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ON DERIVATIONS ASSOCIATED WITH DIFFERENT ALGEBRAIC STRUCTURES IN GROUP ALGEBRAS

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Abstract. The paper is devoted to the comparison of derivation algebras arising in associative and Lie structures of group algebras. We shall prove that an algebra of derivations given by Lie-structure contains an algebra of associative derivations. We will give a description of Lie derivations in terms of the gruppoid associated with an inner action of the group.

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Let G be a discrete group. Denote by $\mathbf{C}[G]$ the group algebra. The algebra $\mathbf{C}[G]$ is associative and has the structure of a Lie algebra with the commutator $[x, y] := xy - yx$.

The purpose of this paper is the comparison of derivations defined by an associative structure and derivations defined by a Lie structure.

The combinatorial structure of a gruppoid Γ introduced in paper [1] generates derivations for an associative structure and for a Lie structure.

Definition 1. A linear mapping $d : \mathbf{C}[G] \to \mathbf{C}[G]$ is called a derivation provided

$$d(uv) = d(u)v + ud(v) \quad \forall u, v \in \mathbf{C}[G]. \tag{1}$$

The derivation d is a derivation with respect to an associative structure. The set of all derivations is a Lie algebra and is denoted by $Der(\mathbf{C}[G])$.

Definition 2. A linear mapping $\partial : \mathbf{C}[G] \to \mathbf{C}[G]$ is called a rate provided

$$\partial([u, v]) = [\partial(u), v] + [u, \partial(v)] \quad \forall u, v \in \mathbf{C}[G]. \tag{2}$$

We denote the linear space of all rates by $Rate(\mathbf{C}[G])$. Note that $Rate(\mathbf{C}[G])$ is not a Lie algebra.

It is a straightforward task to ensure that a derivation is also a rate.

Proposition 1. *A derivation $d \in Der(\mathbf{C}[G])$ is a rate. So, the algebra $Der(\mathbf{C}[G])$ is a subspace of the space $Rate(\mathbf{C}[G])$.*

Proof. It suffices to prove that any derivation $d \in Der(\mathbf{C}[G])$ satisfies condition (2), i.e.

$$d([u, v]) = [d(u), v] + [u, d(v)]. \tag{3}$$

From the linearity and condition (1) it follows that

$$d([u, v]) = d(uv) - d(vu) = d(u)v + ud(v) - d(v)u - vd(u) =$$
$$d(u)v - vd(u) + ud(v) - d(v)u = [d(u), v] + [u, d(v)].$$

Hence, $Der(\mathbf{C}[G]) \subset Rate(\mathbf{C}[G])$. □

For an abelian group G any linear mapping $\tau : \mathbf{C}[G] \to \mathbf{C}[G]$ is a rate. Indeed, since group G is abelian, it is easily seen that

$$0 = d([u,v]) = [d(u),v] + [u,d(v)].$$

For a nontrivial group G the algebra $Der(\mathbf{C}[G])$ can not coincide with the space $Rate(\mathbf{C}[G])$. Let us illustrate this fact by the following example.

Example 1. Define the mapping $\tau : \mathbf{C}[G] \to \mathbf{C}[G]$ over generators $g \in G \subset \mathbf{C}[G]$ as follows:

$$\tau : g \to e.$$

Here e is the neutral element of the group G. Extend the mapping τ by linearity as follows:

$$\tau\left(\sum_{g\in G} \alpha_g g\right) = \left(\sum_{g\in G} \alpha_g\right) e.$$

All the indicated sums are finite.

Let us show that the mapping τ is a rate but not a derivation. Let elements $g_1, g_2 \in G \subset \mathbf{C}[G]$ be generators of the group G. Then we get

$$\tau([g_1,g_2]) = 0 = [\tau(g_1),g_2] + [g_1,\tau(g_2)].$$

So, the mapping τ is a rate. But τ is not a derivation since

$$\tau(g_1g_2) = e,$$

whereas

$$\tau(g_1)g_2 + g_1\tau(g_2) = g_2 + g_1 \neq e.$$

Proposition 2. *The space $Rate(\mathbf{C}[G])$ is a Lie algebra.*

Proof. By ∂_1, ∂_2 denote the following two rates:

$$\begin{aligned} \partial_1([u,v]) &= [\partial_1(u),v] + [u,\partial_1(v)] \quad \forall u,v \in C[G], \\ \partial_2([u,v]) &= [\partial_2(u),v] + [u,\partial_2(v)] \quad \forall u,v \in C[G], \end{aligned}$$

We claim that the commutator $[\partial_1,\partial_2] := \partial_1\partial_2 - \partial_2\partial_1$ is a rate.

$$\begin{aligned} &\partial_2\partial_1([u,v]) = \partial_2\left([\partial_1(u),v] + [u,\partial_1(v)]\right) = \\ &= [\partial_2\partial_1(u),v] + [\partial_1(u),\partial_2(v)] + [\partial_2(u),\partial_1(v)] + [u,\partial_2\partial_1(v)], \end{aligned}$$

$$\begin{aligned} &\partial_1\partial_2([u,v]) = \partial_1\left([\partial_2(u),v] + [u,\partial_2(v)]\right) = \\ &= [\partial_1\partial_2(u),v] + [\partial_2(u),\partial_1(v)] + [\partial_1(u),\partial_2(v)] + [u,\partial_1\partial_2(v)]. \end{aligned}$$

Combining $\partial_2\partial_1([u,v])$ and $\partial_1\partial_2([u,v])$ we get

$$\begin{aligned} &[\partial_2,\partial_1]([u,v]) = \partial_2\partial_1([u,v]) - \partial_1\partial_2([u,v]) = \\ &= [\partial_2\partial_1(u),v] + [\partial_1(u),\partial_2(v)] + [\partial_2(u),\partial_1(v)] + [u,\partial_2\partial_1(v)] - \\ &- \left([\partial_1\partial_2(u),v] + [\partial_2(u),\partial_1(v)] + [\partial_1(u),\partial_2(v)] + [u,\partial_1\partial_2(v)]\right) = \\ &= [[\partial_2,\partial_1](u),v] + [u,[\partial_2,\partial_1](v)]. \end{aligned}$$

□

Propositions 1, 2 and Example 1 imply the following assertion.

Theorem 1. *If G is a discrete group then the algebra of all derivations $Der(\mathbf{C}[G])$ is a Lie subalgebra of the Lie algebra $Rate(\mathbf{C}[G])$. If G is a nontrivial group then the algebra of all derivations $Der(\mathbf{C}[G])$ cannot coincide with the Lie algebra $Rate(\mathbf{C}[G])$.*

This theorem is presumably known in literature but it is important for the following discussion.

Let ∂ be a rate. Then the action of the mapping ∂ on an element $u := \sum\limits_{h\in G} \lambda_u^h h \in \mathbf{C}[G]$ can be written in the following way

$$\partial(u) = \sum_{g\in G}\left(\sum_{h\in G}\partial_h^g \lambda_u^h\right) g, \tag{4}$$

where all coefficients $\partial_h^g \in \mathbf{C}$ are defined by ∂.

Proposition 3. *For any rate ∂ the coefficients ∂_h^g satisfy the condition*

$$\partial_{uv}^g - \partial_{vu}^g = \partial_u^{gv^{-1}} - \partial_u^{v^{-1}g} + \partial_v^{u^{-1}g} - \partial_v^{gu^{-1}} \forall g, u, v \in G. \tag{5}$$

Proof. It suffices to show that Conditions (5) and (2) are equivalent. Let elements $u, v \in G \subset \mathbf{C}[G]$ be generators. Using (4) we get by (2) that

$$\partial([u,v]) = \partial(uv) - \partial(vu) = \sum_{g\in G}(\partial_{uv}^g - \partial_{vu}^g)g. \tag{6}$$

Therefore, we obtain the following formula for the commutator $[\partial(u), v]$

$$[\partial(u), v] = \left[\sum_{g\in G}\partial_u^g g, v\right] = \sum_{g\in G}\partial_u^g gv - \sum_{g\in G}\partial_u^g vg. \tag{7}$$

In the same way we get that

$$[u, \partial(v)] = \left[u, \sum_{g\in G}\partial_v^g g\right] = \sum_{g\in G}\partial_v^g ug - \sum_{g\in G}\partial_v^g gu. \tag{8}$$

Combining (7) and (8) we obtain

$$\sum_{g\in G}(\partial_{uv}^g - \partial_{vu}^g)g = \sum_{g\in G}\left(\partial_u^g gv - \partial_u^g vg + \partial_v^g ug - \partial_v^g gu\right). \tag{9}$$

Since the series in the left-hand side is equal to the series in the right-hand side, we obtain

$$\partial_{uv}^g - \partial_{vu}^g = \partial_u^{gv^{-1}} - \partial_u^{v^{-1}g} + \partial_v^{u^{-1}g} - \partial_v^{gu^{-1}}, \tag{10}$$

for all elements $u, v, g \in G$. In the left-hand side and in the right-hand side we have elements of a group algebra, so they are equal if the coefficients at the appropriate group elements are equal. □

Let us note that Proposition 3 is a necessary condition. To turn it into a sufficient condition we need an assumption on the finiteness of sums in formula (4).

Definition 3 (A local finiteness condition)**.** A mapping $\partial : \mathbf{C}[G] \to \mathbf{C}[G]$ is called locally finite if for each $y \in G$ the coefficient ∂_y^x is equal to zero, for almost all (except of finite number) group elements $x \in G$.

This means that we can get a theorem with necessary and sufficient conditions. Related questions were discussed in [1].

Theorem 2. *A mapping $\partial : \mathbf{C}[G] \to \mathbf{C}[G]$ is a rate if and only if it is locally finite and satisfies condition (5).*

Proof. The local finiteness condition is equivalent to the fact that the map ∂ is well-defined on the group algebra $\mathbf{C}[G]$ (see [1]). Condition (5) for the mapping ∂ is equivalent to condition (2) according to Proposition 3. $\square$

Now we can discuss a combinatorial description of rates in terms of paper [1]. Recall the definition of a gruppoid Γ from paper [1].

Objects of a gruppoid Γ are group elements $Obj(\Gamma) = \{G\}$. Morphisms are pairs of group elements $Mor(\Gamma) = \{G\} \times \{G\}$. Define the composition of morphisms $\xi = (a, b), \zeta = (c, d)$ satisfying the condition $cd^{-1} = b^{-1}a$ as follows

$$\zeta \circ \xi := (da, db).$$

The defined structure Γ is a gruppoid (see [2]).

Define the mapping χ_∂ for a rate ∂ as follows

$$\chi_\partial((g, h)) := \partial_h^g.$$

Given elements $u, v \in G$, consider the morphisms

$$\phi_1^g = (gv^{-1}, u), \psi_1^g = (v^{-1}g, u), \phi_2^g = (u^{-1}g, v), \psi_2^g = (gu^{-1}, v). \tag{11}$$

In this notation, the following equalities are valid

$$\phi_1^g \circ \phi_2^g = (g, uv), \quad \psi_2^g \circ \psi_1^g = (g, vu).$$

Hence, formula (5) takes the following form

$$\chi_\partial(\phi_1^g \circ \phi_2^g) - \chi_\partial(\psi_2^g \circ \psi_1^g) = \chi_\partial(\phi_1^g) + \chi_\partial(\phi_2^g) - \chi_\partial(\psi_1^g) - \chi_\partial(\psi_2^g). \tag{12}$$

Finally, we obtain

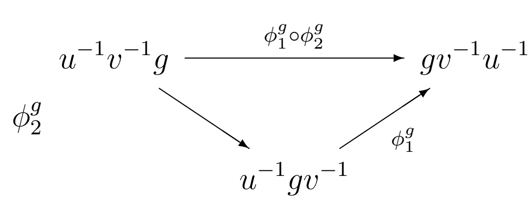
$$\chi_\partial(\phi_1^g \circ \phi_2^g) - \chi_\partial(\phi_1^g) - \chi_\partial(\phi_2^g) = \chi_\partial(\psi_2^g \circ \psi_1^g) - \chi_\partial(\psi_2^g) - \chi_\partial(\psi_1^g). \tag{13}$$

Note that in the special case in which ∂ is a derivation (not just a rate), by paper [1] we get the following stronger condition

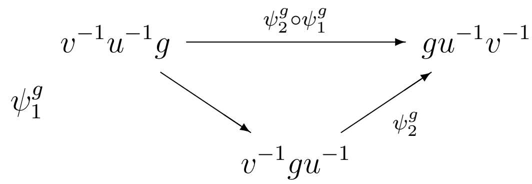
$$\chi_\partial(\phi_1^g \circ \phi_2^g) - \chi_\partial(\phi_1^g) - \chi_\partial(\phi_2^g) = 0.$$

In this case the mapping χ_∂ is called a character (see [1]).

Assume that elements u, v, g do not commute, then we have the following commutative diagrams



(*)



(**)

Let us note that if some elements u, v, g commute with each other, then some of the objects of the diagrams coincide.

We can reformulate Proposition 3 in terms of morphism and diagrams $(*), (**)$. Condition (3) is equivalent to the condition

$$\chi\partial(\phi_1^g \circ \phi_2^g) - \chi\partial(\phi_2^g) - \chi\partial(\phi_1^g) = \chi\partial(\psi_2^g \circ \psi_1^g) - \chi\partial(\psi_2^g) - \chi\partial(\psi_1^g). \tag{14}$$

An arbitrary map $\chi : Mor(\Gamma) \to \mathbf{C}$ is called locally finite, if for each element $h \in G$ we have $\chi(g, h) = 0$ for all $g \in G$ for almost all elements $x \in G$. A locally finite mapping generates a linear mapping $\partial_\chi : \mathbf{C}[G] \to \mathbf{C}[G]$ defined as follows

$$\partial_\chi(u) = \sum_{g \in G} \left(\sum_{h \in G} \partial_h^g \lambda_u^h \right) g, \tag{15}$$

Here $\partial_h^g := \chi(g, h)$.

Summarizing the above we get a theorem in gruppoid terms which follows from Theorem 2. This theorem is important for description of Lie rates.

Theorem 3. *A locally finite mapping ∂_χ defined by formula (15) defines a rate, if and only if for each elements $g, u, v \in G$ for morphisms $\phi_1^g, \phi_2^g, \psi_1^g, \psi_2^g$ defined by formula (11) the following condition is satisfied*

$$\chi(\phi_1^g \circ \phi_2^g) - \chi(\phi_2^g) - \chi(\phi_1^g) = \chi(\psi_2^g \circ \psi_1^g) - \chi(\psi_2^g) - \chi(\psi_1^g).$$

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References

[1] A.A. Arutyunov, A.S. Mishchenko, A.I. Shtern, *Derivations of group algebras.* Fundament. i Prikl. Matem. 21 (2016), no. 6, 65вЂ“78 (in Russian).

[2] S. MacLane, *Categories for the working mathematician.* Springer-Verlag 1978.

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