial^2}{\partial z\partial\overline{z}}\log e(z)\cdot dz\wedge d\overline{z} = -\frac{1}{2}\frac{\partial^2}{\partial z\partial\overline{z}}\log e(z)\,dx\wedge dy \qquad (z=x+yi).$$

The space $\mathcal{H}$ may be identified with the space of holomorphic functions s on Ω that satisfy

$$\|s\|^2_{\mathcal{H}} = \int_\Omega |s(z)|^2\,e(z)\,\omega(z) < +\infty.$$

In other words, $\mathcal{H}$ is the Bergman space $A^2(\Omega, w)$, where

$$w(z) = -\frac{1}{2}e(z)\,\frac{\partial^2}{\partial z \partial \overline{z}} \log e(z). \tag{35}$$

In the definition of θ, we may take $q = 1 \in \mathbf{C} \simeq L_x$. Then e_q will be precisely our reproducing kernel of the space $A^2(\Omega, w)$, and so

$$\theta(x) = e(x) \cdot \|K(\,\cdot\,, \overline{x})\|^2_{A^2(\Omega,w)}.$$

Thus, the constancy of θ is equivalent to

$$e(x) \cdot K(x, \overline{x}) \equiv \text{ const} \qquad \text{throughout } \Omega. \tag{36}$$

Example 3. Take $\Omega = M = \mathbf{D}$ and $e(z) = \frac{2}{\alpha+2}(1 - z\overline{z})^{\alpha+2}$. Then

$$w(z) = -\frac{1}{2}e(z)\frac{\partial^2}{\partial z \partial \overline{z}} \log e(z) = (1 - z\overline{z})^{\alpha},$$

which is the familiar weight from example 1; hence,

$$\theta(z) = e(z)K(z, \overline{z}) = \frac{2}{\alpha + 2} \cdot \frac{\alpha + 1}{\pi},$$

and (36) is satisfied.

Example 4. $\Omega = M = \mathbf{C}$ and $e(z) = \frac{2}{\alpha}e^{-\alpha z\overline{z}}$. Then

$$w(z) = -\frac{1}{2}e(z)\frac{\partial^2}{\partial z \partial \overline{z}} \log e(z) = -\frac{1}{\alpha}e^{-\alpha z\overline{z}} \cdot (-\alpha) = e^{-\alpha z\overline{z}},$$

which is the weight from example 2; and

$$\theta(z) = e(z)K(z, \overline{z}) = \frac{2}{\alpha} \cdot \frac{1}{2\pi}$$

is again a constant function.

Now we have the following result.

Proposition. *Conditions* (35) *and* (36) *imply our condition* (32).

Proof. We have

$$\begin{aligned}\frac{\partial^2}{\partial z\partial\bar z}\log K(z,\bar z) &= -\tfrac{\partial^2}{\partial z\partial\bar z}\log e(z) && \text{by (36)}\\ &= \tfrac{2w(z)}{e(z)} && \text{by (35)}\\ &= \tfrac{2}{\theta}\cdot w(z)K(z,\bar z) && \text{by (36),}\end{aligned}$$

so

$$\frac{\kappa(z)}{w(z)}=\frac{2}{\theta}\equiv \text{ const,}$$

which completes the proof.

Problem. When is the converse of the latter proposition also valid ? That is, under what conditions do (35) and (32) imply (36) ?

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