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On a four-dimensional hyperbolic manifold with finite volume

I. S. Gutsul

Abstract. In article [1] the authors construct and classify all the hyperbolic space-forms H^n/Γ where Γ is a torsion-free subgroup of minimal index in the congruence two subgroup Γ_2^n for $n = 3, 4$. In the present paper some hyperbolic 3- and 4-manifolds are constructed that are absent in [1].

Mathematics subject classification: 51M10, 53C25.

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In the works [1] and [2] some four-dimensional hyperbolic manifolds with finite volume were constructed. They were obtained by identifying the faces of the regular 24-cells in H^4 with all vertices being on the absolute. The present article is devoted to the construction of a four-dimensional hyperbolic manifold with finite volume by identifying the faces of a four-dimensional hyperbolic polyhedron with all the vertices being on the absolute. This polyhedron is not regular and its construction is non-trivial.

1. The construction of a four-dimensional polyhedron

In the four-dimensional space H^4 consider the regular 24-cells R . As it is known this polyhedron has 24 three-dimensional faces, 96 two-dimensional faces, 96 edges, and 24 vertices. The three-dimensional faces of the polyhedron are regular octahedra, two-dimensional faces are regular triangles. Inscribe in the polyhedron R a three-dimensional sphere S^3 . Denote its radius by r . If we begin to enlarge the radius r of the sphere S^3 , the polyhedron R will increase, but its dihedral angles at the two-dimensional faces will decrease. Continuing the process, we ultimately come to the case when for some r_0 all the vertices of the polyhedron R become infinitely removed, i. e. they get out on the absolute. In this case the three-dimensional faces are regular octahedra with all the vertices being on the absolute. Then the dihedral angles at the two-dimensional faces will be equal to $\pi/2$. Indeed, consider a three-dimensional horosphere centered at a vertex of the polyhedron R . Choose the radius of the horosphere such that the horosphere intersects only one-dimensional edges of the polyhedron which go to the center of the horosphere. Then the intersection of the horosphere and the polyhedron R is a cube. But the dihedral angles at the two-dimensional faces of the polyhedron R are equal to the dihedral angles at the edges of the obtained cube. Since the metric on the horosphere is Euclidean,

the dihedral angles of the cube are equal to $\pi/2$, i. e. the dihedral angles at the two-dimensional faces of the polyhedron R are equal to $\pi/2$. If we continue to enlarge the radius of the three-dimensional sphere, the vertices of the polyhedron R will get out on the absolute. We obtain a four-dimensional polyhedron R_2 with all the vertices being infinitely removed. The polyhedron R_2 has three-dimensional faces of two kinds: 24 cubes with all the vertices being on the absolute and 24 truncated octahedra with all the vertices being infinitely removed. The polyhedron R_2 has 94 infinitely removed vertices, 288 one-dimensional edges, two-dimensional faces of two kinds: 144 squares with all the vertices being on the absolute and 96 triangles with all the vertices being infinitely removed. Dihedral angles at two-dimensional faces of this polyhedron are of two kinds: dihedral angles at the squares are equal to $\pi/2$, dihedral angles at the triangles are equal to $\pi/3$, both facts can be easily proved. Label infinitely removed vertices of the polyhedron R_2 by the numbers from 1 to 94. Write all three-dimensional faces of the obtained polyhedron. First write cubes with all the vertices being on the absolute:

$K_{25}(1, 7, 8, 6, 30, 16, 15, 22)$	$K_{26}(13, 21, 91, 39, 5, 1, 2, 12)$
$K_{27}(7, 2, 10, 3, 28, 17, 19, 44)$	$K_{28}(3, 9, 4, 8, 24, 26, 49, 35)$
$K_{29}(4, 6, 5, 11, 54, 33, 31, 40)$	$K_{30}(18, 17, 27, 68, 92, 14, 15, 23)$
$K_{31}(9, 10, 12, 11, 52, 48, 42, 41)$	$K_{32}(13, 20, 64, 38, 36, 16, 14, 93)$
$K_{33}(18, 19, 21, 20, 67, 71, 45, 46)$	$K_{34}(22, 23, 25, 24, 34, 29, 94, 73)$
$K_{35}(50, 26, 25, 96, 69, 51, 28, 27)$	$K_{36}(29, 30, 31, 32, 75, 95, 36, 37)$
$K_{37}(74, 56, 35, 34, 32, 76, 55, 33)$	$K_{38}(37, 38, 39, 40, 59, 77, 65, 58)$
$K_{39}(66, 46, 91, 58, 57, 63, 43, 41)$	$K_{40}(45, 70, 51, 44, 42, 43, 60, 47)$
$K_{41}(47, 48, 49, 50, 72, 61, 53, 56)$	$K_{42}(53, 62, 78, 55, 54, 52, 57, 59)$
$K_{43}(61, 60, 86, 89, 90, 62, 63, 87)$	$K_{44}(66, 67, 64, 65, 85, 87, 81, 80)$
$K_{45}(70, 71, 81, 86, 83, 69, 68, 79)$	$K_{46}(83, 82, 88, 89, 72, 96, 73, 74)$
$K_{47}(78, 90, 85, 77, 75, 76, 88, 84)$	$K_{48}(92, 94, 82, 79, 80, 93, 95, 84)$

The polyhedron R_2 has also 24 truncated octahedra with all the vertices being on the absolute. Write these faces:

$O_1(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$
$O_2(15, 16, 30, 22, 23, 14, 36, 29, 92, 93, 95, 94)$
$O_3(1, 7, 15, 16, 13, 2, 17, 14, 20, 21, 19, 18)$
$O_4(7, 8, 22, 15, 17, 3, 24, 23, 25, 26, 28, 27)$
$O_5(8, 6, 30, 22, 24, 4, 31, 29, 34, 35, 33, 32)$
$O_6(6, 1, 16, 30, 36, 13, 5, 31, 37, 38, 39, 40)$
$O_7(12, 10, 42, 41, 91, 2, 44, 43, 45, 46, 21, 19)$
$O_8(10, 9, 48, 42, 44, 3, 49, 47, 50, 51, 28, 26)$
$O_9(9, 11, 52, 48, 49, 4, 54, 53, 55, 56, 35, 33)$
$O_{10}(11, 12, 41, 52, 54, 57, 91, 5, 39, 40, 59, 58)$
$O_{11}(62, 61, 60, 63, 57, 53, 47, 43, 52, 48, 42, 41)$
$O_{12}(13, 21, 91, 39, 38, 20, 46, 58, 65, 66, 67, 64)$

$O_{13}(17, 19, 44, 28, 27, 18, 45, 51, 69, 68, 71, 70)$
 $O_{14}(72, 74, 73, 96, 50, 56, 34, 25, 26, 24, 35, 49)$
 $O_{15}(78, 76, 75, 77, 59, 55, 32, 37, 40, 54, 33, 31)$
 $O_{16}(18, 20, 67, 71, 68, 14, 64, 81, 80, 79, 92, 93)$
 $O_{17}(25, 27, 69, 96, 73, 83, 68, 23, 92, 94, 82, 79)$
 $O_{18}(38, 37, 77, 65, 64, 36, 75, 85, 84, 80, 93, 95)$
 $O_{19}(45, 46, 67, 71, 70, 43, 66, 81, 86, 60, 63, 87)$
 $O_{20}(32, 34, 74, 76, 75, 29, 73, 88, 82, 84, 95, 94)$
 $O_{21}(50, 51, 69, 96, 83, 70, 47, 72, 89, 86, 60, 61)$
 $O_{22}(55, 56, 74, 76, 78, 53, 72, 88, 90, 62, 61, 89)$
 $O_{23}(58, 59, 77, 65, 66, 57, 78, 85, 90, 62, 63, 87)$
 $O_{24}(89, 88, 85, 81, 83, 86, 90, 87, 80, 84, 79, 82)$

2. The construction of a four-dimensional hyperbolic manifold

Indicate motions (isometries) that identify faces of the polyhedron:

$\varphi_1 : (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$
 $(89, 88, 85, 81, 83, 86, 90, 87, 80, 84, 79, 82);$

$\varphi_2 : (15, 16, 30, 22, 23, 14, 36, 29, 92, 93, 95, 94)$
 $(62, 61, 60, 63, 57, 53, 47, 43, 52, 48, 42, 41);$

$\varphi_3 : (1, 7, 15, 16, 13, 2, 17, 14, 20, 21, 19, 18)$
 $(55, 56, 74, 76, 78, 53, 72, 88, 90, 62, 61, 89);$

$\varphi_4 : (7, 8, 22, 15, 17, 3, 24, 23, 25, 26, 28, 27)$
 $(58, 59, 77, 65, 66, 57, 78, 85, 90, 62, 63, 87);$

$\varphi_5 : (8, 6, 30, 22, 24, 4, 31, 29, 34, 35, 33, 32)$
 $(45, 46, 67, 71, 70, 43, 66, 81, 86, 60, 63, 87);$

$\varphi_6 : (6, 1, 16, 30, 36, 13, 5, 31, 37, 38, 39, 40)$
 $(50, 51, 69, 96, 83, 70, 47, 72, 89, 86, 60, 61);$

$\varphi_7 : (12, 10, 42, 41, 91, 2, 44, 43, 45, 46, 21, 19)$
 $(32, 34, 74, 76, 75, 29, 73, 88, 82, 84, 95, 94);$

$\varphi_8 : (10, 9, 48, 42, 44, 3, 49, 47, 50, 51, 28, 26)$
 $(38, 37, 77, 65, 64, 36, 75, 85, 84, 80, 93, 95);$

$\varphi_9 : (9, 11, 52, 48, 49, 4, 54, 53, 55, 56, 35, 33)$
 $(18, 20, 67, 71, 68, 14, 64, 81, 80, 79, 92, 93);$

$\varphi_{10} : (11, 12, 41, 52, 54, 57, 91, 5, 39, 40, 59, 58)$
 $(25, 27, 69, 96, 73, 83, 68, 23, 92, 94, 82, 79);$

$$\begin{aligned}
 \varphi_{11} : & \quad (13, 21, 91, 39, 38, 20, 46, 58, 65, 66, 67, 64) \\
 & \quad (72, 74, 73, 96, 50, 56, 34, 25, 26, 24, 35, 49); \\
 \varphi_{12} : & \quad (17, 19, 44, 28, 27, 18, 45, 51, 69, 68, 71, 70) \\
 & \quad (78, 76, 75, 77, 59, 55, 32, 37, 40, 54, 33, 31); \\
 \varphi_{13} : & \quad (1, 7, 8, 6, 30, 16, 15, 22) \\
 & \quad (66, 46, 91, 58, 57, 63, 43, 41); \\
 \varphi_{14} : & \quad (13, 21, 91, 39, 5, 1, 2, 12) \\
 & \quad (66, 67, 64, 65, 85, 87, 81, 80); \\
 \varphi_{15} : & \quad (7, 2, 10, 3, 28, 17, 19, 44) \\
 & \quad (50, 26, 25, 96, 69, 51, 28, 27); \\
 \varphi_{16} : & \quad (3, 9, 4, 8, 24, 26, 49, 35) \\
 & \quad (83, 82, 88, 89, 72, 96, 73, 74); \\
 \varphi_{17} : & \quad (4, 6, 5, 11, 54, 33, 31, 40) \\
 & \quad (74, 56, 35, 34, 32, 76, 55, 33); \\
 \varphi_{18} : & \quad (18, 17, 27, 68, 92, 14, 15, 23) \\
 & \quad (45, 70, 51, 44, 42, 43, 60, 47); \\
 \varphi_{19} : & \quad (9, 10, 12, 11, 52, 48, 42, 41) \\
 & \quad (13, 20, 64, 38, 36, 16, 14, 93); \\
 \varphi_{20} : & \quad (18, 19, 21, 20, 67, 71, 45, 46) \\
 & \quad (70, 71, 81, 86, 83, 69, 68, 79); \\
 \varphi_{21} : & \quad (22, 23, 25, 24, 34, 29, 94, 73) \\
 & \quad (61, 60, 86, 89, 90, 62, 63, 87); \\
 \varphi_{22} : & \quad (29, 30, 31, 32, 75, 95, 36, 37) \\
 & \quad (53, 62, 78, 55, 54, 52, 57, 59); \\
 \varphi_{23} : & \quad (37, 38, 39, 40, 59, 77, 65, 58) \\
 & \quad (78, 90, 85, 77, 75, 76, 88, 84); \\
 \varphi_{24} : & \quad (47, 48, 49, 50, 72, 61, 53, 56) \\
 & \quad (92, 94, 82, 79, 80, 93, 95, 84).
 \end{aligned}$$

Consider cycles of two-dimensional faces of the polyhedron R_2 . As the dihedral angles at the quadrangular faces are equal to $\pi/2$, in order that the cycles of these faces be inessential each of them must contain four faces. Write cycles of these faces. We will present cycles of faces as follows: write a face, then write the motion that transfers this face into another face, then again a face, again a motion, and so on.

$$\begin{aligned}
 & (O_1 \cap K_{25})(1, 7, 8, 6) \varphi_{13} (O_{12} \cap K_{39})(66, 46, 91, 58) \varphi_{11} (O_{14} \cap K_{34}) \\
 & (24, 34, 73, 25) \varphi_{21} (O_{24} \cap K_{43})(89, 90, 87, 86) \varphi_1^{-1} (O_1 \cap K_{25})(1, 7, 8, 6);
 \end{aligned}$$

$$\begin{aligned}
& (O_3 \cap K_{25})(1, 7, 15, 16) \varphi_{13} (O_{19} \cap K_{39})(66, 46, 43, 63) \varphi_5^{-1} (O_5 \cap K_{29}) \\
& (31, 6, 4, 33) \varphi_{17} (O_{22} \cap K_{37})(55, 56, 74, 76) \varphi_3^{-1} (O_3 \cap K_{25})(1, 7, 15, 16); \\
& (O_6 \cap K_{25})(1, 6, 30, 16) \varphi_{13} (O_{23} \cap K_{39})(66, 58, 57, 63) \varphi_4^{-1} (O_4 \cap O_{27}) \\
& (17, 7, 3, 28) \varphi_{15} (O_{21} \cap K_{35})(51, 50, 96, 69) \varphi_6^{-1} (O_6 \cap K_{25})(1, 6, 30, 16); \\
& (O_5 \cap K_{25})(8, 6, 30, 22) \varphi_{13} (O_{10} \cap K_{39})(91, 58, 57, 41) \varphi_{10} (O_{17} \cap K_{45}) \\
& (68, 79, 83, 69) \varphi_{20}^{-1} (O_{19} \cap K_{33})(45, 46, 67, 71) \varphi_5^{-1} (O_5 \cap K_{25})(8, 6, 30, 22); \\
& (O_4 \cap K_{25})(8, 7, 15, 22) \varphi_{13} (O_7 \cap K_{39})(91, 46, 43, 41) \varphi_7 (O_{20} \cap K_{47}) \\
& (75, 84, 88, 76) \varphi_{23}^{-1} (O_{23} \cap K_{38})(59, 58, 65, 77) \varphi_4^{-1} (O_4 \cap K_{25})(8, 7, 15, 22); \\
& (O_2 \cap K_{25})(15, 16, 30, 22) \varphi_{13} (O_{11} \cap K_{39})(43, 63, 57, 41) \varphi_2^{-1} (O_2 \cap K_{34}) \\
& (29, 22, 23, 94) \varphi_{21} (O_{11} \cap K_{43})(62, 61, 60, 63) \varphi_2^{-1} (O_2 \cap K_{25})(15, 16, 30, 22); \\
& (O_1 \cap K_{26})(1, 2, 12, 5) \varphi_{14} (O_{24} \cap K_{44})(87, 81, 80, 85) \varphi_1^{-1} (O_1 \cap K_{28}) \\
& (8, 4, 9, 3) \varphi_{16} (O_{24} \cap K_{46})(89, 88, 82, 83) \varphi_1^{-1} (O_1 \cap K_{26})(1, 2, 12, 5); \\
& (O_3 \cap K_{26})(1, 2, 21, 13) \varphi_{14} (O_{19} \cap K_{44})(87, 81, 67, 66) \varphi_5^{-1} (O_5 \cap K_{36}) \\
& (32, 29, 30, 31) \varphi_{22} (O_{22} \cap K_{42})(55, 53, 62, 78) \varphi_3^{-1} (O_3 \cap K_{26})(1, 2, 21, 13); \\
& (O_6 \cap K_{26})(1, 5, 39, 13) \varphi_{14} (O_{23} \cap K_{44})(87, 85, 65, 66) \varphi_4^{-1} (O_4 \cap K_{30}) \\
& (27, 23, 15, 17) \varphi_{18} (O_{21} \cap K_{40})(51, 47, 60, 70) \varphi_6^{-1} (O_6 \cap K_{26})(1, 5, 39, 13); \\
& (O_7 \cap K_{26})(91, 21, 2, 12) \varphi_{14} (O_{16} \cap K_{44})(64, 67, 81, 80) \varphi_9^{-1} (O_9 \cap K_{42}) \\
& (54, 52, 53, 55) \varphi_{22}^{-1} (O_{20} \cap K_{36})(75, 95, 29, 32) \varphi_7^{-1} (O_7 \cap K_{26})(91, 21, 2, 12); \\
& (O_{12} \cap K_{26})(91, 21, 13, 39) \varphi_{14} (O_{12} \cap K_{44})(64, 67, 66, 65) \varphi_{11} (O_{14} \cap K_{28}) \\
& (49, 35, 24, 26) \varphi_{16} (O_{14} \cap K_{46})(73, 74, 72, 96) \varphi_{11}^{-1} (O_{12} \cap K_{26})(91, 21, 13, 39); \\
& (O_{10} \cap K_{26})(91, 12, 5, 39) \varphi_{14} (O_{18} \cap K_{44})(64, 80, 85, 65) \varphi_8^{-1} (O_8 \cap K_{40}) \\
& (44, 51, 47, 42) \varphi_{18}^{-1} (O_{17} \cap K_{30})(68, 27, 23, 92) \varphi_{10}^{-1} (O_{10} \cap K_{26})(91, 12, 5, 3); \\
& (O_1 \cap K_{27})(2, 7, 3, 10) \varphi_{15} (O_{14} \cap K_{35})(26, 50, 96, 25) \varphi_{11}^{-1} (O_{12} \cap K_{38}) \\
& (65, 38, 39, 58) \varphi_{23} (O_{24} \cap K_{47})(88, 90, 85, 84) \varphi_1^{-1} (O_1 \cap K_{27})(2, 7, 3, 10); \\
& (O_3 \cap K_{27})(2, 7, 17, 19) \varphi_{15} (O_8 \cap K_{35})(26, 50, 51, 28) \varphi_8 (O_{18} \cap K_{48}) \\
& (95, 84, 80, 93) \varphi_{24}^{-1} (O_{22} \cap K_{41})(53, 56, 72, 61) \varphi_3^{-1} (O_3 \cap K_{27})(2, 7, 17, 19); \\
& (O_7 \cap K_{27})(2, 10, 44, 19) \varphi_{15} (O_4 \cap K_{35})(26, 25, 27, 28) \varphi_4 (O_{23} \cap K_{43}) \\
& (62, 90, 87, 63) \varphi_{21}^{-1} (O_{20} \cap K_{34})(29, 34, 73, 94) \varphi_7^{-1} (O_7 \cap K_{27})(2, 10, 44, 19); \\
& (O_8 \cap K_{27})(28, 3, 10, 44) \varphi_{15} (O_{17} \cap K_{35})(69, 96, 25, 27) \varphi_{10}^{-1} (O_{10} \cap K_{31}) \\
& (41, 52, 11, 12) \varphi_{19} (O_{18} \cap K_{32})(93, 36, 38, 64) \varphi_8^{-1} (O_8 \cap K_{27})(28, 3, 10, 44); \\
& (O_{13} \cap K_{27})(28, 17, 19, 44) \varphi_{15} (O_{13} \cap K_{35})(69, 51, 28, 27) \varphi_{12} (O_{15} \cap K_{38}) \\
& (40, 37, 77, 59) \varphi_{23} (O_{15} \cap K_{47})(77, 78, 76, 75) \varphi_{12}^{-1} (O_{13} \cap K_{27})(28, 17, 19, 44); \\
& (O_8 \cap K_{28})(3, 9, 49, 26) \varphi_{16} (O_{17} \cap K_{46})(83, 82, 73, 96) \varphi_{10}^{-1} (O_{10} \cap K_{42}) \\
& (57, 59, 54, 52) \varphi_{22}^{-1} (O_{18} \cap K_{36})(36, 37, 75, 95) \varphi_8^{-1} (O_8 \cap K_{28})(3, 9, 49, 26); \\
& (O_5 \cap K_{28})(4, 8, 24, 35) \varphi_{16} (O_{22} \cap K_{46})(88, 89, 72, 74) \varphi_3^{-1} (O_3 \cap K_{30}) \\
& (14, 18, 17, 15) \varphi_{18} (O_{19} \cap K_{40})(43, 45, 70, 60) \varphi_5^{-1}; (O_5 \cap K_{28})(4, 8, 24, 35);
\end{aligned}$$

$$\begin{aligned}
& (O_9 \cap K_{28})(4, 9, 49, 35) \varphi_{16} (O_{20} \cap K_{46})(88, 82, 73, 74) \varphi_7^{-1} (O_7 \cap K_{40}) \\
& (43, 45, 44, 42) \varphi_{18}^{-1} (O_{16} \cap K_{30})(14, 18, 68, 92) \varphi_9^{-1} (O_9 \cap K_{28})(4, 9, 49, 35); \\
& (O_4 \cap K_{28})(3, 8, 24, 26) \varphi_{16} (O_{21} \cap K_{46})(83, 89, 72, 96) \varphi_6^{-1} (O_6 \cap K_{36}) \\
& (36, 37, 51, 30) \varphi_{22} (O_{23} \cap K_{42})(57, 59, 78, 62) \varphi_4^{-1} (O_4 \cap K_{28})(3, 8, 24, 26); \\
& (O_1 \cap K_{29})(4, 6, 5, 11) \varphi_{17} (O_{14} \cap K_{37})(74, 56, 35, 34) \varphi_{11}^{-1} (O_{12} \cap K_{33}) \\
& (21, 20, 67, 46) \varphi_{20} (O_{24} \cap K_{45})(81, 86, 83, 79) \varphi_1^{-1} (O_1 \cap K_{29})(4, 6, 5, 11); \\
& (O_9 \cap K_{29})(4, 11, 54, 33) \varphi_{17} (O_{20} \cap K_{37})(74, 34, 32, 76) \varphi_7^{-1} (O_7 \cap K_{31}) \\
& (42, 10, 12, 41) \varphi_{19} (O_{16} \cap K_{32})(14, 20, 64, 93) \varphi_9^{-1} (O_9 \cap K_{29})(4, 11, 54, 33); \\
& (O_6 \cap K_{29})(5, 6, 31, 40) \varphi_{17} (O_9 \cap K_{37})(35, 56, 55, 33) \varphi_9 (O_{16} \cap K_{48}) \\
& (92, 79, 80, 93) \varphi_{24}^{-1} (O_{21} \cap K_{41})(47, 50, 72, 61) \varphi_6^{-1} (O_6 \cap K_{29})(5, 6, 31, 40); \\
& (O_{10} \cap K_{29})(5, 11, 54, 40) \varphi_{17} (O_5 \cap K_{37})(35, 34, 32, 33) \varphi_5 (O_{19} \cap K_{43}) \\
& (60, 86, 87, 63) \varphi_{21}^{-1} (O_{17} \cap K_{34})(23, 25, 73, 94) \varphi_{10}^{-1} (O_{10} \cap K_{29})(5, 11, 54, 40); \\
& (O_{15} \cap K_{29})(31, 40, 54, 33) \varphi_{17} (O_{15} \cap K_{37})(55, 33, 32, 76) \varphi_{12}^{-1} (O_{13} \cap K_{33}) \\
& (18, 71, 45, 19) \varphi_{20} (O_{13} \cap K_{45})(70, 69, 68, 71) \varphi_{12} (O_{15} \cap K_{29})(31, 40, 54, 33); \\
& (O_2 \cap K_{30})(14, 15, 23, 92) \varphi_{18} (O_{11} \cap K_{40})(43, 60, 47, 42) \varphi_2^{-1} (O_2 \cap K_{36}) \\
& (29, 30, 36, 95) \varphi_{22} (O_{11} \cap K_{42})(53, 62, 57, 52) \varphi_2^{-1} (O_2 \cap K_{30})(14, 15, 23, 92); \\
& (O_{13} \cap K_{30})(17, 18, 68, 27) \varphi_{18} (O_{13} \cap K_{40})(70, 45, 44, 51) \varphi_{12} (O_{15} \cap K_{36}) \\
& (31, 32, 75, 37) \varphi_{22} (O_{15} \cap K_{42})(78, 55, 54, 59) \varphi_{12}^{-1} (O_{13} \cap K_{30})(17, 18, 68, 27); \\
& (O_1 \cap K_{31})(9, 10, 12, 11) \varphi_{19} (O_{12} \cap K_{32})(13, 20, 64, 38) \varphi_{11} (O_{14} \cap K_{41}) \\
& (72, 56, 49, 50) \varphi_{24} (O_{24} \cap K_{48})(80, 84, 82, 79) \varphi_1^{-1} (O_1 \cap K_{31})(9, 10, 12, 11); \\
& (O_8 \cap K_{31})(9, 10, 42, 48) \varphi_{19} (O_3 \cap K_{32})(13, 20, 14, 16) \varphi_3 (O_{22} \cap K_{47}) \\
& (78, 90, 88, 76) \varphi_{23}^{-1} (O_{18} \cap K_{38})(37, 38, 65, 77) \varphi_8^{-1} (O_8 \cap K_{31})(9, 10, 42, 48); \\
& (O_9 \cap K_{31})(9, 11, 52, 48) \varphi_{19} (O_6 \cap K_{32})(13, 38, 36, 16) \varphi_6 (O_{21} \cap K_{45}) \\
& (70, 86, 83, 69) \varphi_{20}^{-1} (O_{16} \cap K_{33})(18, 20, 67, 71) \varphi_9^{-1} (O_9 \cap K_{31})(9, 11, 52, 48); \\
& (O_{11} \cap K_{31})(41, 42, 48, 52) \varphi_{19} (O_2 \cap K_{32})(93, 14, 16, 36) \varphi_2 (O_{11} \cap K_{41}) \\
& (48, 53, 61, 47) \varphi_{24} (O_2 \cap K_{48})(94, 95, 93, 92) \varphi_2 (O_{11} \cap K_{31})(41, 42, 48, 52); \\
& (O_3 \cap K_{33})(18, 19, 21, 20) \varphi_{20} (O_{19} \cap K_{45})(70, 71, 81, 86) \varphi_5^{-1} (O_5 \cap K_{34}) \\
& (24, 22, 29, 34) \varphi_{21} (O_{22} \cap K_{43})(89, 61, 62, 90) \varphi_3^{-1} (O_3 \cap K_{33})(18, 19, 21, 20); \\
& (O_7 \cap K_{33})(19, 21, 46, 45) \varphi_{20} (O_{16} \cap K_{45})(71, 81, 79, 68) \varphi_9^{-1} (O_9 \cap K_{41}) \\
& (48, 53, 56, 49) \varphi_{24} (O_{20} \cap K_{48})(94, 95, 84, 82) \varphi_7^{-1} (O_7 \cap K_{33})(19, 21, 46, 45); \\
& (O_4 \cap K_{34})(22, 23, 25, 24) \varphi_{21} (O_{21} \cap K_{43})(61, 60, 86, 89) \varphi_6^{-1} (O_6 \cap K_{38}) \\
& (40, 39, 38, 37) \varphi_{23} (O_{23} \cap K_{47})(77, 85, 90, 78) \varphi_4^{-1} (O_4 \cap K_{34})(22, 23, 25, 24); \\
& (O_{10} \cap K_{38})(40, 39, 58, 59) \varphi_{23} (O_{18} \cap K_{47})(77, 85, 84, 75) \varphi_8^{-1} (O_8 \cap K_{41}) \\
& (48, 47, 50, 49) \varphi_{24} (O_{17} \cap K_{48})(94, 92, 79, 82) \varphi_{10} (O_{10} \cap K_{38})(40, 39, 58, 59).
\end{aligned}$$

The dihedral angles at the triangular faces of the polyhedron are equal to $\pi/3$. Therefore a cycle of these faces will be inessential if it contains six faces. Write cycles of these faces:

$$\begin{aligned}
& (O_1 \cap O_3)(1, 2, 7) \varphi_1 (O_{22} \cap O_{24})(89, 88, 90) \varphi_3^{-1} (O_3 \cap O_{16}) \\
& (18, 14, 20) \varphi_9^{-1} (O_1 \cap O_9)(9, 4, 11) \varphi_1 (O_{16} \cap O_{24})(80, 81, 79) \\
& \varphi_9^{-1} (O_9 \cap O_{22})(55, 53, 56) \varphi_3^{-1} (O_1 \cap O_3)(1, 2, 7); \\
& (O_1 \cap O_6)(1, 5, 6) \varphi_1 (O_{21} \cap O_{24})(89, 83, 86) \varphi_6^{-1} (O_6 \cap O_{18}) \\
& (37, 36, 38) \varphi_8^{-1} (O_1 \cap O_8)(9, 3, 10) \varphi_1 (O_{18} \cap O_{24})(80, 85, 84) \\
& \varphi_8^{-1} (O_8 \cap O_{21})(51, 47, 50) \varphi_6^{-1} (O_1 \cap O_3)(1, 5, 6); \\
& (O_1 \cap O_5)(8, 6, 4) \varphi_1 (O_{19} \cap O_{24})(87, 86, 81) \varphi_5^{-1} (O_5 \cap O_{20}) \\
& (32, 34, 29) \varphi_7^{-1} (O_1 \cap O_7)(12, 10, 2) \varphi_1 (O_{20} \cap O_{24})(82, 84, 88) \\
& \varphi_7^{-1} (O_7 \cap O_{19})(45, 46, 43) \varphi_5^{-1} (O_1 \cap O_5)(8, 6, 4); \\
& (O_1 \cap O_4)(8, 7, 3) \varphi_1 (O_{23} \cap O_{24})(87, 90, 85) \varphi_4^{-1} (O_4 \cap O_{17}) \\
& (27, 25, 23) \varphi_{10}^{-1} (O_1 \cap O_{10})(12, 11, 5) \varphi_1 (O_{17} \cap O_{24})(82, 79, 83) \\
& \varphi_{10}^{-1} (O_{10} \cap O_{23})(59, 58, 57) \varphi_4^{-1} (O_1 \cap O_4)(8, 7, 3); \\
& (O_2 \cap O_3)(15, 16, 14) \varphi_2 (O_{11} \cap O_{23})(62, 61, 53) \varphi_3^{-1} (O_3 \cap O_7) \\
& (21, 19, 2) \varphi_7 (O_2 \cap O_{20})(95, 94, 29) \varphi_2 (O_7 \cap O_{11})(41, 42, 43) \\
& \varphi_7 (O_{20} \cap O_{22})(76, 74, 88) \varphi_3^{-1} (O_2 \cap O_3)(15, 16, 14); \\
& (O_2 \cap O_4)(15, 22, 23) \varphi_2 (O_{11} \cap O_{23})(62, 63, 57) \varphi_4^{-1} (O_4 \cap O_8) \\
& (26, 28, 3) \varphi_8 (O_2 \cap O_{18})(95, 93, 36) \varphi_2 (O_8 \cap O_{11})(42, 48, 47) \\
& \varphi_8 (O_{18} \cap O_{23})(65, 77, 85) \varphi_4^{-1} (O_2 \cap O_4)(15, 22, 23); \\
& (O_2 \cap O_4)(22, 29, 30) \varphi_2 (O_{11} \cap O_{19})(63, 43, 60) \varphi_5^{-1} (O_5 \cap O_9) \\
& (33, 4, 35) \varphi_9 (O_2 \cap O_{16})(93, 14, 92) \varphi_2 (O_9 \cap O_{11})(48, 53, 52) \\
& \varphi_9 (O_{16} \cap O_{19})(71, 81, 67) \varphi_5^{-1} (O_2 \cap O_4)(22, 29, 30); \\
& (O_2 \cap O_6)(16, 36, 30) \varphi_2 (O_{11} \cap O_{21})(61, 47, 60) \varphi_6^{-1} (O_6 \cap O_{10}) \\
& (40, 5, 39) \varphi_{10} (O_2 \cap O_{17})(94, 23, 92) \varphi_2 (O_{10} \cap O_{11})(41, 57, 52) \\
& \varphi_{10} (O_{17} \cap O_{21})(69, 83, 96) \varphi_6^{-1} (O_2 \cap O_6)(16, 36, 30); \\
& (O_3 \cap O_6)(1, 16, 13) \varphi_3 (O_{15} \cap O_{22})(55, 76, 78) \varphi_{12}^{-1} (O_3 \cap O_{13}) \\
& (18, 19, 17) \varphi_3 (O_{21} \cap O_{22})(89, 61, 72) \varphi_6^{-1} (O_6 \cap O_{15})(37, 40, 31) \\
& \varphi_{12}^{-1} (O_{13} \cap O_{21})(51, 69, 70) \varphi_6^{-1} (O_3 \cap O_6)(1, 16, 13); \\
& (O_3 \cap O_4)(15, 17, 7) \varphi_3 (O_{14} \cap O_{22})(74, 72, 56) \varphi_{11}^{-1} (O_3 \cap O_{12}) \\
& (21, 13, 20) \varphi_3 (O_{22} \cap O_{23})(62, 78, 90) \varphi_4^{-1} (O_4 \cap O_{14})(26, 24, 25) \\
& \varphi_{11}^{-1} (O_{12} \cap O_{23})(65, 66, 58) \varphi_4^{-1} (O_3 \cap O_4)(15, 17, 7); \\
& (O_4 \cap O_5)(8, 22, 24) \varphi_4 (O_{15} \cap O_{23})(59, 77, 78) \varphi_{12}^{-1} (O_4 \cap O_{13}) \\
& (27, 28, 17) \varphi_4 (O_{19} \cap O_{23})(87, 63, 66) \varphi_5^{-1} (O_5 \cap O_{15})(32, 33, 31) \\
& \varphi_{12}^{-1} (O_{13} \cap O_{19})(45, 71, 70) \varphi_5^{-1} (O_4 \cap O_5)(8, 22, 24); \\
& (O_5 \cap O_{14})(34, 35, 24) \varphi_5 (O_{19} \cap O_{21})(86, 60, 70) \varphi_6^{-1} (O_6 \cap O_{12}) \\
& (38, 39, 13) \varphi_{11} (O_{14} \cap O_{21})(50, 96, 72) \varphi_6^{-1} (O_5 \cap O_6)(6, 30, 31) \\
& \varphi_5 (O_{12} \cap O_{19})(46, 67, 66) \varphi_{11} (O_5 \cap O_{14})(34, 35, 24); \\
& (O_7 \cap O_8)(10, 42, 44) \varphi_7 (O_{14} \cap O_{20})(34, 74, 73) \varphi_{11}^{-1} (O_7 \cap O_{12}) \\
& (46, 21, 91) \varphi_7 (O_{18} \cap O_{20})(84, 95, 75) \varphi_8^{-1} (O_8 \cap O_{14})(50, 26, 49) \\
& \varphi_{11}^{-1} (O_{12} \cap O_{18})(38, 65, 64) \varphi_8^{-1} (O_7 \cap O_8)(10, 42, 44);
\end{aligned}$$

$$\begin{aligned}
& (O_7 \cap O_{10})(12, 41, 91) \varphi_7 (O_{15} \cap O_{20})(32, 76, 75) \varphi_{12}^{-1} (O_7 \cap O_{13}) \\
& (45, 19, 44) \varphi_7 (O_{17} \cap O_{20})(82, 94, 73) \varphi_{10}^{-1} (O_{10} \cap O_{15})(59, 40, 54) \\
& \varphi_{12}^{-1} (O_{13} \cap O_{17})(27, 69, 68) \varphi_{10}^{-1} (O_7 \cap O_{10})(12, 41, 91); \\
& (O_8 \cap O_9)(9, 48, 49) \varphi_8 (O_{15} \cap O_{18})(37, 77, 75) \varphi_{12}^{-1} (O_8 \cap O_{13}) \\
& (51, 28, 44) \varphi_8 (O_{16} \cap O_{18})(80, 93, 64) \varphi_9^{-1} (O_9 \cap O_{15})(55, 33, 54) \\
& \varphi_{12}^{-1} (O_{13} \cap O_{16})(18, 71, 68) \varphi_9^{-1} (O_8 \cap O_9)(9, 48, 49); \\
& (O_9 \cap O_{10})(54, 52, 11) \varphi_9 (O_{12} \cap O_{16})(64, 67, 20) \varphi_{11} (O_9 \cap O_{14}) \\
& (49, 35, 56) \varphi_9 (O_{16} \cap O_{17})(68, 92, 79) \varphi_{10}^{-1} (O_{10} \cap O_{12})(91, 39, 58) \\
& \varphi_{11} (O_{14} \cap O_{17})(73, 96, 25) \varphi_{10}^{-1} (O_9 \cap O_{10})(54, 52, 11).
\end{aligned}$$

Finally write cycles of one-dimensional edges of the polyhedron R_2 :

$$\begin{aligned}
& (1, 2) \varphi_{14} (87, 81) \varphi_1^{-1} (8, 4) \varphi_{16} (89, 88) \varphi_3^{-1} (18, 14) \\
& \varphi_{18} (45, 43) \varphi_7 (82, 88) \varphi_{16}^{-1} (9, 4) \varphi_1 (80, 81) \\
& \varphi_{14}^{-1} (12, 2) \varphi_7 (32, 29) \varphi_{22} (55, 53) \varphi_3^{-1} (1, 2); \\
& (1, 5) \varphi_{14} (87, 85) \varphi_1^{-1} (8, 3) \varphi_{16} (89, 83) \varphi_6^{-1} (37, 36) \\
& \varphi_{22} (59, 57) \varphi_{10} (82, 83) \varphi_{16}^{-1} (9, 3) \varphi_1 (80, 85) \\
& \varphi_{14}^{-1} (12, 5) \varphi_{10} (27, 23) \varphi_{18} (51, 47) \varphi_6^{-1} (1, 5); \\
& (1, 6) \varphi_{13} (66, 58) \varphi_{11} (24, 25) \varphi_{21} (89, 86) \varphi_6^{-1} (37, 38) \\
& \varphi_{23} (78, 90) \varphi_3^{-1} (13, 20) \varphi_{19}^{-1} (9, 10) \varphi_1 (80, 84) \\
& \varphi_{24}^{-1} (72, 56) \varphi_3^{-1} (17, 7) \varphi_{15} (51, 50) \varphi_6^{-1} (1, 6); \\
& (1, 7) \varphi_{13} (66, 46) \varphi_{11} (24, 34) \varphi_{21} (89, 90) \varphi_3^{-1} (18, 20) \\
& \varphi_{20} (70, 86) \varphi_6^{-1} (13, 38) \varphi_{19}^{-1} (9, 11) \varphi_1 (80, 79) \\
& \varphi_{24}^{-1} (72, 50) \varphi_6^{-1} (31, 6) \varphi_{17} (55, 56) \varphi_3^{-1} (1, 7); \\
& (1, 16) \varphi_{13} (66, 63) \varphi_5^{-1} (31, 33) \varphi_{17} (55, 76) \varphi_{12}^{-1} (18, 19) \\
& \varphi_{20} (70, 71) \varphi_5^{-1} (24, 22) \varphi_{21} (89, 61) \varphi_6^{-1} (37, 40) \\
& \varphi_{23} (78, 77) \varphi_{12}^{-1} (17, 28) \varphi_{15} (51, 69) \varphi_6^{-1} (1, 16); \\
& (1, 13) \varphi_{14} (87, 66) \varphi_4^{-1} (27, 17) \varphi_{18} (51, 70) \varphi_{12} (37, 31) \\
& \varphi_{22} (59, 78) \varphi_4^{-1} (8, 24) \varphi_{16} (89, 72) \varphi_3^{-1} (18, 17) \\
& \varphi_{18} (45, 70) \varphi_{12} (32, 31) \varphi_{22} (55, 78) \varphi_3^{-1} (1, 13); \\
& (2, 10) \varphi_{15} (26, 25) \varphi_4 (62, 90) \varphi_{21}^{-1} (29, 34) \varphi_5 (81, 86) \varphi_{20}^{-1} \\
& (21, 20) \varphi_{11} (74, 56) \varphi_{17}^{-1} (4, 6) \varphi_5 (43, 46) \varphi_{13}^{-1} \\
& (15, 7) \varphi_4 (65, 58) \varphi_{23} (88, 84) \varphi_1^{-1} (2, 10); \\
& (2, 7) \varphi_{15} (26, 50) \varphi_8 (95, 84) \varphi_{24}^{-1} (53, 56) \varphi_9 (81, 79) \\
& \varphi_{20}^{-1} (21, 46) \varphi_{11} (74, 34) \varphi_{17}^{-1} (4, 11) \varphi_9 (14, 20) \\
& \varphi_{19}^{-1} (42, 10) \varphi_8 (65, 38) \varphi_{23} (88, 90) \varphi_1^{-1} (2, 7); \\
& (2, 19) \varphi_{15} (26, 28) \varphi_4 (62, 63) \varphi_{21}^{-1} (29, 94) \varphi_2 (43, 41) \\
& \varphi_{13}^{-1} (15, 22) \varphi_4 (65, 77) \varphi_{23} (88, 76) \varphi_3^{-1} (14, 16) \\
& \varphi_{19}^{-1} (42, 48) \varphi_2^{-1} (95, 93) \varphi_{24}^{-1} (53, 61) \varphi_3^{-1} (2, 19);
\end{aligned}$$

$(2, 21) \varphi_{14} (81, 67) \varphi_5^{-1} (29, 30) \varphi_{22} (53, 62) \varphi_2^{-1} (14, 15)$
 $\varphi_{18} (43, 60) \varphi_5^{-1} (4, 35) \varphi_{16} (88, 74) \varphi_7^{-1} (43, 42)$
 $\varphi_{18}^{-1} (14, 92) \varphi_2 (53, 52) \varphi_{22}^{-1} (29, 95) \varphi_7^{-1} (2, 21);$

$(3, 7) \varphi_{15} (96, 50) \varphi_{11}^{-1} (39, 38) \varphi_{23} (85, 90) \varphi_4^{-1} (23, 25)$
 $\varphi_{21} (60, 86) \varphi_5^{-1} (35, 34) \varphi_{17}^{-1} (5, 11) \varphi_1 (83, 79)$
 $\varphi_{20}^{-1} (67, 46) \varphi_5^{-1} (30, 6) \varphi_{13} (57, 58) \varphi_4^{-1} (3, 7);$

$(3, 10) \varphi_{15} (96, 25) \varphi_{11}^{-1} (39, 58) \varphi_{23} (85, 84) \varphi_8^{-1} (47, 50)$
 $\varphi_{24} (92, 79) \varphi_9^{-1} (35, 56) \varphi_{17}^{-1} (5, 6) \varphi_1 (83, 86)$
 $\varphi_{20}^{-1} (67, 20) \varphi_9^{-1} (51, 11) \varphi_{19} (36, 38) \varphi_8^{-1} (3, 10);$

$(3, 26) \varphi_{16} (83, 96) \varphi_6^{-1} (36, 30) \varphi_{22} (57, 62) \varphi_2^{-1} (23, 15)$
 $\varphi_{18} (47, 60) \varphi_6^{-1} (5, 39) \varphi_{14} (85, 65) \varphi_8^{-1} (47, 42)$
 $\varphi_{18}^{-1} (23, 92) \varphi_2 (57, 52) \varphi_{22}^{-1} (36, 95) \varphi_8^{-1} (3, 26);$

$(3, 28) \varphi_{15} (96, 69) \varphi_{10}^{-1} (52, 41) \varphi_{19} (36, 93) \varphi_2 (47, 48)$
 $\varphi_{24} (92, 94) \varphi_{10}^{-1} (39, 40) \varphi_{23} (85, 77) \varphi_4^{-1} (23, 22)$
 $\varphi_{21} (60, 61) \varphi_2^{-1} (30, 16) \varphi_{13} (57, 63) \varphi_4^{-1} (3, 28);$

$(4, 33) \varphi_{17} (74, 76) \varphi_3^{-1} (15, 16) \varphi_{13} (43, 63) \varphi_2^{-1} (29, 22)$
 $\varphi_{21} (62, 61) \varphi_3^{-1} (21, 19) \varphi_{20} (81, 71) \varphi_9^{-1} (53, 48)$
 $\varphi_{24} (95, 94) \varphi_2 (42, 41) \varphi_{19} (14, 93) \varphi_9^{-1} (4, 33);$

$(5, 40) \varphi_{17} (35, 33) \varphi_5 (60, 63) \varphi_{21}^{-1} (23, 94) \varphi_2 (57, 41)$
 $\varphi_{13}^{-1} (30, 22) \varphi_5 (67, 71) \varphi_{20} (83, 69) \varphi_6^{-1} (36, 16)$
 $\varphi_{19}^{-1} (52, 48) \varphi_2^{-1} (92, 93) \varphi_{24}^{-1} (47, 61) \varphi_6^{-1} (5, 40);$

$(6, 8) \varphi_{13} (58, 91) \varphi_{13} (25, 73) \varphi_{21} (86, 87) \varphi_5^{-1}$
 $(34, 32) \varphi_{17}^{-1} (11, 54) \varphi_9 (20, 64) \varphi_{19}^{-1} (10, 12) \varphi_1 (84, 82)$
 $\varphi_{24}^{-1} (56, 49) \varphi_9 (79, 68) \varphi_{20}^{-1} (46, 45) \varphi_5^{-1} (6, 8);$

$(7, 8) \varphi_{13} (46, 91) \varphi_7 (84, 75) \varphi_{23}^{-1} (58, 59) \varphi_{10} (79, 82)$
 $\varphi_{24}^{-1} (50, 49) \varphi_{11}^{-1} (38, 64) \varphi_{19}^{-1} (11, 12) \varphi_{10} (25, 27)$
 $\varphi_{15}^{-1} (10, 44) \varphi_7 (34, 73) \varphi_{21} (90, 87) \varphi_1^{-1} (7, 8);$

$(8, 22) \varphi_{13} (91, 41) \varphi_7 (75, 76) \varphi_{23}^{-1} (59, 77) \varphi_{12}^{-1} (27, 28)$
 $\varphi_{15}^{-1} (44, 19) \varphi_7 (73, 94) \varphi_{21} (87, 63) \varphi_5^{-1} (32, 33)$
 $\varphi_{17}^{-1} (54, 40) \varphi_{12}^{-1} (68, 69) \varphi_{20}^{-1} (45, 71) \varphi_5^{-1} (8, 22);$

$(9, 49) \varphi_{16} (82, 73) \varphi_7^{-1} (45, 44) \varphi_{18}^{-1} (18, 68) \varphi_{12} (55, 54)$
 $\varphi_{22}^{-1} (32, 75) \varphi_7^{-1} (12, 91) \varphi_{14} (80, 64) \varphi_8^{-1} (51, 44)$
 $\varphi_{18}^{-1} (27, 68) \varphi_{12} (59, 54) \varphi_{22}^{-1} (37, 75) \varphi_8^{-1} (9, 49);$

$(9, 48) \varphi_{19} (13, 16) \varphi_3 (78, 76) \varphi_{23}^{-1} (37, 77) \varphi_{12}^{-1} (51, 28)$
 $\varphi_{15}^{-1} (17, 19) \varphi_3 (72, 61) \varphi_{24} (80, 93) \varphi_9^{-1} (55, 33)$
 $\varphi_{17}^{-1} (31, 40) \varphi_{12}^{-1} (70, 69) \varphi_{20}^{-1} (18, 71) \varphi_9^{-1} (9, 48);$

$(12, 41) \varphi_{19} (64, 93) \varphi_9^{-1} (54, 33) \varphi_{17} (32, 76) \varphi_{12}^{-1} (45, 19)$
 $\varphi_{20} (68, 71) \varphi_9^{-1} (49, 48) \varphi_{24} (82, 94) \varphi_{10}^{-1} (59, 40)$
 $\varphi_{23} (75, 77) \varphi_{12}^{-1} (44, 28) \varphi_{15} (27, 69) \varphi_{10}^{-1} (12, 41);$

$(13, 21) \varphi_{14} (66, 67) \varphi_{11} (24, 35) \varphi_{16} (72, 74) \varphi_3^{-1} (17, 15)$
 $\varphi_{18} (70, 60) \varphi_6^{-1} (13, 39) \varphi_{14} (66, 65) \varphi_{11} (24, 26)$
 $\varphi_{16} (72, 96) \varphi_6^{-1} (31, 30) \varphi_{22} (78, 62) \varphi_3^{-1} (13, 21);$
 $(95, 75) \varphi_{22} (52, 54) \varphi_9 (67, 64) \varphi_{14}^{-1} (21, 91) \varphi_{11} (74, 73)$
 $\varphi_{16}^{-1} (35, 49) \varphi_9 (92, 68) \varphi_{18} (42, 44) \varphi_8 (65, 64)$
 $\varphi_{14}^{-1} (39, 91) \varphi_{11} (96, 73) \varphi_{16}^{-1} (26, 49) \varphi_8 (95, 75).$

As each cycle contains 12 edges, we have shown that cycles of one-dimensional edges are inessential, too. Thus we have shown that identifying the faces of the polyhedron R_2 by the motions $\varphi_1, \varphi_2, \dots, \varphi_{24}$, the cycles both of two-dimensional faces and one-dimensional edges are inessential. Therefore the group Γ generated by the motions $\varphi_1, \varphi_2, \dots, \varphi_{24}$ does not contain elements of finite order, i. e. Γ is torsion-free. Then the quotient space of the space H^4 by the group Γ is a four-dimensional hyperbolic manifold M with finite volume which is not closed. The manifold M has four cusps, i. e. four ends of the form $T^3 \times [0, \infty)$, where T^3 is a three-dimensional torus.

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