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CORRELATION ASYMPTOTICS AND WITTEN LAPLACIANS

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§0. Introduction

In an earlier work [HS2], B. Helffer and the author studied correlations associated with the measure $e^{-2\phi(x)/h}dx$ on $\mathbb{R}^m$, where ϕ is a smooth convex function, with special attention to the limit as m tends to infinity. We observed (under suitable assumptions) that the expectation $\langle u \rangle$ of a function $u(x)$ can be obtained by solving the equation

$$u - \langle u \rangle = (-h^2\Delta + 2\nabla\phi \cdot h\partial_x)w, \qquad (0.1)$$

with w growing not too fast near infinity. This equation was mainly treated by means of maximum principles as in [S1] (originally inspired by an appendix in Singer-Wong-Yau-Yau [SiWYY]). See also [S2]. Actually, we did not work very much with (0.1), but rather with differentiated versions of the same equation. L^2-methods played only a minor role in [HS2], but we did notice that (0.1) becomes a Schrödinger equation after conjugation (see (1.8) below). Using integration by parts and the results about (0.1) just mentioned, we were able to get exponential decay upper bounds on the correlations $\mathrm{Cor}(x_j, x_k)$ of x_j and x_k, for large $|j-k|$ and, consequently, large m. In [S2] these technique and results were further improved and even some "critical" (non-uniformly convex) cases were studied, and we then established the power decay of the correlations.

In September 1993, R. Minlos gave a very inspiring lecture at a meeting at the Euler Institute, which included results on the asymptotics of correlations for Ising type models with high temperature. The reference Zhizhina-Minlos [ZM], was also inspiring, although I was unable to understand the details.

In the present paper we use more L^2-methods and avoid excessive use of the maximum principle. A new observation (at least for the present author) is that the operator in (0.1) is a conjugated version of a Witten Laplacian in degree 0 (see [W], [HS3]) and that the differentiated versions of (0.1) at least in some cases involve Witten Laplacians $\Delta_\phi^{(\ell)}$ of higher degree ℓ . In [S3], such a Laplacian of degree 1 was implicitly used to get a lower bound for the gap between the first and the second eigenvalue of a Schrödinger operator. As is clear for instance from [HS2], such a lower bound is closely related to lower bounds for the rate of the exponential decay of certain correlations. A very natural idea (present in a different form in [ZM]) is that in order to get more precise asymptotic results on the correlations, one must analyze the spectrum of the Witten Laplacian in degree 0 a little above the first spectral gap, and the main achievement of the present paper is in making a step in that direction. (Much more complete results are perfectly conceivable and might be the object of future investigations.)

The plan of the paper is as follows: In Section 1 we recall some standard facts about Witten Laplacians and establish formula (1.11) for the correlations, with ϕ satisfying only some rather weak condition. A less explicit form of (1.11) was introduced in [HS2] and used in [S2]; it involves the Laplacian $\Delta_\phi^{(1)}$ which is the main object of study in the present paper. In Section 1, we use (1.11) to give a short and easy proof of a slightly special case of the FKG-inequalities. This proof is reminiscent of a similar one in Herbst-Pitt [HeP] and improves one in [HS2]; the latter proofs worked only under a more restricive convexity assumption.

In Section 2 we study the lower part of the spectrum of $\Delta_\phi^{(1)}$ via a so-called Grushin problem reduction, and in Section 3 we add exponential estimates to this reduction. The end result here is Theorem 3.1, which permits us to discuss the asymptotic properties of $(\Delta_\phi^{(1)})^{-1}$. An important object is the $m \times m$-matrix $E_{-+}(0)$, which later will be of convolution type. Starting with Section 2, we work in the semiclassical limit, i.e., we assume that h is small.

In Section 4, we study the inverses of certain convolution matrices, and this section can be viewed as a partial improvement of a similar one in [S2]. Similar considerations are clearly present in [ZM].

In Section 5 we combine the earlier results in the case where ϕ is strictly convex and invariant under cyclic permutation of coordinates (now indexed over $(\mathbb{Z}/N\mathbb{Z})^d$, so that $m = N^d$). The main result of the paper is Theorem 5.1 which describes the asympotics of the exponential decay of the correlations between x_j and x_k as $|j-k| \to \infty$. Since we have not really studied the thermodynamical limit for the correlations, the theorem allows for small $\mathcal{O}(h^{1/2})$ oscillations, which should not be there in realistic models. (In principle, the thermodynamical limit could be taken, but would probably involve estimates for $\Delta_\phi^{(2)}$, which would lengthen the paper considerably.) As an example, we treat the Kac potential

$$2\phi = \sum_{j \in \mathbb{Z}/N\mathbb{Z}} \frac{x_j^2}{4} - \operatorname{ch} \sqrt{\frac{\nu}{2}}(x_j + x_{j+1})$$

in the strictly convex case where $\nu > 1/4$. Our result shows, in particular, that the lower bounds on the rates of decay obtained in [HS2, S2] are roughly sharp.

We are greatful to R. Minlos for giving the original impulse to this work. Discussions with T. Spencer in Spring 1993 were also very useful and stimulated us to try to go beyond the methods used then. Thanks are also due to B. Helffer for discussions and for indicating the reference [HeP], as well as to I. M. Sigal and V. Bach for discussions. Finally, I would like to thank V. Buslaev and the staff of the Euler Institute for organizing a very stimulating and pleasant meeting.

§1. Some general results

In this section the dimension m is fixed. Let $\phi \in C^\infty(\mathbb{R}^m;\mathbb{R})$ be a function satisfying for some $\delta \in]0,1]$, the conditions

$$|\phi'(x)| = \mathcal{O}(\langle x\rangle^\delta),\ \phi'(x)\cdot x \sim |x|^{1+\delta} \quad \text{for } |x| \gg 1,$$
$$\phi''(x) = \mathcal{O}(1),\ \phi^{(\alpha)} = \mathcal{O}(\langle x\rangle^{N(\alpha)}),\ |\alpha| > 2, \quad \text{where } \langle x\rangle = \sqrt{1+x^2}. \tag{H1}$$

Observe that $\phi(x) \sim |x|^{1+\delta}$, $|x| \gg 1$. In what follows we assume that $0 < h \leqslant 1$. Adding an h-dependent constant to ϕ, we may assume that

$$\int e^{-2\phi(x)/h} dx = 1. \tag{1.1}$$

We introduce the Witten complex (cf Witten [W], Helffer-Sjöstrand [HS3]),

$$d_\phi = e^{-\phi/h} h d e^{\phi/h} = hd + d\phi^\wedge = \sum_1^m Z_j dx_j^\wedge, \tag{1.2}$$

where $d = \sum_1^m \partial_{x_j} dx_j^\wedge$ denotes the exterior differentiation, and $Z_j = h\partial_{x_j} + \partial_{x_j}\phi = e^{-\phi/h} h\partial_{x_j} e^{\phi/h}$ can be viewed as an annihilation operator. Let $d_\phi^* = e^{\phi/h} h d^* e^{-\phi/h} = \sum_1^m Z_j^* dx_j^\rfloor$ be the formal complex adjoint. The Witten Laplacian is the Hodge Laplacian associated with d_ϕ:

$$\Delta_\phi = d_\phi^* d_\phi + d_\phi d_\phi^*. \tag{1.3}$$

We note that $d_\phi \Delta_\phi = \Delta_\phi d_\phi$, $d_\phi^* \Delta_\phi = \Delta_\phi d_\phi^*$. If we let $\Delta_\phi^{(\ell)}$ be the restriction of Δ_ϕ to forms of degree ℓ, we obtain more precisely, the relations $d_\phi \Delta_\phi^{(\ell)} = \Delta_\phi^{\ell+1} d_\phi$, $d_\phi^* \Delta_\phi^{(\ell+1)} = \Delta_\phi^{(\ell)} d_\phi^*$. We have

$$\Delta_\phi^{(0)} = d_\phi^* d_\phi = \sum Z_j^* Z_j = -h^2\Delta + \|d\phi\|^2 - h\Delta\phi, \tag{1.4}$$

where Δ is the standard Laplacian. More generally, we have

$$\begin{aligned}\Delta_\phi &= \sum\sum(Z_j Z_k^* dx_j^\wedge dx_k^\rfloor + Z_k^* Z_j dx_k^\rfloor dx_j^\wedge)\\ &= \sum\sum Z_k^* Z_j (dx_j^\wedge dx_k^\rfloor + dx_k^\rfloor dx_j^\wedge) + [Z_j, Z_k^*] dx_j^\wedge dx_k^\rfloor\\ &= \sum Z_j^* Z_j + 2h \sum\sum (\partial_{x_j}\partial_{x_k}\phi) dx_j^\wedge dx_k^\rfloor\\ &= \Delta_\phi^{(0)} \otimes I + 2h\sum\sum(\partial_{x_j}\partial_{x_k}\phi)dx_j^\wedge dx_k^\rfloor.\end{aligned}$$

In particular, identifying the 1-forms with $\mathbb{R}^m$-valued functions, we get

$$\Delta_\phi^{(1)} = \Delta_\phi^{(0)} \otimes I + 2h\phi''. \tag{1.5}$$

Also if we identify the 2-forms with functions valued in the space of real antisymmetric $(m \times m)$ matrices, we have

$$\Delta_\phi^{(2)} u = \Delta_\phi^{(0)} u + 2h(\phi'' \circ u + u \circ \phi''). \tag{1.6}$$

The operators $\Delta_\phi^{(\ell)}$, $0 \leqslant \ell \leqslant 2$, were used in [S3, HS2, S2], though the link with the Witten formalism was unnoticed.

Since $(\Delta_\phi^{(\ell)} u|u) \geqslant 0$ for $u \in C_0^\infty(\mathbb{R}^m; \wedge^\ell \mathbb{R}^m)$, we can define $\Delta_\phi^{(\ell)}$ as a selfadjoint operator by taking the Friedrichs extension. Using (H1), we also see that $\Delta_\phi^{(\ell)}$ has a discrete spectrum contained in $[0, \infty[$. The lowest eigenvalue of $\Delta_\phi^{(0)}$ is zero and a corresponding eigenfunction is $e^{-\phi/h}$, since this function is annihilated by d_ϕ. This eigenvalue is simple, for if u is another eigenfunction associated with the same eigenvalue, then $0 = (\Delta_\phi^{(0)} u|u) = \|d_\phi u\|^2$; hence, $d_\phi u = 0$, which means precisely that u is a multiple of the function $e^{-\phi/h}$.

Next, we show that the lowest eigenvalue of $\Delta_\phi^{(1)}$ is > 0. Indeed, assume that $u \in \mathcal{D}(\Delta_\phi^{(1)})$ and that $\Delta_\phi^{(1)} u = 0$. Then we have $d_\phi u = 0$, $d_\phi^* u = 0$. From the first equation we see that $e^{\phi/h} u = h d e^{\phi/h} v$, $u = d_\phi v$, where v is given by the formula

$$v(x) = \frac{1}{h}\int_0^x e^{-(\phi(x)-\phi(y))/h} u(y) \quad \text{(line integral)}. \tag{1.7}$$

From (H1) we derive (by using weighted Lithner-Agmon estimates as in [HS1]) that $u \in \mathcal{S}(\mathbb{R}^m)$, then, using (1.7) and (H1) once more, we see that $v \in \mathcal{S}(\mathbb{R}^m)$. Since $d_\phi^* u = 0$, we get $\Delta_\phi^{(0)} v = 0$, which implies that $v = \mathrm{Const}.e^{-\phi/h}$. Consequently, $u = d_\phi v = 0$.

Remark 1.1. If $\lambda > 0$ is the smallest eigenvalue of $\Delta_\phi^{(1)}$, then the gap between the first eigenvalue ($= 0$) and the second one is $\geqslant \lambda$. Indeed, let $\mu > 0$ be the second eigenvalue of $\Delta_\phi^{(0)}$, and let $u \in \mathcal{S}(\mathbb{R}^m)$ be a corresponding eigenfunction. Then $0 \neq d_\phi u \in \mathcal{S}(\mathbb{R}^m; \mathbb{R}^m)$ is an eigenform of $\Delta_\phi^{(1)}$ with eigenvalue μ; so, $\mu \geqslant \lambda$. To complete the remark, we notice that if $-h^2\Delta + V$ is a given Schrödinger operator with lowest eigenvalue ν and corresponding eigenfunction $e^{-\phi/h}$, then $-h^2\Delta + V - \nu = \Delta_\phi^{(0)}$. (Cf. [S3].)

Assume that $u \in L^2_{\text{loc}}(\mathbb{R}^m)$ and $e^{-\phi/h}u \in L^2$; we write $e^{-\phi/h}u = (e^{-\phi/h}u|e^{-\phi/h})e^{-\phi/h} + \Delta_\phi^{(0)}(\widetilde{u})$ for some $\widetilde{u} \in \mathcal{D}(\Delta_\phi^{(0)})$. This identity can also be written in the form

$$e^{-\phi/h}u = \langle u \rangle e^{-\phi/h} + \Delta_\phi^{(0)}\widetilde{u} \quad \text{for some } \widetilde{u} \in \mathcal{D}(\Delta_\phi^{(0)}), \tag{1.8}$$

where $\langle u \rangle = \int u(x)e^{-2\phi(x)/h}dx$ is the expectation value of u with respect to the normalized measure $e^{-2\phi/h}dx$.

Let $u, v \in H^1_{\text{loc}}(\mathbb{R}^m)$ satisfy $e^{-\phi/h}u$, $e^{-\phi/h}v \in L^2$, $d_\phi e^{-\phi/h}u$, $d_\phi e^{-\phi/h}v \in L^2$. We consider the correlation

$$\text{Cor}(u,v) = \langle (u - \langle u \rangle)(v - \langle v \rangle) \rangle = (e^{-\phi/h}(u - \langle u \rangle)|e^{-\phi/h}(v - \langle v \rangle)). \tag{1.9}$$

Let $\widetilde{u}$ be a solution of (1.8). Then

$$\text{Cor}(u,v) = (d_\phi \widetilde{u}|d_\phi(e^{-\phi/h}v)). \tag{1.10}$$

Applying d_ϕ to (1.8), we get $d_\phi(e^{-\phi/h}u) = \Delta_\phi^{(1)} d_\phi \widetilde{u}$. Here we know that the LHS is in L^2 and that $d_\phi \widetilde{u}$ is in L^2. In the case where u has compact support, we can use weighted L^2 estimates to see that $d_\phi \widetilde{u}$ is in the domain of $\Delta_\phi^{(1)}$; hence, $d_\phi \widetilde{u} = (\Delta_\phi^{(1)})^{-1} d_\phi(e^{-\phi/h}u)$. This leads to the formula

$$\begin{aligned}\text{Cor}(u,v) &= ((\Delta_\phi^{(1)})^{-1} d_\phi(e^{-\phi/h}u)|d_\phi(e^{-\phi/h}v)) \\ &= h^2((\Delta_\phi^{(1)})^{-1} e^{-\phi/h}du|e^{-\phi/h}dv).\end{aligned} \tag{1.11}$$

By a density argument it is easy to eliminate the assumption that u is compactly supported in this formula. (In [HS2, S2], (1.11) was used in a less explicit form.)

This formula, will be used for recovering a slightly special case of the FKG inequalities ([FKG], see also [C]) on the positivity of correlations in the ferromagnetic case where

$$\partial_{x_j}\partial_{x_k}\phi \leqslant 0, \quad j \neq k. \tag{1.12}$$

As a preparation, we first consider the heat equation

$$(\partial_t + (-\Delta + V))u = 0, \ t \geqslant 0, \quad u(0,x) = v(x), \tag{1.13}$$

on $M \stackrel{\text{def}}{=} (\mathbb{R}/R\mathbb{Z})^m$, where $V = (v_{j,k}(x))_{1\leqslant j,k\leqslant \tilde{m}}$ is a real symmetric matrix depending smoothly on x and satisfying $v_{j,k} \leqslant 0$ for $j \neq k$.

Lemma 1.2. *If $v \geqslant 0$ componentwise, then the same is true for u.*

Proof. Writing the heat equation for $e^{\lambda t}u$, we see that the $v_{j,j}$ can be replaced by $v_{j,j} - \lambda$; choosing λ sufficiently large, we may assume that $v_{j,j} < 0$. We may also assume that $v > 0$ componentwise. It follows that $u(t,x) > 0$ componentwise for $0 \leqslant t \leqslant T$ for some sufficiently small $T > 0$. For $0 \leqslant t \leqslant T$, let $\gamma(t) = \inf_x \min_j u_j(t,x)$. For some fixed $t \in [0,T]$, let $j_0(t)$, $x_0(t)$ have the property that $\gamma(t) = u_{j_0(t)}(t, x_0(t))$. Consider (1.13) at the corresponding point:

$$\partial_t u_{j_0}(t,x_0) - \Delta u_{j_0}(t,x_0) + v_{j_0,j_0}(x_0)u_{j_0}(t,x_0) + \sum_{k\neq j_0} v_{j_0,k}(x_0)u_k(t,x_0) = 0.$$

Since the second and the fourth terms are $\leqslant 0$ while the third one is < 0, it follows that $(\partial_t u_{j_0}(t,x_0) > 0$. Consequently, $\gamma(t)$ is an increasing function; iterating this argument, we see that $u > 0$ for $0 \leqslant t < \infty$. •

If, in addition, we assume that $0 < \inf \sigma(-\Delta + V)$, then if $v \in L^2(M;\mathbb{R}^{\tilde{m}})$ and $v \geqslant 0$ componentwise, the same is true for $u = (-\Delta + V)^{-1}v$. Indeed, we have $u = \int_0^\infty e^{-t(-\Delta+V)}v\,dt$, so it suffices to apply the lemma.

Now let ϕ satisfy (H1) and (1.12). We observe that an arbitrarily large compact region in $\mathbb{R}^m$ can be viewed as a subset of a torus M as above with R sufficiently large and that $\Delta_\phi^{(1)}$ coincides on this region with an operator $h^2(-\Delta + V(x;h))$ satisfying the assumptions of the preceding lemma. Using weighted L^2 estimates and passing to the limit, we then deduce from the lemma that if $v \in L^3(\mathbb{R}^m;\mathbb{R}^m)$, $v \geqslant 0$ componentwise, then the same is true for $u = (\Delta_\phi^{(1)})^{-1}v$.

Proposition 1.3. *Assume that ϕ satisfies* (H1) *and* (1.12)*. If u, v satisfy the assumptions of* (1.11) *and, in addition, $\partial_{x_j}u \geqslant 0$, $\partial_{x_j}v \geqslant 0$ for $1 \leqslant j \leqslant m$, then* $\operatorname{Cor}(u,v) \geqslant 0$.

Proof. It suffices to note that $e^{-\phi/h}du$ and $e^{-\phi/h}dv$ are $\geqslant 0$ componentwise; hence, so also is $(\Delta_\phi^{(1)})^{-1}(e^{-\phi/h}du)$, in accordance with the discussion above. The nonnegativity is then immediate from (1.11). •

The above approach is reminiscent of the one of Herbst-Pitt [HeP], which is more probabilistic. See, in particular, Theorem 1.6 and Corollary 1.7 in their paper. In [HS2] we obtained a similar result in the case where ϕ is convex, using more heavily the maximum principle.

§2. The Grushin problem and consequences for the spectrum of $\Delta_\phi^{(1)}$

We keep the assumption (H1). We replace the set of indices $\{1,\dots,m\}$ by some finite set Γ; Γ will vary in some class of sets. Eventually, Γ will be a "torus" $(\mathbb{Z}/R\mathbb{Z})^d$ or a subset of $\mathbb{Z}^d$. Unless otherwise specified, all estimates and assumptions will be uniform with respect to Γ, and ϕ will of course depend on Γ: $\phi(x) = \phi_\Gamma(x_\Gamma)$, $x_\Gamma = (x_j)_{j\in\Gamma}$. Nowever, we do not require uniformity with respect to Γ in the assumption (H1). We assume that, uniformly in Γ,

$$\phi''(x) = \mathcal{O}(1) \text{ in } \mathcal{L}(\ell^2(\Gamma),\ell^2(\Gamma)) \text{ and } \phi''(x) \geqslant r_\phi > 0, \tag{H2}$$

where r_ϕ is some constant independent of Γ. In what follows we work with $0 < h \leqslant h_0$ for some sufficiently small $h_0 > 0$. From (H2) and (1.5) it follows that

$$(\Delta_\phi^{(1)} u|u) \geqslant 2hr_0\|u\|^2, \tag{2.1}$$

for some $r_0 \geqslant r_\phi$. (We may allow r_0 to depend on h as long as $r_0 = \mathcal{O}(1)$.) In accordance with the discussion in Section 1, we know that the first spectral gap of $\Delta_\phi^{(0)}$ is $\geqslant 2hr_0$. If u is a 1-form orthogonal to $e^{-\phi/h}dx_j$ for all $j\in\Gamma$, then

$$(\Delta_\phi^{(1)}u|u) = ((\Delta_\phi^{(0)}\otimes I)u|u) + 2h(\phi''u|u) \geqslant 2h(r_0+r_\phi)\|u\|^2. \tag{2.2}$$

Let $R_+ : L^2(\mathbb{R}^\Gamma) \to \ell^2(\Gamma)$ be defined by $R_+u(j) = (u|e^{-\phi/h}dx_j) = (u_j|e^{-\phi/h})$, $j\in\Gamma$ (with $u = \sum_{j\in\Gamma} u_j dx_j$). Let $R_- = R_+^*$, we consider the Grushin problem

$$(\Delta_\phi^{(1)} - z)u + R_-u_- = v,\ R_+u = v_+,\ u\in\mathcal{D}(\Delta_\phi^{(1)}),\ v\in L^2,\ u_-, v_+\in\ell^2, \tag{2.3}$$

where we shall always assume that $z = \mathcal{O}(h)$.

Proposition 2.1. *For every $C > 0$, if $h \leqslant h_0$ and $h_0 > 0$ is small enough, then the problem* (2.3) *is well posed for $z \leqslant 2h(r_0 + r_\phi - 1/C)$, and we have the following a priori estimate:*

$$h\|u\| + \|u_-\| \leqslant \mathcal{O}(1)(\|v\| + h\|v_+\|). \tag{2.4}$$

Proof. The operator

$$\mathcal{P}(z) = \begin{pmatrix} \Delta_\phi^{(1)} - z & R_- \\ R_+ & 0 \end{pmatrix}$$

is selfadjoint with domain $\mathcal{D}(\Delta_\phi^{(1)})\times\ell^2$; so, it suffices to establish the *a priori* estimate (2.4). First, we consider the case where $v_+ = 0$ in (2.3). Then $u \perp e^{-\phi/h}dx_j$, $j\in\Gamma$, and (2.2) holds. Moreover, $(u|R_-u_-) = (R_+u|u_-) = 0$; taking the scalar product of the first equation in (2.3) with u, we get

$$(v|u) = ((\Delta_\phi^{(1)} - z)u|u) \geqslant (2h(r_0 + r_\phi) - z)\|u\|^2 \geqslant \frac{h}{\mathcal{O}(1)}\|u\|^2,$$

whence

$$h\|u\| \leqslant \mathcal{O}(1)\|v\|. \tag{2.5}$$

Next, we take the scalar product with R_-u_- in (2.3) and use the fact that

$$((\Delta_\phi^{(1)} - z)u|R_-u_-) = ((2h\phi'' - z)u|R_-u_-) = \mathcal{O}(h)\|u\|\|u_-\|,$$

because

$$((\Delta_\phi^{(0)} \otimes I)u|R_-u_-) = (u|(\Delta_\phi^{(0)} \otimes I)R_-u_-) = 0$$

and $\|R_-u_-\|^2 = \|u_-\|^2$. We get

$$\|u_-\|^2 \leqslant \|v\|\|u_-\| + \mathcal{O}(h)\|u\|\|u_-\|,$$

and, by (2.5),

$$\|u_-\| \leqslant \mathcal{O}(1)\|v\|. \tag{2.6}$$

So, we have (2.4) in the special case where $v_+ = 0$.

Now we consider (2.3) in full generality. Since $R_+R_- = I$, we get

$$\begin{aligned}(\Delta_\phi^{(1)} - z)(u - R_-v_+) + R_-u_- &= v + (z - 2h\phi'')R_-v_+\\ R_+(u - R_-v_+) &= 0,\end{aligned} \tag{2.7}$$

which is of the type (2.3) with $v_+ = 0$; so, in accordance with the estimates already established, we have

$$h\|u - R_-v_+\| + \|u_-\| \leqslant \mathcal{O}(1)(\|v\| + h\|v_+\|). \tag{2.8}$$

Using the inequality

$$h\|u\| \leqslant h\|u - R_-v_+\| + h\|R_-v_+\| = h\|u - R_-v_+\| + h\|v_+\|,$$

we arrive at (2.4) in full generality. •

Let $u = Ev + E_+v_+$, $u_- = E_-v + E_{-+}v_+$ be the solution of (2.3). We shall derive semiclassical approximations for the operators E_+, E_-, E_{-+}. As an approximate solution to (2.3) with $v = 0$, we try

$$u^0 = R_-v_+, \quad u_-^0 = (z - 2h\langle\phi''\rangle)v_+. \tag{2.9}$$

If u, u_- is the corresponding exact solution, then

$$\begin{gathered}(\Delta_\phi^{(1)} - z)(u^0 - u) + R_-(u_-^0 - u_-) = 2h(\phi'' - \langle\phi''\rangle)R_-v_+, \\ R_+(u^0 - u) = 0.\end{gathered} \tag{2.10}$$

The above proposition implies that

$$h\|u^0 - u\| + \|u_-^0 - u_-\| \leqslant \mathcal{O}(1)h\|(\phi'' - \langle\phi''\rangle)R_-v_+\|, \tag{2.11}$$

and we need to estimate the right hand side. Straightforward computation gives

$$\|(\phi'' - \langle\phi''\rangle)R_-v_+\|^2 = \sum\sum m_{j,k}v_+(j)v_+(k), \tag{2.12}$$

where

$$m_{j,k} = \sum_\nu m_{j,k,\nu}, \tag{2.13}$$

$$m_{j,k,\nu} \doteq \operatorname{Cor}(\partial_{x_\nu}\partial_{x_j}\phi|\partial_{x_\nu}\partial_{x_k}\phi) = h^2((\Delta_\phi^{(1)})^{-1}e^{-\phi/h}d\partial_{x_\nu}\partial_{x_j}\phi|e^{-\phi/h}d\partial_{x_\nu}\partial_{x_k}\phi). \tag{2.14}$$

Here we have used (1.11). Now, assume that

$$\|e^{-\phi/h}d(\partial_{x_\nu}\partial_{x_j}\phi)\| \leqslant m_{\nu,j}^\Gamma, \quad \text{with } (m_{\nu,j}^\Gamma) = \mathcal{O}(1) : \ell^2(\Gamma) \to \ell^2(\Gamma). \tag{H3}$$

Then from (2.1), (2.12)-(2.14) we deduce that

$$\|(\phi'' - \langle\phi''\rangle)R_-v_+\|^2 \leqslant \mathcal{O}(h)\|v_+\|^2. \tag{2.15}$$

Using this in (2.11), we arrive at the following statement.

Proposition 2.2. *Under the above assumptions, for* $|z| = \mathcal{O}(h)$, $z \leqslant 2h(r_0 + r_\phi - 1/C)$, *where* $0 < h \leqslant h_0$ *with* $h_0 > 0$ *sufficiently small depending on* C, *we have*

$$\|E_{-+} - (z - 2h\langle\phi''\rangle)\|_{\mathcal{L}(\ell^2,\ell^2)} = \mathcal{O}(h^{3/2}), \tag{2.16}$$

$$\|E_+ - R_-\|_{\mathcal{L}(\ell^2,L^2)} = \mathcal{O}(h^{1/2}), \tag{2.17}$$

$$\|E_- - R_+\|_{\mathcal{L}(L^2,\ell^2)} = \mathcal{O}(h^{1/2}). \tag{2.18}$$

Here (2.18) follows from (2.17) by duality. Let $\lambda_{\min}(\langle\phi''\rangle) \geqslant r_\phi$ be the smallest eigenvalue of $\langle\phi''\rangle$. If $\lambda_{\min}(\langle\phi''\rangle) < r_0 + r_\phi - 1/\mathcal{O}(1)$, we see that E_{-+} (hence, also $\Delta_\phi^{(1)} - z$) is bijective for $z \leqslant 2h\lambda_{\min}(\langle\phi''\rangle) - \mathcal{O}(h^{3/2})$. For $h \leqslant h_0$ small enough, we can therefore replace r_0 by $\lambda_{\min}(\langle\phi''\rangle) - 1/\mathcal{O}(1)$ in (2.1), which is of interest if $\lambda_{\min}(\langle\phi''\rangle) \geqslant r_0 + 1/\mathcal{O}(1)$. The conclusion is as follows:

We have (2.1), (2.2) for $r_0 = \lambda_{\min}(\langle\phi''\rangle) - 1/C$ provided that $0 < h \leqslant h_0$ with $h_0 > 0$ sufficiently small depending on C. (2.19)

Now we assume, instead, that $\lambda_{\min}(\langle\phi''\rangle) \geqslant r_0 + r_\phi - 1/C$ for some large constant $C > 0$ with $1/C \leqslant r_\phi/2$. Then we can only say that $E_{-+}(z)$ is bijective for $z \leqslant 2h(r_0 + r_\phi - \frac{1}{C} - \frac{1}{\mathcal{O}(1)})$ if $h > 0$ is sufficiently small. Then r_0 can be replaced by $r_0 + r_\phi - \frac{1}{C} - \frac{1}{\mathcal{O}(1)}$, and after iterating this argument finitely many times, we reach the preceding case and, hence, also (2.19).

To complete the picture, we want to replace $\langle\phi''\rangle$ by the semiclassical limit $\phi''(0)$, which is independent of h. Assume for simplicity that

$$\phi(x) \text{ attains its minimum at } x = 0. \tag{H4}$$

In order to apply the maximum principle as in [HS2, S1,2], we start with considering

$$ue^{-\phi/h} = \langle u\rangle e^{-\phi/h} + \Delta_\phi^{(0)}(e^{-\phi/h}\widetilde{u}), \tag{2.20}$$

where u is assumed to be of class C^2 with all the derivatives of u of order $\leqslant 2$ bounded. Relation (2.20) can be rewritten in the form

$$\frac{1}{2h}(u - \langle u\rangle) = \nabla\phi \cdot \partial_x\widetilde{u} - \frac{h}{2}\Delta\widetilde{u}. \tag{2.21}$$

We assume that

For some fixed $\delta > 0$, $\phi''(x)$ satisfies the maximum principle (mpδ) (see [S2]) with $B = \ell^\infty$ (hence, also with B replaced by $B^* = \ell^1$). (H5)

If

$$\phi''' \in L^\infty, \tag{2.22}$$

then, as shown in [HS2] (see the argument prior to Theorem 2.2 in [S2]), (2.21) has a solution with ∇u, $\nabla^2 u$ bounded; formal application of the maximum principle to this solution is allowed in a way that we shall now recall.

Take the gradient of (2.21):

$$\frac{1}{2h}\nabla u = \left(\nabla\phi \cdot \partial_x - \frac{h}{2}\Delta + \nabla^2\phi\right)\nabla\widetilde{u}. \tag{2.23}$$

Then, by the maximum principle and (mpδ),

$$\|\nabla\widetilde{u}\|_B \leqslant \frac{1}{2h\delta}\sup_x \|\nabla u\|_B, \tag{2.24}$$

and similarly with B replaced by B^*. Next, we take the gradient of (2.23):

$$\begin{aligned}\frac{1}{2h}\nabla^2 u &= \left(\nabla\phi \cdot \partial_x - \frac{h}{2}\Delta\right)\nabla^2\widetilde{u}\\ &\quad + \nabla^2\phi \circ \nabla^2\widetilde{u} + \nabla^2\widetilde{u}\circ\nabla^2\phi + \langle\nabla^3\phi, \nabla\widetilde{u}\rangle,\end{aligned} \tag{2.25}$$

where the last term is the contraction of the symmetric 3-tensor $\nabla^3\phi$ by the 1-vector $\nabla\widetilde{u}$. We want to estimate $\|\nabla^2\widetilde{u}\|_{\mathcal{L}(B,B^*)} = \|\nabla^2\widetilde{u}\|_{(B\otimes B)^*}$. Assume that the supremum M of this quantity is attained at a point $x = x_0$, and let $t \in B$, $s \in B$ be normalized vectors satisfying

$$\langle\nabla^2\widetilde{u}(x_0)t, s\rangle = M = \|\nabla^2\widetilde{u}(x_0)t\|_{B^*}\|s\|_B = \|t\|_B\|\nabla^2\widetilde{u}(x_0)s\|_{B^*}.$$

(By a density argument as in [HS2, S1,2], we can assume that the supremum is actually attained.) Applying (2.25) to t and taking the scalar product with s, at $x = x_0$ we obtain

$$\begin{aligned}&\frac{1}{2h}\langle\nabla^2 u\, t, s\rangle\\ &\qquad \geqslant \langle\nabla^2\phi\circ\nabla^2\widetilde{u}\, t, s\rangle + \langle t, \nabla^2\phi\circ\nabla^2\widetilde{u}\, s\rangle - \langle\nabla^3\phi, \nabla\widetilde{u}\otimes t\otimes s\rangle\\ &\qquad \geqslant 2\delta M - \langle\nabla^3\phi, \nabla\widetilde{u}\otimes t\otimes s\rangle.\end{aligned} \tag{2.26}$$

This gives

$$\begin{aligned}&\sup_x \|\nabla^2\widetilde{u}\|_{\mathcal{L}(B,B^*)}\\ &\quad \leqslant \frac{1}{4\delta h}\sup_x \|\nabla^2 u\|_{\mathcal{L}(B,B^*)} + \frac{1}{2\delta}\sup_x \|\nabla^3\phi\|_{(B^*\otimes B\otimes B)^*}\sup_x \|\nabla\widetilde{u}\|_{B^*}\\ &\quad \leqslant \frac{1}{4\delta h}(\sup_x \|\nabla^2 u\|_{\mathcal{L}(B,B^*)} + \frac{1}{\delta}\sup_x \|\nabla^3\phi\|_{(B^*\otimes B\otimes B)^*}\sup_x \|\nabla u\|_{B^*}),\end{aligned}$$

where we have also used (2.24) with B replaced by B^*. By the trace lemma (see [S3]) we have

$$\Delta\widetilde{u} = |\mathrm{tr}\nabla^2\widetilde{u}| \leqslant \|\nabla^2\widetilde{u}\|_{\mathcal{L}(B,B^*)} \quad (B = \ell^\infty),$$

and if we putting $x = 0$ in (2.21), we get

$$|u(0) - \langle u\rangle| \leqslant \frac{h}{4\delta}(\sup_x \|\nabla^2 u\|_{\mathcal{L}(\ell^\infty,\ell^1)} + \frac{1}{\delta}\sup_x \|\nabla^3\phi\|_{(\ell^1\otimes\ell^\infty\otimes\ell^\infty)^*}\sup_x \|\nabla u\|_{\ell^1}). \tag{2.27}$$

In particular,

$$\begin{aligned} &|\partial_{x_j}\partial_{x_k}\phi(0) - \langle\partial_{x_j}\partial_{x_k}\phi\rangle| \\ &\quad\leqslant \frac{h}{4\delta}(\sup_x \|\nabla^2\partial_{x_j}\partial_{x_k}\phi\|_{(\ell^\infty\otimes\ell^\infty)^*} \\ &\qquad + \frac{1}{\delta}\sup_x \|\nabla^3\phi\|_{(\ell^1\otimes\ell^\infty\otimes\ell^\infty)^*}\sup_x \|\nabla\partial_{x_j}\partial_{x_k}\phi\|_{\ell^1}). \end{aligned} \tag{2.28}$$

Now we introduce the following assumptions:

$$\|\nabla^3\phi\|_{(\ell^1\otimes\ell^\infty\otimes\ell^\infty)^*} = \mathcal{O}(1), \tag{H6}$$

$$\|\nabla^2\partial_{x_j}\partial_{x_k}\phi\|_{(\ell^\infty\otimes\ell^\infty)^*},\ \|\nabla\partial_{x_j}\partial_{x_k}\phi\|_{\ell^1} \leqslant m_{j,k}, \tag{H7}$$

where $(m_{j,k}) = \mathcal{O}(1)\colon \ell^2 \to \ell^2$. It follows that

$$\phi'' - \langle\phi''\rangle = \mathcal{O}(h)\colon \ell^2 \to \ell^2. \tag{2.29}$$

The above discussion can be summarized as follows.

Theorem 2.3. *Assume* (H1-3). *For every constant $C > 0$, the Grushin problem* (2.3) *is well posed for $-Ch \leqslant z \leqslant 2h(\lambda_{\min}(\langle\phi''\rangle) + r_\phi - \frac{1}{C})$ provided that $h \leqslant h_0$, with $h_0 > 0$ sufficiently small, depending on C. If $u = Ev + E_+v_+$, $u_- = E_-v + E_{-+}v_+$ denotes the solution, then*

$$\begin{aligned} \|E\|_{\mathcal{L}(L^2,L^2)} &= \mathcal{O}\Big(\frac{1}{h}\Big), \\ \|E_+\|_{\mathcal{L}(\ell^2,L^2)},\ \|E_-\|_{\mathcal{L}(L^2,\ell^2)} &= \mathcal{O}(1), \\ \|E_{-+}\|_{\mathcal{L}(\ell^2,\ell^2)} &= \mathcal{O}(h), \end{aligned} \tag{2.30}$$

$$\|E_+ - R_-\|_{\mathcal{L}(\ell^2,L^2)},\ \|E_- - R_+\|_{\mathcal{L}(L^2,\ell^2)} = \mathcal{O}(h^{1/2}), \tag{2.31}$$

$$\|E_{-+} - (z - 2h\langle\phi''\rangle)\|_{\mathcal{L}(\ell^2,\ell^2)} = \mathcal{O}(h^{3/2}). \tag{2.32}$$

Under the additional assumptions (H4-7), *we can replace* $\langle\phi''\rangle$ *by* $\phi''(0)$ *above.*

Using the formulas $\partial_z E_{-+} = E_- E_+$, $R_+ R_- = I$ and (2.30), (2.31), we get

$$\partial_z E_{-+} = I + \mathcal{O}(h^{1/2}). \tag{2.33}$$

Let $\lambda_1 \leqslant \cdots \leqslant \lambda_m$, where $m = \sharp\Gamma$, be the eigenvalues of $\langle\phi''\rangle$. Then by (2.32) and minimax, the eigenvalues of E_{-+}, arranged in decreasing order, are of the form

$$\mu_k(z) = z - 2h\lambda_k + \mathcal{O}(h^{3/2}). \tag{2.34}$$

They are Lipschitz continuous functions of z, and (2.33) implies that

$$\partial_z \mu_k(z) = 1 + \mathcal{O}(h^{1/2}). \tag{2.35}$$

The zeros of $\det(E_{-+}(z))$ are then satisfying the values z_k given by $\mu_k(z_k) = 0$, so that $z_k = 2h\lambda_k + \mathcal{O}(h^{3/2})$. These are the eigenvalues of $\Delta_\phi^{(1)}$ below $2h(\lambda_1 + r_\phi - \frac{1}{\mathcal{O}(1)})$. Observe that the nonvanishing eigenvalues of $\Delta_\phi^{(0)}$ in the interval $]-\infty, 2h(\lambda_1 + r_\phi - \frac{1}{\mathcal{O}(1)})[$ are among the z_k.

§3. Weighted estimates

Below, considering classes of weights $\rho\colon \Gamma \to]0,\infty[$, we shall always assume that, uniformly with respect to ρ in such a class, we have

$$\rho\phi''(x)\rho^{-1},\ \rho^{-1}\phi''(x)\rho = \mathcal{O}(1) = \mathcal{L}(\ell^2(\Gamma), \ell^2(\Gamma)). \tag{H8}$$

Let $\mathcal{R}_0$ be a class of weights as above satisfying $\rho \in \mathcal{R}_0 \Rightarrow \frac{1}{\rho} \in \mathcal{R}_0$ and such that, for every $\rho \in \mathcal{R}_0$,

$$(\rho\phi''(x)\rho^{-1}t|t) \geqslant \Big(\frac{1}{\mathcal{O}(1)} - r_0\Big)\|t\|^2, \quad t \in \mathbb{R}^\Gamma\ (\mathcal{O}(1) = \text{ some positive constant}), \tag{H9}$$

where now (cf. the preceding section) $r_0 = \lambda_{\min}(\langle\phi''\rangle) - \frac{1}{C}$, with $C > 0$ arbitrarily large and h sufficiently small depending on C. Using weights of class $\mathcal{R}_0$, we shall obtain weighted estimates for the problem (2.3) with $z = 0$ for $h \leqslant h_0$ and $h_0 > 0$ sufficiently small.

As before, we start with the case where $v_+ = 0$, multiply the first equation by ρ and take the scalar product with ρu (defined as $\sum \rho(j) u_j dx_j$). Since $R_\pm$ and ρ commute, we have $(\rho R_- u_- | \rho u) = (\rho u_- | \rho R_+ u) = 0$; consequently, $(\rho\Delta_\phi^{(1)} u | \rho u) = (\rho v | \rho u)$. Here the LHS is equal to $((\Delta_\phi^{(0)} \otimes I)\rho u|\rho u) + 2h(\rho\phi''\rho^{-1}\rho u|\rho u)$; since $\rho u \perp e^{-\phi/h}dx_j$ for every j, this expression is bounded from below by

$$2hr_0\|\rho u\|^2 + 2h\Big(\frac{1}{\mathcal{O}(1)} - r_0\Big)\|\rho u\|^2 = \frac{2h}{\mathcal{O}(1)}\|\rho u\|^2.$$

Thus, we get the estimate

$$h\|\rho u\| \leqslant \mathcal{O}(1)\|\rho v\| \tag{3.1}$$

for the solutions of (2.3) (with $z=0$) in the case where $v_+ = 0$. In order to estimate ρu_-, we multiply the 1st equation in (2.3) by ρ and take the scalar product with $\rho R_- u_-$. As before, we obtain

$$|(\rho\Delta_\phi^{(1)}u|\rho R_-u_-)| = |2h(\rho\phi''\rho^{-1}\rho u|R_-\rho u_-)| \leqslant \mathcal{O}(1)h\|\rho u\|\|\rho u_-\|,$$

whence

$$\|\rho u_-\|^2 = \|\rho R_- u_-\|^2 \leqslant \|\rho v\|\|\rho u_-\| + \mathcal{O}(1)h\|\rho u\|\|\rho u_-\|.$$

Using (3.1), we get

$$\|\rho u_-\| \leqslant \mathcal{O}(1)\|\rho v\|. \tag{3.2}$$

We now drop the assumption $v_+ = 0$ (keeping all the time the assumption $z = 0$) and apply (3.1), (3.2) to the problem (2.7):

$$h\|\rho(u - R_-v_+)\| + \|\rho u_-\| \leqslant \mathcal{O}(1)(\|\rho v\| + h\|\rho v_+\|).$$

(Here we also use (H8).) Using the triangle inequality, we obtain

$$h\|\rho u\| + \|\rho u_-\| \leqslant \mathcal{O}(1)(\|\rho v\| + h\|\rho v_+\|), \tag{3.3}$$

which implies that

$$\begin{gathered} \rho E(0)\rho^{-1} = \mathcal{O}(\frac{1}{h}), \\ \rho E_+(0)\rho^{-1},\ \rho E_-(0)\rho^{-1} = \mathcal{O}(1), \\ \rho E_{-+}(0)\rho^{-1} = \mathcal{O}(h), \end{gathered} \tag{3.4}$$

in the various spaces of L^2, ℓ^2 bounded operators.

So far in this section, we have only used the assumptions (H1, 2, 8, 9). Since we want to have weighted semiclassical approximations, we introduce a weighted analogue of (H3), namely,

$$\|e^{-\phi/h}d\partial_{x_\nu}\partial_{x_j}\phi\| \leqslant m_{\nu,j}^\Gamma \quad \text{with } (\rho(\nu)m_{\nu,j}^\Gamma\rho(j)^{-1}) = \mathcal{O}(1)\colon \ell^2(\Gamma) \to \ell^2(\Gamma), \tag{H10}$$

uniformly in $\rho \in \mathcal{R}_0$. Repeating the discussion prior to Proposition 2.2, we get the estimate

$$h\|\rho(u^0 - u)\| + \|\rho(u^0_- - u_-)\| \leqslant \mathcal{O}(1)h^{3/2}\|\rho v_+\| \tag{3.5}$$

for solutions of (2.3) with $v = 0$ (and $z = 0$). Here $u^0 = R_- v_+$, $u^0_- = -2h\langle\phi''\rangle v_+$. In other words, we have

$$\begin{aligned} \|\rho(E_+ - R_-)\rho^{-1}\|_{\mathcal{L}(\ell^2, L^2)} &= \mathcal{O}(h^{1/2}), \\ \|\rho(E_- - R_+)\rho^{-1}\|_{\mathcal{L}(L^2, \ell^2)} &= \mathcal{O}(h^{1/2}), \\ \|\rho(E_{-+} + 2h\langle\phi''\rangle)\rho^{-1}\|_{\mathcal{L}(\ell^2, \ell^2)} &= \mathcal{O}(h^{3/2}). \end{aligned} \tag{3.6}$$

Here the second estimate follows from the first one by duality (replace ρ with ρ^{-1}).

Finally, we would like to replace $\langle\phi''\rangle$ by $\phi''(0)$ in the last estimate in (3.6), where, of course, we assume (H4). Applying the results of Section 2, we see that if the assumptions (H5), (H6) are fulfilled and

$$\|\nabla^2 \partial_{x_j} \partial_{x_k} \phi\|_{(\ell^\infty \otimes \ell^\infty)^*}, \|\nabla \partial_{x_j} \partial_{x_k} \phi\|_{\ell^1} \leqslant m_{j,k}, \tag{H11}$$

where $(\rho(j) m_{j,k} \rho(k)^{-1}) = \mathcal{O}(1) : \ell^2 \to \ell^2$, uniformly in $\rho \in \mathcal{R}_0$, then

$$\rho(\phi''(0) - \langle\phi''\rangle)\rho^{-1} = \mathcal{O}(h) : \ell^2 \to \ell^2, \tag{3.7}$$

and then we can replace $\langle\phi''\rangle$ by $\phi''(0)$ in (3.6). (Observe that (H11) implies (H10).) Summarizing, we have

Theorem 3.1. *Under the assumptions* (H1, 2, 8, 9), *we have* (3.4) *uniformly in* $\rho \in \mathcal{R}_0$, *where* $u = E(0)v + E_+(0)v_+$, $u_- = E_-(0)v + E_{-+}(0)v_+$ *is the solution of* (2.3) *with* $z = 0$. *Under the additional assumption* (H10), *we have* (3.6) *for* $z = 0$, *and if we also assume* (H4, 5, 6, 11), *then* $\langle\phi''\rangle$ *may be replaced by* $\phi''(0)$ *in* (3.6).

§4. Certain convolution equations

Let $0 \leqslant v \in \ell^1(\mathbb{Z}^d)$ be an even function on $\mathbb{Z}^d$ such that

$$\|v\|_{\ell^1} < 1, \tag{4.1}$$

and

$$\text{the smallest subgroup of } \mathbb{Z}^d \text{ containing } \operatorname{supp} v \text{ is } \mathbb{Z}^d. \tag{4.2}$$

We are interested in the convolution operator $(\delta_0 - v)*$, where δ_0 is the characteristic function of $\{0\}$. In addition, we assume that

$$v(k) = \mathcal{O}(1)e^{-|k|/C} \quad \text{for some } C > 0. \tag{4.3}$$

For $x \in \mathbb{R}^d$, put

$$F(y) = \sum_k v(k)\,\mathrm{ch}(k \cdot y) = \sum_k v(k) e^{k \cdot y}, \tag{4.4}$$

and let $\Omega \subset \mathbb{R}^d$ be the largest open set where $F < \infty$. For $y_1, y_2 \in \Omega$, $t_1, t_2 \geqslant 0$, $1 = t_1 + t_2$, from Hölder's inequality we obtain

$$\begin{aligned} &F(t_1 y_1 + t_2 y_2) \\ &\quad = \sum v(k)^{t_1} (e^{k \cdot y_1})^{t_1} v(k)^{t_2} (e^{k y_2})^{t_2} \\ &\quad \leqslant F(y_1)^{t_1} F(y_2)^{t_2}; \end{aligned} \tag{4.5}$$

so, Ω is convex and $\log F$ is a convex function on Ω. Then $F = e^{\log F}$ is also convex and, hence, locally bounded. Let $\widetilde{\Omega} \Subset \Omega$ be open; we choose $\delta > 0$ such that $\widetilde{\Omega} + B(0, \delta) \Subset \Omega$. For $y \in \widetilde{\Omega}$, we write

$$v(k) e^{k \cdot y} \leqslant e^{-|k|\delta} \sup_{y \in \widetilde{\Omega} + B(0,\delta)} v(k) e^{k \cdot y} \leqslant e^{-|k|\delta} \sup_{\widetilde{\Omega} + B(0,\delta)} F, \tag{4.6}$$

where $B(0, \delta)$ denotes the open ball of radius δ centered at 0. It follows that (4.4) converges uniformly on $\widetilde{\Omega}$ as a geometric series; consequently, $F(y)$ is of class C^∞ and even analytic, and we can differentiate repeatedly under the sign of summation.

Observe that

$$\langle F''(y), t \otimes t \rangle = \sum v(k) e^{k \cdot y} (t \cdot k)^2.$$

Since (4.2) implies that $\operatorname{supp} v$ is not contained in any hyperplane in $\mathbb{R}^d$, we see that F is strictly convex in the sense that $\langle F''(y), t \otimes t \rangle > 0$ for $y \in \Omega$, $0 \neq t \in \mathbb{R}^d$. We can even show that $\phi = \log F$ is strictly convex. Indeed,

$$\begin{aligned} \langle \phi'', t \otimes t \rangle &= \frac{1}{F} \langle F'', t \otimes t \rangle - \frac{\langle F', t \rangle^2}{F^2} \\ &= \frac{1}{F} \sum v(k) (\mathrm{ch}\, y \cdot k)(t \cdot k)^2 - \Big(\frac{1}{F} \sum v(k) \mathrm{sh}(y \cdot k) t \cdot k \Big)^2. \end{aligned}$$

Consequently, writing

$$|v(k) \mathrm{sh}(y \cdot k) t \cdot k| \leqslant (v(k) |\mathrm{sh}(y \cdot k)|)^{1/2} (v(k) |\mathrm{sh}(y \cdot k)| (t \cdot k)^2)^{1/2},$$

we get

$$\langle \phi'', t \otimes t \rangle \geqslant \frac{1}{F} \sum v(k) (\mathrm{ch}(y \cdot k) - |\mathrm{sh}(y \cdot k)|)(t \cdot k)^2 > 0,$$

provided that $t \neq 0$.

Assume that

$$\underline{\lim}_{y \to \partial\Omega} F(y) > 1. \tag{4.7}$$

We note that (4.7) holds (with $\Omega = \mathbb{R}^d$) if v has compact support or, more generally, if $v(k) = \mathcal{O}(1)e^{-C|k|}$ for every $C > 0$. Then using also the fact that $F(0) = \|v\|_{\ell^1} < 1$, we see that the set

$$\Sigma \stackrel{\text{def}}{=} \{y \in \Omega; F(y) = 1\} \tag{4.8}$$

is the smooth boundary of a strictly convex relatively compact domain $D \Subset \Omega$.

Now we consider the inverse Fourier transform $\hat{v}$ of v:

$$\begin{aligned}\widehat{v}(z) &= \sum_k v(k)e^{ik\cdot z} = \sum_k v(k)\cos(k\cdot z)\\ &= \sum v(k)(\operatorname{ch}(k\cdot y)\cos(k\cdot x) - i\operatorname{sh}(k\cdot y)\sin(k\cdot x)),\end{aligned}$$

where $z = x+iy$. $\widehat{v}$ can be regarded as a holomorphic function on $\mathbb{T}^d + i\Omega$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. We observe that

$$\Re\widehat{v}(x+iy) = \sum v(k)\operatorname{ch}(k\cdot y)\cos(k\cdot x) \leqslant F(y). \tag{4.9}$$

Equality in (4.9) occurs precisely when $v(k) > 0 \Rightarrow k\cdot x \in 2\pi\mathbb{Z}$. For such an x, we also have $k\cdot x \in 2\pi\mathbb{Z}$ for all k lying in the group generated by supp$v$, i.e., for all $k \in \mathbb{Z}^d$, by (4.2). Hence, $x = 0$ and

$$\text{equality in (4.9) occurs precisely for } x = 0 \text{ (in } \mathbb{T}^d). \tag{4.10}$$

(Conversely, (4.10) implies (4.2); see the appendix.) We conclude that the hypersurface $1 - \widehat{v}(z) = 0$ in $\mathbb{T}^d + i\Omega$ intersects $\mathbb{T}^d + i\overline{D}$ precisely along $i\Sigma$. We want to study the asymptotics of the convolution kernel E given by

$$E(k) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \frac{e^{ik\cdot x}}{1-\widehat{v}(x)}dx \tag{4.11}$$

as $k \to \infty$; E satisfies $(\delta_0 - v) * E = \delta_0$. If $y \in D$, we can replace $\mathbb{T}^d$ by $\mathbb{T}^d + iy$ and gain a factor $e^{-y\cdot k}$ from the exponential in (4.11). For a given $k \in \mathbb{Z}^d \setminus \{0\}$, we are then interested in finding $y = y(k) \in \overline{D}$ with $y\cdot k$ as large as possible. Clearly, $y(k)$ is the unique point on Σ for which the exterior unit normal is equal to $|k|^{-1}k$. The corresponding value of $y\cdot k$ is

$$y(k)\cdot k = \sup_{y\in\overline{D}} y\cdot k = H_D(k), \tag{4.12}$$

where $H_D(k)$ is the support function of D. We observe that

$$H_D(k) \text{ is convex, continuous. On } \mathbb{R}^d \setminus \{0\},\ H_D \text{ is smooth, } > 0, \text{ positively homogeneous of degree 1, and strictly convex in the non-radial direction: } \langle H_D''(k), t \otimes t\rangle > 0 \text{ if } t, k \text{ are linearly independent.} \tag{4.13}$$

By Stokes' formula, we may replace $\mathbb{T}^d$ in (4.11) by the complex contour $z = x + i(1-\epsilon)y(k)$, $x \in \mathbb{T}^d$, for any small $\epsilon > 0$. Away from $x = 0$, we may deform the contour further, replacing it by $z = x + ir(x)y(k)$, where $r(x)$ satisfies the following conditions: $r(0) < 1$, $1 - \epsilon \leqslant r(x)$, and $r(x) = 1 + 2\epsilon_0$ outside some arbitrarily small neighborhood V of $x = 0$. Here $\epsilon_0 > 0$ is sufficiently small, depends on V, and is independent of k. We obtain

$$E(k) = \frac{1}{(2\pi)^d} \int\limits_{x\in V,\ z=x+ir(x)y(k)} e^{ik\cdot z}\frac{1}{1-\widehat{v}(z)}dz + \mathcal{O}(1)(e^{-(1+\epsilon_0)H_D(k)}). \tag{4.14}$$

In order to understand the integral here, we make a real rotation of the z-coordinates so that $k/|k|$ becomes $(0,\dots,1)$, and look at the hypersurface $1 = \widehat{v}$ in the new coordinates. At the point $iy(k)$, we have $\partial_{y_d}(\widehat{v}(iy)) = s$ for some $s = s(k/|k|) > 0$, $\partial_{y'}(\widehat{v}(iy)) = 0$, $\partial_y^2(\widehat{v}(iy)) > 0$, so that

$$\partial_{z_d}\widehat{v} = \frac{s}{i},\ \ \partial_{z'}\widehat{v} = 0,\ \ \partial_z^2\widehat{v} < 0 \quad (s = \partial_{y_d}F(y(k))) \tag{4.15}$$

(still at $z = iy(k)$). The implicit function theorem shows that near $iy(k)$ our hypersurface is of the form $z_d = f(z')$, $f = f(k/|k|, z')$, so that $iy_d(k) = f(iy'(k))$. Differentiating the equation $\widehat{v}(z', f(z')) = 1$, we obtain

$$\frac{\partial\widehat{v}}{\partial z'} + \frac{\partial\widehat{v}}{\partial z_d}\frac{\partial f}{\partial z'} = 0,$$

so that

$$\frac{\partial f}{\partial z'}(iy'(k)) = 0. \tag{4.16}$$

Differentiating once more, at $iy(k)$ we get

$$\frac{\partial\widehat{v}}{\partial z_d}\frac{\partial^2 f}{\partial z'^2} + \frac{\partial^2\widehat{v}}{\partial z'^2} = 0,$$

whence

$$\frac{\partial^2 f}{\partial z'^2}(iy'(k)) = -\frac{i}{s}\frac{\partial^2 \widehat{v}}{\partial z'^2}(iy(k)). \tag{4.17}$$

Using Taylor expansion, we see that the hypersurface takes the form

$$\begin{aligned} z_d &= f(z') \\ &= iy_d(k) \\ &\quad + \frac{i}{2s}\langle -(\partial_{z'}^2 \widehat{v})(iy(k))(z' - iy'(k)), (z' - iy'(k))\rangle \\ &\quad + \mathcal{O}(|z' - iy'(k)|^3). \end{aligned} \tag{4.18}$$

In the integral in (4.14), we may replace the contour by $z = x + i(y'(k), r(x)y_d(k))$, and first consider the resulting z_d-integral for a fixed (small) x'. If $\Im f(x' + iy'(k)) \geqslant (1 + \frac{3}{2}\epsilon_0)y_d(k)$, then, by contour deformation we establish that the contribution to the integral is $\mathcal{O}(1)e^{-(1+\epsilon_0)H_D(k)}$. If $\Im f(x' + iy'(k)) \leqslant (1 + \frac{3}{2}\epsilon_0)y_d(k)$, applying contour deformation and residues, we obtain

$$\begin{aligned} E(k) &= \frac{i}{(2\pi)^{d-1}} \int\limits_{x'\in V'} \frac{e^{i|k|f(x'+iy'(k))}}{-\partial_{z_d}\widehat{v}(x' + iy'(k), f(x' + iy'(k)))} dx' \\ &\quad + \mathcal{O}(e^{-(1+\epsilon_0)H_D(k)}), \end{aligned} \tag{4.19}$$

where V' is a small real neighborhood of 0. Using (4.15), (4.18), and the fact that $\widehat{v}(iy) = F(y)$, we can evaluate (4.19) by the method of stationary phase; this leads to the formula

$$E(k) = \frac{(\partial_{y_d} F(y(k)))^{\frac{d-3}{2}}}{\sqrt{\det \partial_{y'}^2 F(y(k))}} \frac{(1 + \mathcal{O}(\frac{1}{|k|}))}{(2\pi|k|)^{\frac{d-1}{2}}} e^{-H_D(k)}, \quad |k| \to \infty, \tag{4.20}$$

(we recall that $y(k)$ is the unique point on $F(y) = 1$ where ∇F is a positive multiple of k and that we use rotated coordinates, so that $\partial_{y'} F(y(k)) = 0$).

Next, we examine the behaviour of our convolution operator in weighted spaces. Let $R > 1$ be some number satisfying

$$D_R \stackrel{\text{def}}{=} \{y \in \Omega; F(y) \leqslant R\} \quad \text{is compact.} \tag{4.21}$$

For $F(0) < r \leqslant R$, we define D_r as in (4.21). Since v and F are even, the support functions $H_{D_r}(k) = \sup_{y\in D_r} k \cdot y$ are norms (strictly increasing with r); we let $\widetilde{d}_r(k, \ell) = H_{D_r}(k - \ell)$ be the corresponding distance. If $\rho\colon \mathbb{Z}^d \to]0, \infty[$ has the property

$$\frac{\rho(k)}{\rho(\ell)} \leqslant e^{\widetilde{d}_R(k,\ell)}, \tag{4.22}$$

then $\|\rho(v*)\rho^{-1}\|_{\mathcal{L}(\ell^p,\ell^p)}$ is finite and bounded by a constant independent of ρ. Indeed it suffices to estimate

$$\sup_k \sum_\ell \frac{\rho(k)}{\rho(\ell)} v(k-\ell) \quad \text{and} \quad \sup_\ell \sum_k \frac{\rho(k)}{\rho(\ell)} v(k-\ell).$$

Since $\frac{\rho(k)}{\rho(\ell)} = e^{y(k,\ell)\cdot(k-\ell)}$ for some $y(k,\ell) \in D_R$, we have

$$\begin{aligned}\sum_\ell \frac{\rho(k)}{\rho(\ell)} v(k-\ell) &= \sum_\ell v(k-\ell) e^{y(k,\ell)\cdot(k-\ell)} \\ &\leqslant \sum_\ell e^{-|k|\delta} \sup_{D_R+B(0,\delta)} F < \infty\end{aligned}$$

(if $\delta > 0$ is sufficiently small), by (4.6). The second expression $\sup_\ell \sum_k \cdots$ can be estimated similarly.

In what following, we shall always assume that ρ satisfies (4.22). Also, we note that

$$\|\rho(v * -(1_{B(0,L)}v)*)\rho^{-1}\|_{\mathcal{L}(\ell^p,\ell^p)} \to 0, \quad L \to \infty, \tag{4.23}$$

uniformly with respect to ρ (satisfying (4.22)).

Proposition 4.1. *Let* $F(0) < r \leqslant R$. *If* ρ *satisfies* (4.22) *and if, for some* $y(k) \in D_r$,

$$\max\left(\left|\log\frac{\rho(k)}{\rho(\ell)} - y(k)\cdot(k-\ell)\right|, \left|\log\frac{\rho(k)}{\rho(\ell)} - y(\ell)\cdot(k-\ell)\right|\right) \leqslant \delta \tag{4.24}$$

for $|k-\ell| \leqslant L$, *then*

$$\left\|\frac{1}{2}(\rho(v*)\rho^{-1} + \rho^{-1}(v*)\rho)\right\|_{\mathcal{L}(\ell^p,\ell^p)} \leqslant r + \epsilon(\delta, L), \quad 1 \leqslant p \leqslant \infty, \tag{4.25}$$

where $\epsilon(\delta, L) \to 0$ *as* $\delta, 1/L \to 0$.

Proof. In accordance with the observation (4.23), we may replace v by $1_{B(0,L)}v$ (this can only decrease the corresponding function F). We need to estimate

$$\sup_k \sum_\ell \frac{1}{2}\left(\frac{\rho(k)}{\rho(\ell)} + \frac{\rho(\ell)}{\rho(k)}\right) v(k-\ell)$$

and the corresponding $\sup_\ell \sum_k \cdots$; these expressions are equal by symmetry. By (4.24), we have

$$\frac{1}{2}\left(\frac{\rho(k)}{\rho(\ell)} + \frac{\rho(\ell)}{\rho(k)}\right) \leqslant e^\delta \operatorname{ch}(y(k)\cdot(k-\ell)),$$

so

$$
\begin{aligned}
&\sum_{\ell} \frac{1}{2}\left(\frac{\rho(k)}{\rho(\ell)}+\frac{\rho(\ell}{\rho(k)}\right) v(k-\ell) \\
&\quad \leqslant e^{\delta} \sum_{\ell} v(k-\ell) \operatorname{ch}(y(k)\cdot(k-\ell))=e^{\delta} F(y(k)) \\
&\quad \leqslant e^{\delta} r,
\end{aligned}
$$

and the proposition follows. •

We shall use the following consequence.

Corollary 4.2. *Under the assumption* (4.25), *we have*

$$
(\rho(I-v*)t|\rho t) \geqslant (1-r-\epsilon(\delta, L))\|\rho t\|_{\ell^2}^2, \quad t \in \ell^2(\mathbb{Z}^d;\mathbb{R}). \tag{4.26}
$$

Proof. Replacing t by $s=\rho t$, we see that

$$
\begin{aligned}
&((I-\rho(v*)\rho^{-1})s|s) \\
&\quad =\left(\left(I-\frac{1}{2}(\rho(v*)\rho^{-1}+\rho^{-1}(v*)\rho)\right)s|s\right) \\
&\quad \geqslant \left(1-\left\|\frac{1}{2}(\rho(v*)\rho^{-1}+\rho^{-1}(v*)\rho)\right\|_{\mathcal{L}(\ell^2,\ell^2)}\right)\|s\|^2. \quad \bullet
\end{aligned}
$$

The above results will be applied to convolution equations on $(\mathbb{Z}/N\mathbb{Z})^d$ with N large. Let v be a function on $(\mathbb{Z}/N\mathbb{Z})^d$; we put

$$
\mathcal{F}_N^{-1} v(x)=\sum_{k\in(\mathbb{Z}/N\mathbb{Z})^d} v(k) e^{ik\cdot x}, \quad x \in \left(\frac{2\pi}{N}\mathbb{Z}/2\pi\mathbb{Z}\right)^d. \tag{4.27}
$$

Since

$$
\frac{1}{N^d}\sum_{x\in(\frac{2\pi}{N}\mathbb{Z}/2\pi\mathbb{Z})^d} e^{i(\ell-k)\cdot x}=\delta(\ell-k), \quad \ell, k \in \mathbb{Z}/N\mathbb{Z},
$$

we see that $\mathcal{F}_N^{-1}$ is the inverse of

$$
\mathcal{F}_N u(k)=\frac{1}{N^d}\sum_{x\in(\frac{2\pi}{N}\mathbb{Z}/2\pi\mathbb{Z})^d} u(x)\, e^{-ik\cdot x}, \quad k \in (\mathbb{Z}/N\mathbb{Z})^d. \tag{4.28}
$$

Up to a power of 2π, this can be viewed as a Riemann sum for the corresponding integral over $(\mathbb{R}/2\pi\mathbb{Z})^d$. As $N \to \infty$, relations (4.27,28) converge to the standard formulas for the duality $\mathbb{Z}^d \leftrightarrow (\mathbb{R}/2\pi\mathbb{Z})^d$:

$$\mathcal{F}^{-1}v(x) = \sum_{k\in\mathbb{Z}^d} v(k)e^{ik\cdot x}, \quad x \in (\mathbb{R}/2\pi\mathbb{Z})^d, \tag{4.29}$$

$$\mathcal{F}u(k) = \frac{1}{(2\pi)^d} \int_{(\mathbb{R}/2\pi\mathbb{Z})^d} e^{-ik\cdot x}u(x)dx, \; k \in \mathbb{Z}^d. \tag{4.30}$$

A more precise relation is given by a Poisson formula that we shall derive. Let f be a function on $(\mathbb{R}/2\pi\mathbb{Z})^d$; look at

$$F(h) = \frac{1}{N^d} \sum_{x\in(\frac{2\pi}{N}\mathbb{Z}/2\pi\mathbb{Z})^d} f\left(x + \frac{h}{N}\right), \quad h \in (\mathbb{R}/2\pi\mathbb{Z})^d.$$

An easy calculation shows that the Fourier coefficients of this $(2\pi\mathbb{Z})^d$-periodic function are given by

$$\mathcal{F}F(\ell) = \mathcal{F}f(N\ell),$$

and the Fourier inversion formula for $h = 0$ gives the following Poisson formula:

$$\frac{1}{N^d} \sum_{x\in(\frac{2\pi}{N}\mathbb{Z}/2\pi\mathbb{Z})^d} f(x) = \sum_{\ell\in\mathbb{Z}^d} \mathcal{F}f(N\ell).$$

Replacing here $f(x)$ by $f(x)e^{-ik\cdot x}$, $k \in \mathbb{Z}^d$, and using the relation $\mathcal{F}(f(x)e^{-ik\cdot x})(N\ell) = (\mathcal{F}f)(N\ell + k)$, we obtain

$$\mathcal{F}_N f(k) = \sum_{\ell\in\mathbb{Z}^d} \mathcal{F}f(k + N\ell). \tag{4.31}$$

This is slightly incoherent, since k on the left should rather be viewed as an element of $(\mathbb{Z}/N\mathbb{Z})^d$. If $\pi_N \colon \mathbb{Z}^d \to (\mathbb{Z}/N\mathbb{Z})^d$ is the natural projection, for $k \in (\mathbb{Z}/N\mathbb{Z})^d$ we get:

$$\mathcal{F}_N f(k) = \sum_{\{\widetilde{k}\in\mathbb{Z}^d;\pi_N\widetilde{k}=k\}} \mathcal{F}f(\widetilde{k}). \tag{4.32}$$

Let $v = v_N \geqslant 0$ be an even function on $(\mathbb{Z}/N\mathbb{Z})^d$, and let $\widetilde{v} = \widetilde{v}_N$ be the unique (even) function on $\mathbb{Z}^d$ with $\operatorname{supp}\widetilde{v}_N \subset \{k \in \mathbb{Z}^d; \|k\|_{\ell^\infty} \leqslant \frac{N}{2}\}$ such that

$\widetilde{v}_N(k_1) = \widetilde{v}_N(k_2)$ if $\pi_N(k_1) = \pi_N(k_2)$ and satisfying $v_N(k) = \sum_{\pi_N^{-1}(k)} \widetilde{v}_N(\widetilde{k})$. (Observe that this definition simplifies in the case where N is odd.)

Let F_N be the function corresponding to $\widetilde{v}_N$, defined as in (4.4). Since $\widetilde{v}_N$ has compact support, F_N is defined everywhere. Let $\Omega \subset \mathbb{R}^d$ be a convex open set, symmetric with respect to the origin; we assume that

$$F_N = \mathcal{O}(1) \quad \text{in } \Omega. \tag{4.33}$$

We continue to use the convention that all estimates should be uniform with respect to N; later we shall replace the index set $\Gamma = \{1, 2, \dots, m\}$ by $\Gamma = (\mathbb{Z}/N\mathbb{Z})^d$.) We assume that

$$F_N(0) \leqslant 1 - \frac{1}{\mathcal{O}(1)}, \tag{4.34}$$

$$\begin{gathered} D_r = D_{r,N} = \{y \in \Omega; F_N(y) \leqslant r\},\ r \leqslant R, \text{ are compact in } \Omega, \\ \text{uniformly with respect to } r \text{ and } N. \end{gathered} \tag{4.35}$$

Here $R > 1$ is some fixed number. We also assume that

$$\begin{gathered} \text{there exists a compact set } K \subset \mathbb{Z}^d \text{ such that the group generated by} \\ K \text{ is equal to } \mathbb{Z}^d \text{ and, for sufficiently large } N,\ \widetilde{v}_N \geqslant \frac{1}{\mathcal{O}(1)} \text{ on } K. \end{gathered} \tag{4.36}$$

From now on we restrict the discussion to sufficiently large values of N. Under these uniform assumptions, we see that the F_N are uniformly in $C^\infty(\Omega)$ that D_r, $r \leqslant R$, are uniformly strictly convex, and that all the said above about the inverse of $I - v*$ and the weighted estimates for this operator applies uniformly to the convolution operator $I - \widetilde{v}_N *$. The inverse convolution kernel $\widetilde{E}_N$ with $(\delta_0 - \widetilde{v}_N) * \widetilde{E}_N = \delta_0$ has the following asymptotics uniform with respect to N:

$$\widetilde{E}_N(k) = \frac{(\partial_{y_d} F_N(y_N(k)))^{\frac{d-3}{2}}}{\sqrt{\det \partial_{y'} F_N(y_N(k))}} \frac{(1 + \mathcal{O}(\frac{1}{|k|}))}{(2\pi|k|)^{\frac{d-1}{2}}} e^{-H_{D,N}(k)}, \quad |k| \to \infty, \tag{4.37}$$

where $y_N(k)$, (y', y_d) (now depending on N) are defined as before. $H_{D,N}$ and $1/H_{D,N}$ are uniformly bounded in $C^\infty(\mathbb{R}^d \setminus \{0\})$, convex, positively homogeneous of degree 1, and uniformly convex in the non-radial directions.

As before, we have the distances

$$\widetilde{d}_r(k, \ell) = \widetilde{d}_{r,N}(k, \ell) = H_{D_r,N}(k - \ell)$$

on $\mathbb{Z}^d$; now we also introduce the corresponding distance on $(\mathbb{Z}/N\mathbb{Z})^d$:

$$d_{r,N}(k, \ell) = \inf_{\substack{\pi_N(\widetilde{k}) = k, \\ \pi_N(\widetilde{\ell}) = \ell}} \widetilde{d}_{r,N}(\widetilde{k}, \widetilde{\ell}). \tag{4.38}$$

The convolution operator $I - v_N *$ on $(\mathbb{Z}/N\mathbb{Z})^d$ has the inverse kernel $E_N(k)$ given by the formula

$$E_N(k) = \mathcal{F}_N\left(\frac{1}{1-\mathcal{F}_N^{-1}v_N}\right)(k). \tag{4.39}$$

Here $\mathcal{F}_N^{-1}v_N(x) = \mathcal{F}^{-1}\widetilde{v}_N(x)$, $x \in (\frac{2\pi}{N}\mathbb{Z}/2\pi\mathbb{Z})^d$; applying (4.32), we get

$$E_N(k) = \sum_{\widetilde{k}\in\pi_N^{-1}(k)} \widetilde{E}_N(\widetilde{k}). \tag{4.40}$$

(A different way to obtain this formula is to observe that $\pi_{N*}(\widetilde{u}*\widetilde{v}) = (\pi_{N*}\widetilde{u})*(\pi_{N*}\widetilde{v})$, where π_{N*} is the natural direct image associated with π_N.) From (4.37) we deduce the estimate

$$E_N(k) = \mathcal{O}(|k|^{\frac{1-d}{2}}e^{-d_{1,N}(0,k)}). \tag{4.41}$$

In certain regions, we can get asymptotics. Assume that, for some fixed $\delta > 0$,

$$d_{1,N}(0,k) = H_{D,N}(\widehat{k}) \leqslant (1+\delta)^{-1}H_{D,N}(\widetilde{k}), \quad \pi_N(\widetilde{k}) = \pi_N(\widehat{k}), \quad \widetilde{k} \neq \widehat{k},$$

with some $\widehat{k} \in \pi_N^{-1}(k)$. For such values of k, (4.40), (4.37) yield k

$$E_N(k) = \frac{(\partial_{y_d}F_N(y_N(\widehat{k})))^{\frac{d-3}{2}}}{\sqrt{\det\partial_{y'}F_N(y_N(\widehat{k}))}}\frac{(1+\mathcal{O}(\frac{1}{|\widehat{k}|}))}{(2\pi|k|)^{\frac{d-1}{2}}}e^{-H_{D,N}(\widehat{k})}, \quad |k| \to \infty. \tag{4.42}$$

To complete this section, we extend the weighted estimates to the discrete torus case. If $\rho\colon (\mathbb{Z}/N\mathbb{Z})^d \to]0,\infty[$ satisfies

$$\frac{\rho(k)}{\rho(\ell)} \leqslant e^{d_R(k,\ell)} \tag{4.43}$$

(we shall suppress the subscript N quite often), then $\|\rho(v*)\rho^{-1}\|_{\mathcal{L}(\ell^p,\ell^p)} = \mathcal{O}(1)$ (uniformly in N, ρ). Indeed, let $\widetilde{\rho} = \rho\circ\pi_N$, and let $\widetilde{v}$ be the special lift of v to $\mathbb{Z}^d$ discussed above. Then, for example, for $\pi_N(\widetilde{k}) = k$ we have

$$\sum_{\ell}\frac{\rho(k)}{\rho(\ell)}v(k-\ell) = \sum_{\widetilde{\ell}\in\mathbb{Z}^d}\frac{\widetilde{\rho}(\widetilde{k})}{\widetilde{\rho}(\widetilde{\ell})}\widetilde{v}(\widetilde{k}-\widetilde{\ell}),$$

which can be estimated as after (4.22), since $\widetilde{\rho}$ satisfies (4.22).

We also have (4.23) on $(\mathbb{Z}/N\mathbb{Z})^d$, where it is understood that both N and L tend to ∞ in such a way that $L < \frac{N}{2}$, so that $B(0,L)$ regarded as a subset of $(\mathbb{Z}/N\mathbb{Z})^d$ is in a natural bijective correspondence with $B(0,L)$ as a subset of $\mathbb{Z}^d$. The condition (4.24) becomes meaningful under the same convention, and we have the corresponding extension of Proposition 4.1. Indeed define $\widetilde{v}$ and $\widetilde{\rho}$ as before. The natural lift of $1_{B(0,L)}v$ is $1_{B(0,L)}\widetilde{v}$; so, we may assume (as in the proof of Proposition 4.1) that v and $\widetilde{v}$ have their support in $B(0,L)$. Then, for example, for $\pi_N(\widetilde{k}) = k$ we have

$$\sum_{\ell} \frac{1}{2}\left(\frac{\rho(k)}{\rho(\ell)} + \frac{\rho(\ell)}{\rho(k)}\right) v(k-\ell) = \sum_{\widetilde{\ell}\in\mathbb{Z}^d} \frac{1}{2}\left(\frac{\widetilde{\rho}(\widetilde{k})}{\widetilde{\rho}(\widetilde{\ell})} + \frac{\widetilde{\rho}(\widetilde{\ell})}{\widetilde{\rho}(\widetilde{k})}\right)\widetilde{v}(\widetilde{k}-\widetilde{\ell}),$$

which can be estimated as in the proof of Proposition 4.1.

§5. Asymptotics of the correlations

Let $\Gamma = (\mathbb{Z}/N\mathbb{Z})^d$, where d is fixed and N is sufficiently large. We write $x = x_\Gamma \in \mathbb{R}^\Gamma$. Let $\phi \in C^\infty(\mathbb{R}^\Gamma;\mathbb{R})$. We shall make a series of assumptions about ϕ, implying the appropriate assumptions used earlier in this paper. If $\gamma \in \Gamma$ and $x \in \mathbb{R}^\Gamma \simeq \ell^2(\Gamma)$, we put $(\tau_\gamma x)_j = x_{j-\gamma}$ Assume that

$$\phi \text{ satisfies the last condition from (H1) and } \phi = \tau_\gamma^*\phi,\ \ \gamma\in\Gamma, \tag{G1}$$

where $\tau_\gamma^*\phi(x) = \phi(\tau_\gamma x)$. Then

$$\partial_{x_j}\phi(x) = (\partial_{x_{j+\gamma}}\phi)(\tau_\gamma x),\quad \partial_{x_j}\partial_{x_k}\phi(x) = (\partial_{x_{j+\gamma}}\partial_{x_{k+\gamma}}\phi)(\tau_\gamma x). \tag{5.1}$$

Assume that

$$\phi'(0) = 0 \quad\text{and}\quad \phi'' \geqslant r_\phi > 0, \tag{G2}$$

for some fixed constant $r_\phi > 0$. From the second parts of (G2) and (G1) it follows that ϕ has a unique critical point x_c and that x_c is of the form $(t,\dots,t)$. The first part of (G2) only says that $t = 0$, which could have been achieved by translating all the coordinates by the same amount. It follows from (5.1) that $\phi''(0)$ is a convolution matrix. Without loss of generality, we may assume that $\phi''(0) = I - v*$, where v is a function on $(\mathbb{Z}/N\mathbb{Z})^d$ vanishing at 0. We assume that $\phi''(0)$ is of ferromagnetic type:

$$\partial_{x_j}\partial_{x_k}\phi(0) \leqslant 0 \quad \text{if } j \neq k. \tag{G3}$$

In other words, v is assumed to be a nonnegative function. Observe that v is even; from (G2) it follows that

$$\|v\|_{l^1} \leqslant 1 - r_\phi.$$

We make all the assumptions of Section 4 about v: Let $\widetilde{v}$ be the lift of v to $\mathbb{Z}^d$ determined by the conditions

$$(\pi_N)_*\widetilde{v} = v,\ \operatorname{supp}\widetilde{v} \subset \left\{k \in \mathbb{Z}^d\ ;\ |k|_\infty \leqslant \tfrac{N}{2}\right\},\ \widetilde{v}(k_1) = \widetilde{v}(k_2) \text{ if } |k_j|_\infty \leqslant \tfrac{N}{2},\ \pi_N(k_1) = \pi_N(k_2). \tag{5.2}$$

Here π_N is the natural projection $\mathbb{Z}^d \to (\mathbb{Z}/N\mathbb{Z})^d$ and π_{N*} is the corresponding direct image. We assume that

there is a bounded set $K \subset \mathbb{Z}^d$ independent of Γ such that, for N sufficiently large, we have $v \geqslant \frac{1}{\mathcal{O}(1)}$ on K, and $\widehat{K} = \mathbb{Z}^d$. (G4)

Here $\widehat{K}$ denotes the smallest subgroup of $\mathbb{Z}^d$ containing K. Let $F(y) = F_\Gamma(y) = \sum \widetilde{v}(k)\operatorname{ch} y \cdot k$, $y \in \mathbb{R}^d$. We assume that

there is an open neighborhood $\Omega \subset \mathbb{R}^d$ of 0 independent of N such that $F\ (= F_\Gamma) = \mathcal{O}(1)$ on Ω. (G5)

Observe that $\|v\|_{\ell^1} = \|\widetilde{v}\|_{\ell^1} = F(0)$,

$$\|v\|_{\ell^1} \leqslant 1 - \frac{1}{\mathcal{O}(1)}. \tag{G6}$$

We also assume that

there is a number $R > 1$, independent of Γ, such that $D_R \overset{\text{def}}{=} \{y \in \Omega;\ F(y) \leqslant R\}$ is uniformly compact. (G7)

For $F(0) < r \leqslant R$, we also introduce $D_r = \{y \in \Omega; F(y) \leqslant r\}$. From Section 4 it follows that F is uniformly bounded in $C^\infty(\Omega)$, and uniformly strictly convex on D_R. The whole discussion of that section now applies to the convolution operator $\phi''(0)^{-1}$ without any further assumptions. In particular, if ρ: $(\mathbb{Z}/N\mathbb{Z})^d \to]0,\infty[$ satisfies

$$\frac{\rho(k)}{\rho(\ell)} \leqslant e^{d_R(k,\ell)}, \quad k,\ell \in (\mathbb{Z}/N\mathbb{Z})^d, \tag{5.3}$$

where $d_r = d_{r,N}$ is the distance defined in Section 4, then

$$\|\rho\phi''(0)\rho^{-1}\|_{\mathcal{L}(\ell^p,\ell^p)} = \mathcal{O}(1), \quad 1 \leqslant p \leqslant \infty, \tag{5.4}$$

uniformly with respect to N, ρ. After decreasing R to some new fixed value > 1 (if needed), we assume that

$$\|\rho\phi''(x)\rho^{-1}\|_{\mathcal{L}(\ell^p,\ell^p)} = \mathcal{O}(1), \quad 1 \leqslant p \leqslant \infty, \tag{G8}$$

uniformly with respect to Γ, $x \in \mathbb{R}^\Gamma$, and ρ as in (5.3). Before continuing, we check which of the hypotheses from Sections 1-3 are fulfilled. (G7) implies (H4). (G8) implies (H8) for ρ as in (5.3). From (G7), (G2), we get (H2), and from (H2), (G1), we get (H1) with $\delta = 1$. After decreasing R to a new fixed value > 1 (if needed), we assume that

we have (H11) uniformly for ρ as in (5.3), as well as (H5), (H.6). (G9)

Observe that (H11) implies (H7,10) and that (H10) implies (H3). Since from (G1-8), we get (H1-8,10,11), Theorem 2.3 is entirely available. We put

$$r_0 = 1 - \|v\|_{\ell^1}; \tag{5.5}$$

this is the smallest eigenvalue of $\phi''(0)$ and, hence, also $\mathcal{O}(h^{1/2})+$ the smallest eigenvalue of $\langle\phi''(0)\rangle$. We assume (cf (H9)) possibly after decreasing R to a new value $\geqslant 1 + \frac{1}{\mathcal{O}(1)}$, that

if ρ is a weight on $(\mathbb{Z}/N\mathbb{Z})^d$ and $\widetilde{\rho}$ ($= \rho \circ \pi_N$) satisfies $\sup_{|k-\ell|\leqslant L} \max(|\log \frac{\widetilde{\rho}(k)}{\widetilde{\rho}(\ell)} - y(k)\cdot(k)\ell)|, |\log \frac{\widetilde{\rho}(k)}{\widetilde{\rho}(\ell)} - y(\ell)\cdot(k-\ell)|) \leqslant \delta$, with $y(k) \in D_R$, for some sufficiently small δ, $1/L$, then $(\rho\phi''(x)\rho^{-1}t,t) \geqslant (\frac{1}{\mathcal{O}(1)} - r_0)\|t\|^2$. (G10)

(Here it is assumed that $L \leqslant N/2$.) Since we may decrease R when going from (5.3) to (G10), we may assume that the ρ's in (G10) satisfy (5.3). We recall (see Section 4) that for $x = 0$ we have

$$(\rho\phi''(0)\rho^{-1}t,t) \geqslant (1 - R - \epsilon(\delta, 1/L))\|t\|^2,$$

where $\epsilon(\delta, 1/L) \to 0$ as $\delta, 1/L \to 0$. So, for $R < 1 + r_0$, we get (G10) from the following stronger assumption:

$$\phi''_{j,k}(x) \leqslant 0 \text{ for } j \neq k \text{ and } \phi''_{j,k}(0) \leqslant \phi''_{j,k}(x) \text{ for all } j, k. \tag{G10$_{\text{strong}}$}$$

When ($G10_{\mathrm{strong}}$) holds, we may drop (H5) in (G9), since (H5) will hold automatically. The remaining hypothesis (H9) is fulfilled for ρ as in (G10); therefore, Theorem 3.1 is available and we have

$$\|\rho(E_{-+}+2h\phi''(0))\rho^{-1}\|_{\mathcal{L}(\ell^2,\ell^2)}=\mathcal{O}(h^{3/2}), \tag{5.6}$$

$$\|\rho(E_{+}-R_{-})\rho^{-1}\|_{\mathcal{L}(\ell^2,L^2)}=\mathcal{O}(h^{1/2}), \tag{5.7}$$

$$\|\rho(E_{-}-R_{+})\rho^{-1}\|_{\mathcal{L}(L^2,\ell^2)}=\mathcal{O}(h^{1/2}), \tag{5.8}$$

uniformly for ρ satisfying (G.10) if δ and $1/L$ are sufficiently small. Moreover, from (G1) it follows that E_{-+} is a convolution.

After decreasing Ω and R from (5.7) we see that $-\frac{1}{2h}E_{-+}$ is the convolution with some kernel δ_0-w, where

$$\widehat{\widetilde{w}}=\widehat{\widetilde{v}}+\mathcal{O}(h^{1/2})\quad\text{in } \mathbb{T}^d+i\Omega, \tag{5.9}$$

and $\widetilde{w},\widetilde{v}$ are the lifts to $\mathbb{Z}^d$ defined in Section 4.

Then analyzing the inverse of $(\delta_0-\widetilde{w})*$ and of $(\delta_0-w)*$ as in Section 4, we get

$$\begin{aligned}(\delta_0-\widetilde{w}*)^{-1}(\widetilde{k},\widetilde{\ell})=&\frac{(\partial_{y_d}F_N(y_N(\widetilde{k}-\widetilde{\ell})))^{\frac{d-3}{2}}}{\sqrt{\det\partial^2_{y'}F_N(y_N(\widetilde{k},\widetilde{\ell}))}}\\&\times\frac{(1+\mathcal{O}(h^{1/2})+\mathcal{O}(\frac{1}{|\widetilde{k}-\widetilde{\ell}|}))}{(2\pi|\widetilde{k}-\widetilde{\ell}|)^{(d-1)/2}}e^{-H_{D,N,h}(\widetilde{k}-\widetilde{\ell})}\end{aligned} \tag{5.10}$$

(cf. (4.37)), where $H_{D,N,h}(k)=H_{D,N}(k)+\mathcal{O}(h^{1/2}|k|)$ has the same properties as $H_{D,N}$. Let $d_{1,N,h}$ be the corresponding distance on $(\mathbb{Z}/N\mathbb{Z})^d$. We have

$$E_{-+}^{-1}(k,\ell)=\mathcal{O}(1)h^{-1}|k-\ell|^{(1-d)/2}e^{-d_{1,N,h}(k,\ell)}, \tag{5.11}$$

and if we are in a region where, for some $\widehat{k}\in\pi_N^{-1}(k)$, $\widehat{\ell}\in\pi_N^{-1}(\ell)$,

$$d_{1,N}(k,\ell)\leqslant(1+\delta)^{-1}H_{D,N}(\widetilde{k}-\widetilde{\ell})$$

for all $\widetilde{k}\in\pi_N^{-1}(k)$, $\widetilde{\ell}\in\pi_N^{-1}(\ell)$ satisfying $\widetilde{k}-\widetilde{\ell}\neq\widehat{k}-\widehat{\ell}$, then

$$E_{-+}^{-1}(k,\ell)=-\frac{1}{2h}\cdot\text{(same expression as the RHS of (5.10)).} \tag{5.12}$$

Finally, we shall use all the above results in order to study the correlations of two functions. For simplicity, we take $u = x_j$, $v = x_k$. Combining (1.11) with the standard formulas

$$(\Delta_\phi^{(1)})^{-1} = E - E_+ E_{-+}^{-1} E_-, \quad E_+^* = E_-,$$

we get

$$\begin{aligned}\mathrm{Cor}(u,v) &= h^2(Ee^{-\phi/h}du|e^{-\phi/h}dv) \\ &\quad - h^2(E_{-+}^{-1}E_-e^{-\phi/h}du|E_-e^{-\phi/h}dv).\end{aligned} \tag{5.13}$$

We know that $\|\rho E\rho^{-1}\|_{\mathcal{L}(L^2,L^2)} = \mathcal{O}(\frac{1}{h})$ if ρ satisfies the assumptions of (G.10). By varying ρ, it is easily seen that

$$\begin{aligned}&|h^2(Ee^{-\phi/h}dx_j|e^{-\phi/h}dx_k)| \\ &\quad \leqslant \mathcal{O}(1)h|j-k|^{-(d-1)/2}e^{-(1+\delta_0)d_{1,N,h}(j,k)}\end{aligned} \tag{5.14}$$

for some fixed $\delta_0 > 0$. Therefore, we expect that the main contribution in (5.13) comes from the second term. In the study of that term, we first use (5.7) and (5.11) to see that

$$\begin{aligned}&|h^2((E_{-+}^{-1}E_-e^{-\phi/h}dx_j|E_-e^{-\phi/h}dx_k) - (E_{-+}^{-1}R_+e^{-\phi/h}dx_j|R_+e^{-\phi/h}dx_k))| \\ &\quad \leqslant \mathcal{O}(1)h^{3/2}|j-k|^{(1-d)/2}e^{-d_{1,N,h}(j,k)}.\end{aligned} \tag{5.15}$$

Observing, that

$$R_+e^{-\phi/h}du = \langle du\rangle, \tag{5.16}$$

and, in particular, that

$$R_+e^{-\phi/h}dx_j = \delta_j, \tag{5.17}$$

we obtain

$$(E_{-+}^{-1}R_+e^{-\phi/h}dx_j|R_+e^{-\phi/h}dx_k) = E_{-+}^{-1}(j,k). \tag{5.18}$$

Thus, we arrive at the following main result.

Theorem 5.1. *Let $\phi = \phi_\Gamma$ be a family of smooth real valued functions on $\mathbb{R}^\Gamma$ satisfying* (G1-10) (*uniformly*) *for sufficiently large N. Then for sufficiently large N and $1/h$, we have*

$$\mathrm{Cor}(x_j, x_k) = \mathcal{O}(1)h\langle j-k\rangle^{(1-d)/2}e^{-d_{1,N,h}(j,k)}. \tag{5.19}$$

If, in addition, for some $\widehat{k} \in \pi_N^{-1}(k)$, $\widehat{\ell} \in \pi_N^{-1}(\ell)$, *we have* $d_{1,N}(j,k) = H_{D,N}(\widehat{j} - \widehat{k}) \leqslant (1+\delta)^{-1} H_{D,N}(\widetilde{j} - \widetilde{k})$ *for all* $\widetilde{j} \in \pi_N^{-1}(j)$, $\widetilde{k} \in \pi_N^{-1}(k)$ *satisfying* $\widetilde{j} - \widetilde{k} \neq \widehat{j} - \widehat{k}$ *(where* $\delta > 0$ *is fixed), then*

$$\mathrm{Cor}(x_j, x_k) = \frac{h}{2} \cdot \frac{(\partial_{y_d} F_N(y_N(\widehat{k} - \widehat{\ell})))^{(d-3)/2}}{\sqrt{\det \partial^2_{y'} F_N(y_N(\widehat{k} - \widehat{\ell}))}} \times \frac{(1 + \mathcal{O}(h^{1/2}) + \mathcal{O}(\frac{1}{|\widehat{k} - \widehat{\ell}|}))}{(2\pi |\widehat{k} - \widehat{\ell}|)^{(d-1/2}} e^{-H_{D,N,h}(\widehat{k} - \widehat{\ell})}. \tag{5.20}$$

Here $H_{D,N,h}(k) = H_{D,N}(k) + \mathcal{O}(h^{1/2}|k|) \in C^\infty(\mathbb{R}^d \setminus \{0\})$ *is positively homogeneous of degree* 1 *in the variable* k. $H_{D,N}(k)$ *is the norm discussed above and, also, the support function of the strictly convex set* $D_1 = D_{1,\Gamma}$.

Remark. If $\phi''(0) = I - v*$, where the "lift" $\widetilde{v}$ has bounded support and is independent of N, then y_N, F_N, $H_{D,N}$ are independent of N.

Example. As in [HS2], we treat a potential motivated by some work of M. Kac (with $d = 1$):

$$2\phi(x) = \sum_{j \in \mathbb{Z}/N\mathbb{Z}} \frac{x_j^2}{4} - \log \mathrm{ch} \sqrt{\frac{\nu}{2}}(x_j + x_{j+1}), \quad \nu \geqslant 0;$$

we assume that $\nu < 1/4$, which is precisely the case where ϕ is strictly convex. Indeed,

$$\partial^2_{x_j}(2\phi) = \frac{1}{2} - \frac{\nu}{2} \frac{1}{\mathrm{ch}^2 \sqrt{\frac{\nu}{2}}(x_{j-1} + x_j)} - \frac{\nu}{2} \frac{1}{\mathrm{ch}^2 \sqrt{\frac{\nu}{2}}(x_j + x_{j+1})},$$

$$\partial_{x_j} \partial_{x_{j+1}}(2\phi) = -\frac{\nu}{2} \frac{1}{\mathrm{ch}^2 \sqrt{\frac{\nu}{2}}(x_j + x_{j+1})},$$

$$\partial_{x_j} \partial_{x_k}(2\phi) = 0 \quad \text{if } \mathrm{dist}(j,k) \geqslant 2,$$

whence

$$4\phi''(x) \geqslant ((1 - 2\nu)\delta_0 - \nu\delta_{-1} - \nu\delta_1)* = 4\phi''(0).$$

We have

$$\frac{4}{(1-2\nu)} \phi''(0) = (\delta_0 - v)*, \quad v = \frac{\nu}{1-2\nu}\delta_{-1} + \frac{\nu}{1-2\nu}\delta_1,$$

and the lift $\widetilde{v}$ of v to $\mathbb{Z}$ is given by the same expression as v. The corresponding function F, defined on $\Omega = \mathbb{R}$ is given by the formula $F(y) = \frac{2\nu}{1-2\nu} \mathrm{ch}\, y$. It is easy to check that conditions (G1-10) and even ($\mathrm{G10}_{\mathrm{strong}}$), are satisfied; so, Theorem 5.1 applies, now with N, y_N, and $H_{D,N}$ is independent of N. Since we have not performed the thermodynamical limit, we cannot assert, however, that there are no N-dependent oscillations in the $\mathcal{O}(h^{1/2})$-terms. In this case, there is some overlap with Theorem 3.1 in [S2] and (5.19) does not improve the upper bound of that theorem. The lower bound given by Theorem 5.1 seems to be new.

Appendix

Here we prove that (4.10) implies (4.2). First, we recall that if V is a discrete subgoup of $\mathbb{R}^d$, then $V = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_k$, where $e_1, \dots, e_k$ are linearly independent. It is also useful to recall a proof of this fact, by induction on d. For $d = 1$, let $0 \neq e_1 \in V$ be an element with minimal norm. Then $V = \mathbb{Z}e_1$. Assume we have proved the statement for the dimension $d-1$, and let V be a closed discrete subgroup of $\mathbb{R}^d$. Let $e_d \in V$ be minimal in the sense that if $v = \lambda e_d \in V \setminus \{0\}$, then $|\lambda| \geqslant 1$ (necessarily, λ is an integer). We may assume that $e_d = (0, \dots, 1)$ and consider V', the projection of V to $\mathbb{R}^{d-1}$. We claim that V' is discrete. Let v'_j be a bounded family in V', and let $v_j \in V$ be a preimage of v'_j. Replacing v_j by $v_j - \lambda_j e_d$ for suitable $\lambda_j \in \mathbb{Z}$, we may assume that the v_j form a bounded family. Since V is discrete, v_j take only finitely many values; consequently, the same is true for v'_j. Thus, V' is discrete. We can then apply the induction hypothesis to V' to obtain the statement in dimension d.

Let V be a subgroup of $\mathbb{Z}^d$.

Proposition A.1. *If $V \neq \mathbb{Z}^d$, we can find $\theta \in \mathbb{R}^d \setminus (2\pi\mathbb{Z})^d$ such that $v \cdot \theta \in 2\pi\mathbb{Z}$ for all $v \in V$.*

Proof. If $k < d$, it suffices to take $\theta \notin (2\pi\mathbb{Z})^d$ in the orthogonal space of $\mathbb{R} \otimes V$. Let $k = d$. Let $e \in \mathbb{Z}^d \setminus V$; if necessary, we replace e by one of the two generators of $\mathbb{R}e \cap \mathbb{Z}^d$. Then the new e is still off V. If $V = \mathbb{Z}f_1 \oplus \cdots \oplus \mathbb{Z}f_d$, we have $e = \lambda_1 f_1 + \dots \lambda_d f_d$, with $\lambda_j \in \mathbb{Q}$ so $me \in V$ for some $m \in \mathbb{Z}$. Let $m \neq 0$, $me \in V$, with $|m|$ minimal. Clearly $|m| \geqslant 2$. By the above construction, we can find $f_1, \dots, f_d$ with $f_d = me$. Choose $\theta \in \mathbb{R}^d$ with $f_j \cdot \theta = 0$, $1 \leqslant j \leqslant m-1$, $f_d \cdot \theta = 2\pi$. Then $v \cdot \theta \in 2\pi\mathbb{Z}$ for all $v \in V$; since $e \cdot \theta = \frac{2\pi}{m} \notin 2\pi\mathbb{Z}$, we see that $\theta \notin (2\pi\mathbb{Z})^d$. •

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