From Differential Equations to Difference Equations

Ushangi Goginava, Farrukh Mukhamedov [1]

Самый популярный и широко используемый метод решения линейных уравнений первого порядка — применение индукции. Однако существуют методы, которые можно использовать для получения общего решения без применения математической индукции. Также, общие решения можно найти методом, основанном на характеристическом уравнении — для линейных разностных уравнений второго порядка типа Эйлера-Коши. Эти дифференциальные уравнения являются важным аспектом обучения, поскольку они дают фундаментальное сочетание инструментов и интуиции, которые в итоге приводят к уравнениям в частных производных. Последние широко используются для описания разнообразных явлений в естественных науках.

Статья публикуется на английском языке.

The most popular, widely used method in regards to solving first order linear equations is through induction. However there are similar techniques that can be employed to obtain a general solution without the use of mathematical induction. Also, general solutions can be provided by borrowing a method based on the characteristic equation for second order linear difference equations of the Euler-Cauchy type. These differential equations are an important aspect of learning as they provide a fundamental foundation of tools and intuition that lead to partial differential equations which are used to describe phenomena in natural sciences.

1. Introduction

The main fundamental tool of calculus is considered to be the notion of the derivative with many applications in natural sciences and explanations for phenomenas ranging from heat, traffic, waves, etc. Concepts such as acceleration, Newton's Second law, etc will make an appearance in a student's life at some point especially in physics. Hence understanding the key concept of the derivative is imperative in truly grasping the notions presented within the student's other studies which helps in the applications of the derivative in real world systems. Considering some of the most simple real world applications and notions, the main point boils down to the involvement ordinary differential equations and their solutions.

Taking a step further, studying discrete analogous of the derivatives builds intuition on using integration. This can be extended to the case of discrete versions of ordinary differential equations where a heuristic method of separation of variables is employed when solving first order ordinary differential equations. Note, there are many books and resources dedicated in solving and providing solutions to these type of ordinary differential equations (see [3]) where most provide the explicit form of the solution and mathematical induction is employed to prove that the result is a general solution.

Now, the main objective is to employ similar techniques to acquire the general solution for first order linear difference equations without the use of induction and to awaken the intuition of the student in regards to applications in real world systems. These techniques gently lead to the next objective which is to solve and find general solutions of the Euler-Cauchy type second order linear difference equations. This is done through a borrowed method based on the characteristic equation associated with the Euler-Cauchy differential equation.

$$t^2y'' + aty' + by = 0$$

In [5] general solutions of linear difference equations related to a particular equation $y'' + (\lambda/t^2)y = 0$ are presented. Note that the discoveries within this paper may have potential applications in analysis of asymptotic expansion of solution of the Euler type equations. This illustrates the fact that similar

[1] Date: Received: 20.03.2023; Accepted: 30.06.2023.
2010 Mathematics Subject Classification. Primary 39A21; Secondary 39A12.
Key words and phrases. difference equation; Cauchy-Euler equation.

approaches in other literature have not been found, hence cementing the findings of this paper as a new discovery with potential for applications. However, the characteristic equation method has been used to investigate asymptotic expansion of the second order linear difference equations.

$$y_{n+2} + a(n)y_{n+1} + b(n)y_n = 0; \tag{1.1}$$

where $a(n)$ and $b(n)$ have asymptotic expansions of the form

$$a(n) \sim \sum_{k=0}^{\infty} \frac{a_k}{n^k}, \;\; b(n) \sim \sum_{k=0}^{\infty} \frac{b_k}{n^k}$$

Asymptotic solutions to (1.1) are classified by the roots of the characteristic equation

$$\rho^2 + a_0\rho + b_0 = 0.$$

We refer the reader to [1, 2, 7] for more information. It is pointed out that the last characteristic equation is different from what we are presenting in this paper.

2. Discrete Derivative

For convenience think of a sequence of real numbers $\{y_n\}$ as a function defined on the natural numbers $\mathbb{N}$, and as such the discrete derivative Dy_n can be defined to be the sequence obtained as follows:

$$Dy_n = y_{n+1} - y_n, \;\; n \in N.$$

For instance, for the constant sequence $y_n = c$, $n \in \mathbb{N}$, we have $Dy_n = 0$ for all values of n. If we consider the sequence $y_n = n^2$, then $Du_n = (n+1)^2 - n^2 = 2n + 1$. Another example is $y_n = 2^n$. Then, $Dy_n = 2^{n+1} - 2^n = 2^n$, so $Dy_n = y_n$. The literature [6] is recommended to further explore these examples.

Using the same spirit of differential calculus, the discrete derivative of a product of two sequences can be calculated [3, 4]:

$$\begin{aligned} D(x_n y_n) &= x_{n+1}y_{n+1} - x_n y_n \\ &= x_{n+1}y_{n+1} - x_{n+1}y_n + x_{n+1}y_n - x_n y_n \\ &= x_{n+1}(y_{n+1} - y_n) - y_n(x_{n+1} - x_n) \\ &= x_{n+1}Dy_n + y_n Dx_n. \end{aligned} \tag{2.1}$$

For example, $D(n2^n) = (n+1)D(2^n) + nD(n) = (n+1)2^n + n$.

Taking into account $x_{n+1} = Dx_n + x_n$, the product rule (2.1) can be rewritten as follows:

$$D(x_n y_n) = x_n Dy_n + y_n Dx_n + Dx_n Dy_n. \tag{2.2}$$

Now, observe the second order derivative of $\{y_n\}$. Indeed, we have

$$D^2 y_n = D(Dy_n) = Dy_{n+1} - Dy_n = y_{n+2} - 2y_{n+1} + y_n$$

In the coming sections, applications of the discrete derivates are demonstrated. This is applied to the solution of difference equations by employing the methods of the differential equations.

3. First order difference equations

In this section, the methods of integrating factors is demonstrated to illustrate how it can be employed to solve first order linear difference equations.

Consider the first order linear difference equation:

$$y_{n+1} - p_n y_n = r_n, \;\; y_0 = a, \;\; n \geq 0, \tag{3.1}$$

where p_n, r_n are given real numbers. It is assumed that $p_n \neq 0$, for all values of n.

The interest lies in finding a solution of (3.1) such that $y_n \neq 0$ for all $n \in \mathbb{N}$.

Remark 3.1. In [3] the solution of (3.1) is given explicitly which is then proved through induction. However, the solution can be acquired by solving the considered equation directly. In the spirit of the paper, this is demonstrated through the use of integrating factors method borrowed from the ordinary differential equations (ODE).

Now, rewrite (3.1) as follows

$$Dy_n + (1 - p_n)y_n = r_n. \tag{3.2}$$

To solve the last equation, the integrating factor technique is employed.

Assume that $\{F_n\}$ is a sequence, which will be found later on. Then from (3.2), we find

$$F_{n+1}Dy_n + F_{n+1}(1 - p_n)y_n = F_{n+1}r_n. \tag{3.3}$$

Suppose that

$$D(F_ny_n) = F_{n+1}Dy_n + F_{n+1}(1 - p_n)y_n \tag{3.4}$$

Then, due to (2.1), from the last equality

$$F_{n+1}Dy_n + y_nD(F_n) = F_{n+1}Dy_n + F_{n+1}(1 - p_n)y_n$$

is obtained. Hence,

$$y_nD(F_n) = F_{n+1}(1 - p_n)y_n$$

is found. This implies $D(F_n) = F_{n+1}(1 - p_n)$. Hence,

$$F_{n+1} - F_n = F_{n+1}(1 - p_n)$$

therefore,

$$p_nF_{n+1} = F_n \Rightarrow F_{n+1} = \frac{1}{p_n}F_n.$$

Hence,

$$F_n = \frac{1}{\prod_{k=0}^{n-1} p_k}F_0. \tag{3.5}$$

Without loss of generality, we may assume that $F_0 = 1$.

Now, using the assumption (3.4), from (3.3) it follows that

$$D(F_ny_n) = F_{n+1}r_n$$

which yields

$$\sum_{k=0}^{n-1} D(F_ky_k) = \sum_{k=0}^{n-1} F_{k+1}r_k \Rightarrow F_ny_n - y_0 = \sum_{k=0}^{n-1} F_{k+1}r_k$$

so,

$$y_n = \frac{1}{F_n}\left(y_0 + \sum_{k=0}^{n-1} F_{k+1}r_k\right).$$

Now, using (3.5) from the last equality, one gets

$$y_n = \prod_{k=0}^{n-1} p_k \left(y_0 + \sum_{k=0}^{n-1} \frac{r_k}{\prod_{j=0}^{k-1} p_j}\right) = y_0\left(\prod_{k=0}^{n-1} p_k\right) + \sum_{k=0}^{n-1} r_k\left(\prod_{j=k}^{n-1} p_j\right).$$

Hence, the solution of (3.1) has the following form

$$y_n = y_0 \prod_{k=0}^{n-1} p_k + \sum_{k=0}^{n-1} r_k \left(\prod_{k=0}^{n-1} p_j \right), \ n \geq 1. \tag{3.6}$$

Now, we consider a version of the Bernoulli equation:

$$y_n^m - p_n y_{n+1}^m = r_n y_n^m y_{n+1}^m, \ y_0 = a \geq 0, \tag{3.7}$$

where $p_n \neq 0$ and $m \neq 0$.

Denoting

$$u_n = \frac{1}{y_n^m}$$

from (3.8), we obtain

$$u_{n+1} - p_n u_n = r_n; \ u_n = \frac{1}{y_n^m} \tag{3.8}$$

So, by (3.10), one finds

$$u_n = \frac{1}{a^m} \prod_{k=0}^{n-1} p_k + \sum_{k=0}^{n-1} r_k \left(\prod_{j=k}^{n-1} p_j \right), \ n \geq 1. \tag{3.9}$$

This implies

$$y_n^m = \frac{a^m}{\prod_{k=0}^{n-1} p_k + \sum_{k=0}^{n-1} r_k \left(\prod_{j=k}^{n-1} p_j \right)}, \ n \geq 1. \tag{3.10}$$

4. Euler-Cauchy type second order difference equations

This sections aims to show the method employed to solve the second order linear difference equation. This is done through the use of the Euler-Cauchy method which is borrowed from classical differential equations.

Let us first consider the following equation:

$$y_{n+2} + p_n y_{n+1} + q_n y_n = 0, \tag{4.1}$$

where

$$p_n = -2 + \frac{a}{n+2}, \ q_n = 1 - \frac{a}{n+2} + \frac{b}{(n+1)(n+2)}.$$

Now, using the discrete derivative, the equation (4.1) can be rewritten as follows:

$$(n+1)(n+2)D^2 y_n + a(n+1)Dy_n + by_n = 0. \tag{4.2}$$

The last equation similarly looks like the classical Euler-Cauchy equation. Now, using the same strategy as the classical case, the solution is searched in the following form:

$$y_n = A_n^\alpha, \ n \in \mathbb{N}, \ \alpha \in \mathbb{C}, \tag{4.3}$$

where A_n^m is the generalized binomial coefficient given by

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!}, \ \alpha \notin -\mathbb{N}, \tag{4.4}$$

The coefficient A_n^α enjoys the following property [8]:

$$A_{n+1}^{\alpha} - A_n^{\alpha} = A_{n+1}^{\alpha-1} \tag{4.5}$$

Using (4.5), for y_n given by (4.3) we obtain

$$Dy_n = A_{n+1}^{\alpha-1}, \;\; D^2 y_n = A_{n+2}^{\alpha-2} \tag{4.6}$$

Now, by (4.4) one finds

$$Dy_n = \frac{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n)}{(n+1!)} = \frac{\alpha}{n+1} A_n^{\alpha}, \tag{4.7}$$

Similarly, we have

$$D^2 y_n = \frac{\alpha(\alpha-1)}{(n+1)(n+2)} A_n^{\alpha}. \tag{4.8}$$

Substituting these into (4.2), we find

$$\alpha(\alpha-1) + a\alpha + b = 0 \Rightarrow \alpha^2 + (a-1)\alpha + b = 0. \tag{4.9}$$

This equation is called *characteristic equation* of (4.2). Its discriminant equals to $\mathcal{D} := (a-1)^2 - 4b$.

Now, we consider several cases.

Case I. Assume that $\mathcal{D} > 0$, then (4.9) has two distinct real roots, say α_1 and α_2. Then

$$y_n^{(1)} = A_n^{\alpha_1}, \;\; y_n^{(2)} = A_n^{\alpha_2}, \;\; n \in \mathbb{N}.$$

are two solutions of (4.2). We stress that α_1; α_2 should not be negative integer, i.e. α_1, $\alpha_2 \notin -\mathbb{N}$. In this case, general solution of (4.2) has the following form:

$$y_n = c_1 A_n^{\alpha_1} + c_2 A_n^{\alpha_1}, \;\; n \in \mathbb{N}. \tag{4.10}$$

where c_1, c_2 are constants.

Case II. Assume that $\mathcal{D} = 0$, then (4.9) has only real root, say $\alpha_1 = (1-a)/2$. Then

$$Y_n = A_n^{\frac{1-a}{2}}, \;\; n \in \mathbb{N}$$

is a solutions of (4.2). It is supposed that $1 - a \notin -2\mathbb{N}$.

In this case, stress is put on the fact that the second solution of (4.2) is looked in the form $z_n = u_n Y_n$ (this is also borrowed from the theory of linear differential equations), where u_n will be determined later on. By (2.2), one can find that

$$Dz_n = Du_n(Y_n + DY_n) + u_n DY_n \tag{4.11}$$

which yields

$$D^2 z_n = D^2 u_n(Y_n + 2DY_n + D^2 Y_n) + 2Du_n(DY_n + D^2 Y_n) + u_n D^2 Y_n. \tag{4.12}$$

Rewriting (4.2) by

$$D^2 y_n + \frac{a}{n+2} Dy_n + \frac{b}{(n+1)(n+2)} y_n = 0 \tag{4.13}$$

and substituting (4.11),(4.12) into the last one, we obtain

$$D^2 u_n(Y_n + 2DY_n + D^2 Y_n) + 2Du_n(DY_n + D^2 Y_n) + u_n D^2 Y_n +$$
$$+ \frac{a}{n+2}\left[Du_n(Y_n + DY_n) \;\; + \;\; u_n DY_n\right] + \frac{b}{(n+1)(n+2)} u_n Y_n = 0$$

which yields

$$D^2u_n(Y_n+2DY_n+D^2Y_n)+Du_n\left[2D^2Y_n+2DY_n+\frac{a}{n+2}(Y_n+DY_n)\right]+$$

$$+u_n\underbrace{\left[2D^2Y_n+\frac{a}{n+2}Y_n+\frac{b}{(n+1)(n+2)}Y_n\right]}_{=0}=0.$$

Now, denoting $v_n = Du_n$, from the last equation, we have

$$Dv_n\mathcal{A}_n+v_n\mathcal{B}_n=0, \tag{4.14}$$

where

$$\mathcal{A}_n=Y_n+2DY_n+D^2Y_n,\ \ \mathcal{B}_n=2D^2Y_n+2DY_n+\frac{a}{n+2}(Y_n+DY_n). \tag{4.15}$$

We reduce (4.14) to $\mathcal{A}_n(v_{n+1}-v_n)+\mathcal{B}_nv_n=0$ which implies

$$v_{n+1}=v_n\left(1-\frac{\mathcal{B}_n}{\mathcal{A}_n}\right),$$

hence,

$$v_{n+1}=v_0\prod_{k=0}^{n-1}\left(1-\frac{\mathcal{B}_n}{\mathcal{A}_n}\right).$$

Since, we are interested in a particular solution of (4.15), therefore, we may assume that $v_0=1$.

Now, taking into account $Y_n = A_n^{\alpha}$, where $a = 1-2\alpha$, and using (4.6) together with (4.7),(4.8) we obtain

$$\mathcal{A}_k = A_k^{\alpha}+2A_{k+1}^{\alpha-1}+A_{k+2}^{\alpha-2} \tag{4.16}$$

$$=A_k^{\alpha}\left(1+\frac{2_{\alpha}}{k+1}+\frac{\alpha(\alpha-1)}{(k+1)(k+2)}\right) \tag{4.17}$$

$$\mathcal{B}_k=A_k^{\alpha}\left[\frac{2\alpha}{k+1}+\frac{2\alpha(\alpha-1)}{(k+1)(k+2)}+\frac{1-2\alpha}{k+2}\left(1+\frac{\alpha}{k+1}\right)\right] \tag{4.18}$$

Using these equalities and simple calculations, one gets

$$1-\frac{\mathcal{B}_k}{\mathcal{A}_k}=\frac{k+\alpha+1}{k+\alpha+2} \tag{4.19}$$

So,

$$v_n=\prod_{k=0}^{n-1}=\frac{k+\alpha+1}{k+\alpha+2}=\frac{\alpha+1}{1+\alpha+n} \tag{4.20}$$

Due to $Du_n = v_n$, we have

$$\sum_{k=0}^{n-1}Du_n=\sum_{k=0}^{n-1}v_k$$

which implies

$$u_n-u_0=\sum_{k=0}^{n-1}v_k.$$

Without loss of generality, we may assume that $u_0 = 0$. So, by (4.20) we find

$$u_n = \sum_{l=0}^{n-1} \frac{\alpha+1}{1+\alpha+l}$$

Consequently, the second solution of (4.2) has the following form

$$y_n^{(2)} = A_n^{\frac{1-a}{2}} \sum_{l=0}^{n-1} \frac{3-a}{3-a+2l}. \tag{4.21}$$

In this case, general solution of (4.2) has the following form:

$$y_n = c_1 A_n^{\frac{1-a}{2}} + c_2 A_n^{\frac{1-a}{2}} \sum_{l=0}^{n-1} \frac{3-a}{3-a+2l} \quad n \in \mathbb{N}, \tag{4.22}$$

where c_1, c_2 are constants.

Case III. In this case, we consider a situation when $\mathcal{D} > 0$, but (4.9) has two real roots such that $\alpha_1 \notin -\mathbb{N}$ and $\alpha_2 \in -\mathbb{N}$. Then, the first solution of (4.2) is given by

$$Y_n = A_n^{\alpha_1} \quad n \in \mathbb{N}$$

However, the second solution cannot be defined as before. To find it, we are going to use the same argument as shown in Case II. Namely, the second solution of (4.2) is found in the form $z_n = u_n Y_n$, where un will be determined later on. Now, by employing all above calculations, we arrive at

$$v_{n+1} = v_n \left(\frac{n+2-\alpha_1-a}{n+\alpha_1+2} \right).$$

Now, noticing that the value $2-\alpha_1-a$ could be less or equal 0, therefore, we have

$$v_n = \prod_{k \geq [a+\alpha_1-2]+1}^{n-1} \left(\frac{k+2-\alpha_1-a}{k+\alpha_1+2} \right). \tag{4.23}$$

Due to $Du_n = v_n$, we have

$$u_n = \sum_{k=0}^{n-1} v_k.$$

So, by (4.29) we find

$$u_n = \prod_{l=k \geq [a+\alpha_1-2]+2}^{n-1} \prod_{k \geq [a+\alpha_1-2]+1}^{l-1} \left(\frac{k+2-\alpha_1-a}{k+\alpha_1+2} \right)$$

Consequently, the second solution of (4.2) has the following form

$$y_n^{(2)} = A_n^{\alpha_1} \prod_{l=k \geq [a+\alpha_1-2]+2}^{n-1} \prod_{k \geq [a+\alpha_1-2]+1}^{l-1} \left(\frac{k+2-\alpha_1-a}{k+\alpha_1+2} \right) \tag{4.24}$$

where $n \geq a+\alpha_1-2$.

Case IV. Assume that $\mathcal{D} < 0$, then (4.9) has two complex roots, say $\alpha_1 = U + iV$, $\alpha_2 = U - iV$. Then, we can see that $A_n^{\alpha_1} = \overline{A_n^{\alpha_1}}$. Therefore, the corresponding solutions of (4.2) are given by

$$y_n^{(1)} = \frac{A_n^{\alpha_1} + A_n^{\alpha_2}}{2}, \quad y_n^{(1)} = \frac{A_n^{\alpha_1} - A_n^{\alpha_2}}{2i}, \quad n \in \mathbb{N}.$$

Hence, general solution of (4.2) has the following form:

$$y_n = c_1 \frac{A_n^{\alpha_1} + A_n^{\alpha_2}}{2} + c_2 \frac{A_n^{\alpha_1} - A_n^{\alpha_2}}{2i}, \quad n \in \mathbb{N}, \tag{4.25}$$

where, as before, c_1, c_2 are constants.

Example 4.1. Let us consider the following equation:

$$y_{n+2} - \left(2 + \frac{1}{n+1}\right) y_{n+1} + \left(2 + \frac{1}{n+2} + \frac{1}{(n+1)(n+2)}\right) y_n = 0. \tag{4.26}$$

This equation is reduced to

$$(n+1)(n+2)D^2 y_n - (n+1)D y_n + y_n = 0.$$

In this case, $a = -1$, $b = 1$. One can see that its characteristic equation is $\alpha^2 - 2\alpha + 1 = 0$ which means that $\alpha = 1$ is a unique solution. Now, we are in the case II. Hence, the first solution is $y_n^{(1)} = A_n^1 = n+1$. By (4.21), the second solution is given by

$$y_n^{(2)} = (n+1) \sum_{l=0}^{n-1} \frac{1}{2+l}.$$

Example 4.2. Let us consider the following equation:

$$y_{n+2} - \left(2 - \frac{1}{n+1}\right) y_{n+1} + \left(1 - \frac{1}{n+2} - \frac{1}{(n+1)(n+2)}\right) y_n = 0.$$

This equation is reduced to

$$(n+1)(n+2)D^2 y_n + (n+1)D y_n - y_n = 0.$$

In this case, $a = 1$, $b = -1$. The characteristic equation is $\alpha^2 - 1 = 0$ which means that $\alpha_1 = 1$ and $\alpha_2 = -1$. Now, we are in the case III. Hence, the first solution is $y_n^{(1)} = A_n^1 = n+1$. By (4.30), the second solution is given by

$$y_n^{(2)} = (n+1) \sum_{l=2}^{n-1} \prod_{k=1}^{l-1} \left(\frac{k}{k+3}\right) = \frac{3n^2 + 3n - 6}{2n}$$

The last one can be rewritten as follows

$$y_n^{(2)} = \frac{3}{2}(n+1) - 3\frac{1}{n}.$$

Since $n+1$ is the first solution, then we infer that the second solution of the equation is

$$y_n^{(2)} = \frac{1}{n}.$$

Consequently, a general solution of (4.26) has the following form:

$$y_n = c_1(n+1) + c_2 \frac{1}{n}, \quad n \in \mathbb{N}, \tag{4.27}$$

where, as before, c_1, c_2 are constants.

Example 4.3. Now, consider another equation:

$$y_{n+2} - \left(2 - \frac{2}{n+1}\right) y_{n+1} + \left(1 - \frac{2}{n+2} - \frac{2}{(n+1)(n+2)}\right) y_n = 0.$$

This equation is reduced to (4.2) with $a = 2$, $b = -2$. The characteristic equation is $\alpha^2 + \alpha - 2 = 0$ which has $\alpha_1 = 1$ and $\alpha_2 = -2$ roots. Again, by the case III, we find $y_n^{(1)} = n+1$. By (4.30), the second solution can be calculated as follows

$$y_n^{(2)} = \frac{n^3 - n - 24}{2n(n-1)} = \frac{1}{3}(n+1) - 8\frac{1}{n(n-1)}.$$

Since $n+1$ is the first solution, then we infer that the second solution of the equation is

$$y_n^{(2)} = \frac{1}{n(n-1)}.$$

Now, we consider the remaining cases.

Case V. In this case, we consider a situation when $\mathcal{D} > 0$, such that (4.9) has two negative integer roots, i.e. $\alpha_1 = -k$, $\alpha_2 = -m$, where k, $m \in \mathbb{N}$, $k \neq m$. Then, the above considered examples suggest us the solutions of (4.2) have the following forms

$$y_n^{(1)} = \frac{1}{n(n-1)\dots(n-k+1)}, \quad y_n^{(2)} = \frac{1}{n(n-1)\dots(n-m+1)}, \quad n \in \mathbb{N}.$$

Hence, general solution of (4.2) is given by

$$y_n = \frac{c_1}{n(n-1)\dots(n-k+1)} + \frac{c_2}{n(n-1)\dots(n-m+1)}, \quad n \in \mathbb{N}, \tag{4.28}$$

where, as before, c_1, c_2 are constants.

Case VI. This is the last possible case, in which it is assumed that $\mathcal{D} = 0$, and (4.9) has one negative integer root $\alpha = -k$, where $k \in \mathbb{N}$. Then by the case V, one of the solution of (4.2) can be found by

$$Y_n = \frac{1}{n(n-1)\dots(n-k+1)}, \quad n \in \mathbb{N}.$$

Now, to find the second solution of (4.2) we are going to employ the method given in the case II. Namely, the second solution of (4.2) is looked at the form $z_n = u_n Y_n$, where u_n will be determined later on. By using the same argument in Case II, we arrive at

$$v_{n+2} = v_n \left(\frac{n-k+1}{n-k+2}\right).$$

Now, noticing that the value $2 - \alpha_1 - a$ could be less or equal 0, therefore, we have

$$v_n = \prod_{l=k}^{n-1} \frac{l-k+1}{l-k+2} = \frac{1}{n-k+1}. \tag{4.29}$$

Due to $Du_n = v_n$, we have

$$u_n = \sum_{m=k}^{n-1} \frac{l}{m-k+1} = \sum_{t=1}^{n-k} \frac{l}{t}$$

Consequently, the second solution of (4.2) has the following form

$$y_n^{(2)} = \frac{1}{n(n-1)\dots(n-k+1)} \sum_{t=1}^{n-k} \frac{l}{t} \tag{4.30}$$

where $n \geq a + \alpha_1 - 2$.

Concluding remarks

A multitude of ideas stemming from differential equations and their analogous in the discrete difference equations have been shown and explored. This new idea introduces a new outlook and provides a different perspective regarding differential equations with ideas rooted in the theory of linear differential equations and the meaning of the derivative.

A heavy emphasis is expressed regarding the method employed in finding the second solution as knowing the first solution is based on the theory of linear differential equations. However, a formula given in [3] which is used to find the second suction by means of the first one involving Casoratian determinants (see [3, p.91]). Nevertheless its calculation is nearly impossible as it requires a large number of determinants which has a high computational complexity. The beauty of the formulas discovered lie with its simplicity which can be explicitly utilised in finding asymptotic expansion of the solutions. Further development of the provided ideas can be extended for a variety of Euler-Cauchy equations. This is mentioned in hopes that it incites students to search for new findings in the field of discrete difference equations. Lastly, the obtained formulas are a great extension of differential equations and can be employed in training students to further their preparation for maths olympiads and competitions.

References

1. Adams C.R. On the irregular cases of linear ordinary difference equations // Trans. Amer. Math. Soc. - Vol. 30. - 1928. - P. 507-541.

2. Birkhoff G.D. Formal theory of irregular linear drfference equations // Acta Math. - Vol. 54. - 1930. - P. 205-246.

3. Elaydi S. An Introduction to Difference Equations. - Springer Science+Business Media, Inc., 2005.

4. Goldberg S. An Inroduction to Difference Equations. - New York: Wiley, 1958.

5. Hongyo A., Yamaoka N. General solution of second-order linear difference equations of Euler type // Opuscula Math. - Vol. 37. - 2012. - P. 389-402.

6. Strang G. Sums and differences vs. integrals and derivatives // College Math. J. - Vol. 21. - 1990. - P. 20-27.

7. Wong R., Li H. Asymptotic expansions for second-order linear difference equations // J. Comput. Appl. Math. - Vol. 41. - 1992. - P. 65-94.

8. Zygmund A. Trigonemetric Series. - Cambridge Univ. Press, 2002.

Ushangi Goginava,
Mathematical Sciences Department, College of Science,
United Arab Emirates University 15551,
Al-Ain United Arab Emirates.

E-mail: zazagoginava@gmail.com, ugoginava@uaeu.ac.ae

Farrukh Mukhamedov,
Mathematical Sciences Department, College of Science,
United Arab Emirates University 15551,
Al-Ain United Arab Emirates.

E-mail: far75m@gmail.com, farrukh.m@uaeu.ac.ae