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# A SPECTRUM ASSOCIATED WITH MINKOWSKI DIAGONAL continued fraction

Alena Aleksenko[1]

Let $\alpha$ be real irrational number. The function $\mu_\alpha(t)$ is defined as follows.
The Legendre theorem states that if

$$\left|\alpha - \frac{A}{Q}\right| < \frac{1}{2Q^2}, \ \ (A,Q) = 1 \tag{1}$$

then the fraction $\frac{A}{Q}$ is a convergent fraction for the continued fraction expansion of $\alpha$. The converse statement is not true. It may happen that $\frac{A}{Q}$ is a convergent to $\alpha$ but (1) is not valid. One should consider the sequence of the denominators of the convergents to $\alpha$ for which (1) is true. Let this sequence be

$$Q_0 < Q_1 < \cdots < Q_n < Q_{n+1} < \cdots .$$

Then for $\alpha \notin \mathbb{Q}$ the function $\mu_\alpha(t)$ is defined by

$$\mu_\alpha(t) = \frac{Q_{n+1} - t}{Q_{n+1} - Q_n} \cdot ||Q_n\alpha|| + \frac{t - Q_n}{Q_{n+1} - Q_n} \cdot ||Q_{n+1}\alpha||, \quad Q_n \leqslant t \leqslant Q_{n+1}.$$

From the other hand, for every $\nu$ one of the consecutive convergent fractions $\frac{p_\nu}{q_\nu}$, $\frac{p_{\nu+1}}{q_{\nu+1}}$ to $\alpha$ satisfies (1). So either

$$(Q_n, Q_{n+1}) = (q_\nu, q_{\nu+1})$$

for some $\nu$, or

$$(Q_n, Q_{n+1}) = (q_{\nu-1}, q_{\nu+1})$$

for some $\nu$.

Actually the function $\mu_\alpha(t)$ was considered by Minkowski [4]. There exists an alternative geometric definition of $\mu_\alpha(t)$. Some related facts were discussed in [3, 6].

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[1]Department of Mathematics, Aveiro University, Aveiro 3810, Portugal. This work was supported by *FEDER* founds through *COMPETE*–Operational Programme Factors of Competitiveness (“Programa Operacional Factores de Competitividade”) and by Portuguese founds through the *Center for Research and Development in Mathematics and Applications* (University of Aveiro) and the Portuguese Foundation for Science and Technology (“FCT–Fundção para a Ciência e a Tecnologia”), within project PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690

The quantity

$$\mathfrak{m}(\alpha) = \limsup_{t\to+\infty} t\cdot\mu_\alpha(t).$$

was considered in [6]. An explicit formula for the value of $\mathfrak{m}(\alpha)$ in terms of continued fraction expansion for $\alpha$ was proved in [6]. It is as follows. Put

$$\mathfrak{m}_n(\alpha) = \begin{cases} G(\alpha_\nu^*, \alpha_{\nu+2}^{-1}), & \text{if } (Q_n, Q_{n+1}) = (q_{\nu-1}, q_{\nu+1}) \text{ with some } \nu, \\ F(\alpha_{\nu+1}^*, \alpha_{\nu+2}^{-1}), & \text{if } (Q_n, Q_{n+1}) = (q_\nu, q_{\nu+1}) \text{ with some } \nu, \end{cases} \tag{2}$$

where

$$G(x,y) = \frac{x+y+1}{4}, \quad F(x,y) = \frac{(1-xy)^2}{4(1+xy)(1-x)(1-y)}$$

and $\alpha_\nu, \alpha_\nu^*$ come from continued fraction expansion to

$$\alpha = [a_0; a_1, a_2, \dots, a_t, \dots]$$

in such a way:

$$\alpha_\nu = [a_\nu; a_{\nu+1}, ...], \quad \alpha_\nu^* = [0; a_\nu, a_{\nu-1}, ..., a_1].$$

Then

$$\mathfrak{m}(\alpha) = \limsup_{n\to+\infty} \mathfrak{m}_n(\alpha),$$

The specrtum

$$\mathbb{M} = \{m \in \mathbb{R}: \ \exists \alpha \in \mathbb{R}\setminus\mathbb{Q} \text{ such that } m = \mathfrak{m}(\alpha)\}.$$

was studied in [6]. It was proven there that $\mathbb{M} \subset \left[\frac{1}{4}, \frac{1}{2}\right]$ and that $\frac{1}{4}, \frac{1}{2} \in \mathbb{M}$. However no further structure of the spectrum $\mathbb{M}$ is known.

In this paper we consider the spectrum

$$\mathbb{I} = \{m \in \mathbb{R}: \ \exists \alpha \in \mathbb{R}\setminus\mathbb{Q} \text{ such that } \mathfrak{i}(\alpha) = m\},$$

where

$$\mathfrak{i}(\alpha) = \liminf_{n\to\infty} \mathfrak{m}_n(\alpha)$$

(however, compared to $\mathfrak{m}(\alpha)$, this quantity has no clear Diophantine sense).

It is clear that

$$\min \mathbb{I} = \frac{1}{4}, \quad \max \mathbb{I} = \frac{1}{2}.$$

**Theorem.** *There exists positive $\omega_0$ such that*

$$\left[\frac{1}{4}, \omega_0\right] \subset \mathbb{I}.$$

The proof is based ol M. Hall's ideas (see [2]). It uses technique from [5].

**Remark.** *An explicit formula for $\omega_0$ may be obtained from the proof below. It is interesting to get optimal estimates for the value of $\omega_0$.*

We need some well known results.

Recall the definition of a $\tau$-set $\mathcal{F} \subset \mathbb{R}$. The set $\mathcal{F}$ must be of the form

$$\mathcal{F} = \mathcal{S} \setminus \left( \bigcup_{\nu+1}^{\infty} \Delta_\nu \right),$$

where $\mathcal{S} \subset \mathbb{R}$ is a segment, and $\Delta_\nu \subset \mathcal{S}$, $\nu = 1, 2, 3, ...$ is an ordered sequence of disjoint intervals. Moreover for every $t$ if

$$\mathcal{S} \setminus \left( \bigcup_{\nu+1}^{t-1} \Delta_\nu \right) = \bigcup_{j=1}^{r} \mathcal{M}_j$$

is a union of segments $\mathcal{M}_j$ and $\Delta_t \subset \mathcal{M}_{j^*}$ then

$$\mathcal{M}_{j^*} = \mathcal{N}^1 \sqcup \Delta_t \sqcup \mathcal{N}^2,$$

and

$$\min(|\mathcal{N}^1|, |\mathcal{N}^2|) \geqslant \tau |\Delta_t|.$$

Consider the set

$$\mathcal{F}_5 = \{\alpha = [0; b_1, b_2, b_3, ...] : \ b_\nu \leqslant 5 \ \ \forall\, \nu\}$$

consisting of all irrational real numbers from the unit interval $(0, 1)$ with partial quotients bounded by $5$. One can easily see that

$$\begin{cases} A = \min \mathcal{F}_5 = [0; \overline{5, 1}] = \frac{\sqrt{45}-5}{10} = 0.1708^+, \\ B = \max \mathcal{F}_5 = [0; \overline{1, 5}] = \frac{\sqrt{45}-5}{2} = 0.85410^+. \end{cases} \tag{3}$$

Put

$$\mathcal{S}_5 = [A, B] \subset [0, 1].$$

The following lemma comes from the results of the papers [1] or [7].

**Lemma 1.** *The set $\mathcal{F}_5$ is a $\tau$-set with $\tau = \tau_5 = 1.788^+$.*

Let $H(x, y) : \ \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ be a function in two variables of the class $G \in C^1(\mathcal{S} \times \mathcal{S})$. Consider the set

$$\mathcal{J} = \{z \in \mathbb{R} : \ \exists\, x, y \in \mathcal{S} \ \ z = H(x, y)\}.$$

By continuousity argument $\mathcal{J}$ is a segment.

**Lemma 2.** *Suppose that the derivatives $\partial H/\partial x, \partial H/\partial y$ do not take zero values on the box $\mathcal{S} \times \mathcal{S}$. Suppose that $\mathcal{F}$ is $\tau$-set and $\mathcal{S} = [\min \mathcal{F}, \max \mathcal{F}]$. Suppose that*

$$\tau \geqslant \max_{x,y \in \mathcal{S}} \ \max \left( \left| \frac{\partial H/\partial x}{\partial H/\partial y} \right|, \left| \frac{\partial H/\partial y}{\partial H/\partial x} \right| \right). \tag{4}$$

*Then*

$$\{z:\ \exists x,y\in\mathcal{F}\quad such\ that\quad z=H(x,y)\}=\mathcal{J}.$$

Lemma 2 is a staightforward generalization of a result from [5]. We do not give its proof here as the proof follows the argument from [5] word-by-word.

Now we are able to conclude the proof of Theorem.

We consider pairs of integers $(R_1,R_2)$ of the form

$$(R_1,R_2)\ =\ (R,R)\ \text{ or }\ (R,R+1) \tag{5}$$

with $R\geqslant 6$. Consider a function

$$H_{R_1,R_2}(x,y)=F\left(\frac{1}{R_1+x},\frac{1}{R_2+y}\right).$$

For $R_1,R_2$ under consideration the function $H_{R_1,R_2}(x,y)$ decreases both in $x$ and in $y$.

For $0<x,y<1$ put

$$\varphi(x,y)=(1-3x+3xy-x^2y)(1-x).$$

For any $y\in(0,1)$ the function $\varphi(x,y)$ decreases in $x$. For any $x\in(0,1)$ the function $\varphi(x,y)$ increases in $y$. Now

$$\frac{\partial F/\partial y}{\partial F/\partial x}=\frac{\varphi(x,y)}{\varphi(y,x)},$$

and

$$\left|\frac{\partial H_{R_1,R_2}/\partial y}{\partial H_{R_1,R_2}/\partial x}\right|=\frac{\varphi\left(\frac{1}{R_1+x},\frac{1}{R_2+y}\right)}{\varphi\left(\frac{1}{R_2+y},\frac{1}{R_1+x}\right)}\left(\frac{R_1+x}{R_2+y}\right)^2.$$

Easy calculation shows that for $R_1,R_2\geqslant 6$ one has

$$\max_{x,y\in\mathcal{S}_5}\ \max\left(\left|\frac{\partial H_{R_1,R_2}/\partial x}{\partial H_{R_1,R_2}/\partial y}\right|,\left|\frac{\partial H_{R_1,R_2}/\partial y}{\partial H_{R_1,R_2}/\partial x}\right|\right)=$$

$$=\frac{\varphi\left(\frac{1}{R_1+B},\frac{1}{R_2+A}\right)}{\varphi\left(\frac{1}{R_2+A},\frac{1}{R_1+B}\right)}\left(\frac{R_1+B}{R_2+A}\right)^2\leqslant\frac{\varphi\left(\frac{1}{R_1+B},\frac{1}{R_1+A}\right)}{\varphi\left(\frac{1}{R_1+A},\frac{1}{R_1+B}\right)}\left(\frac{R_1+B}{R_1+A}\right)^2\leqslant$$

$$\leqslant\frac{\varphi\left(\frac{1}{6+B},\frac{1}{6+A}\right)}{\varphi\left(\frac{1}{6+A},\frac{1}{6+B}\right)}\left(\frac{6+B}{6+A}\right)^2=1.363^+<\tau_5.$$

Here $A$ and $B$ are defined in (3) and in the last inequalities we use the bounds $6\leqslant R_1\leqslant R_2$ which follows from (5).

We see that for any $R_1, R_2$ under consideration and for $\tau_5$-set $\mathcal{F}_5$ the condition (4) is satisfied. We apply Lemma 2 to see that the image of the set $\mathcal{F}_5 \times \mathcal{F}_5$ under the mapping $H_{R_1,R_2}(x,y)$ is just the segment

$$\mathcal{J}_{R_1,R_2} = [H_{R_1,R_2}(B,B), H_{R_1,R_2}(A,A)].$$

But

$$H_{R,R}(B,B) < H_{R,R+1}(A,A)$$

and

$$H_{R,R+1}(B,B) < H_{R+1,R+1}(A,A).$$

That is why if we put

$$\omega_0 = H_{R_0,R_0}(A,A).$$

with $R_0 \geqslant 6$ we get

$$\bigcup_{R\geqslant R_0} \mathcal{J}_{R,R} \cup \bigcup_{R\geqslant R_0} \mathcal{J}_{R,R+1} = (1/4, \omega_0].$$

Take $m \in (0, \omega_0]$. Then there exists $R_1, R_2$ such that

$$m \in \mathcal{J}_{R_1,R_2}$$

and there exist

$$\beta = [0; b_1, b_2, ..., b_\nu, ...], \quad \gamma = [0; c_1, c_2, ..., c_\nu, ...], \quad \beta, \gamma \in \mathcal{F}_5,$$

such that

$$F\left(\frac{1}{R_1+\alpha}, \frac{1}{R_2+\beta}\right) = m.$$

Now we take

$$\alpha = [0; \underbrace{a_1, R_1, R_2, b_1}_{1}, \underbrace{a_2, a_1, R_1, R_2, b_1, b_2}_{2}, ..., \underbrace{a_\nu, a_{\nu-1}, ..., a_2, a_1, R_1, R_2, b_1, b_2, ..., b_{\nu-1}, b_\nu}_{\nu}, ...].$$

Standard argument shows that for $n_\nu$ defined from

$$\frac{p_{n_\nu}}{q_{n_\nu}} = [0; a_1, R_1, R_2, b_1, a_2, a_1, R_1, R_2, b_1, b_2, ..., a_\nu, a_{\nu-1}, ..., a_2, a_1, R_1]$$

one has

$$\lim_{\nu\to+\infty} F(\alpha^*_{n_\nu}, \alpha^{-1}_{n_\nu+1}) = m.$$

At the same time for $F(\cdot,\cdot)$ and $G(\cdot,\cdot)$ we have

$$\inf_{n\neq n_\nu \,\forall \nu} F(\alpha^*_n, \alpha_{n+1}) > \omega_0$$

and

$$\inf_{n\in\mathbb{Z}_+} G(\alpha^*_n, \alpha^{-1}_{n+2}) > \omega_0,$$

for large $R_0$. So $\mathfrak{i}(\alpha) = m$ and everything is proved. $\Box$

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Department of Mathematics, Aveiro University, Aveiro 3810, Portugal
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