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The topological classification of germs of the maximum and minimax functions of a family of functions in general position

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0. To every continuous real-valued function f defined on a space of a locally trivial fibering p_1: $X \to Y$ with compact fibre we can assign a continuous function given on the base: with a point y of the base we associate the maximal value of f on the fibre $p_1^{-1}(y)$. The resulting function is called the maximum function of f. If the base Y in its turn is a space of a locally trivial fibering with compact fibre p_2: $Y \to Z$, then by taking the minimum function of the maximum function we obtain the minimax function. In this note we prove that any germ both of the maximum and the minimax function of a smooth function in general position is topologically equivalent to a germ of the Morse function.

1. Definitions.
Let $\dim X = n$, $\dim Y = k$, and $\dim Z = m$. A smooth function f: $X \to \mathbf{R}$ is called *a family of functions* in n_1 variables with k parameters, where $n_1 = n - k$. Let M and N be smooth manifolds. The germ of a map H: $M \to N$ at a point $x \in M$ is denoted by H: $(M, x) \to (N, H(x))$ or $[H]_x$. The germs of two functions F: $(M, x) \to (\mathbf{R}, f(x))$ and G: $(N, y) \to (\mathbf{R}, G(y))$ are *topologically equivalent* if there exists a germ of a homeomorphism H: $(M, x) \to (N, y)$ for which $F = G \circ H$. A function $M \to \mathbf{R}$ is called a *C-Morse function* if its germ at any point $x \in M$ is topologically equivalent either to the germ of a linear function or of a Morse function at a singular point (in this case the point x is called *C-singular*, and its index is the index of the corresponding quadratic form). *The direct sum* of two germs F: $(M, x) \to (\mathbf{R}, 0)$ and G: $(N, y) \to (\mathbf{R}, 0)$ is the germ $F \oplus G$: $(M \times N, (x, y)) \to (\mathbf{R}, 0)$, where $(F \oplus G)(u, v) = F(u) + G(v)$. The maximum function and the minimax function of a family f are denoted by $\max(f)$ and $\operatorname{minmax}(f)$ respectively. By $\mathfrak{m}(p)$ we denote the space of smooth germs $(\mathbf{R}^p, 0) \to (\mathbf{R}, 0)$.

2. ***Theorem* 1.** *The maximum function of a family in general position is a C-Morse function.*

This proposition was stated as a conjecture by A.M. Vershik during a discussion of [2], in which it was verified for $k \leqslant 6$; arguments in favour of the general conjecture were given by Arnol'd in [1].

***Theorem* 2.** *The minimax function of a family in general position is a C-Morse function.*

***Theorem* 3.** *The index $\nu(y)$ of a C-singular point y of the function* $\max(f)$ *of a family f in general position satisfies the inequality $\nu(y) \leqslant k - s(y) + 1$, where $s(y)$ is the number of distinct points of the fibre $p_1^{-1}(y)$ at which f attains the value* $\max(f)(y)$.

***Theorem* 4.** *The index $\eta(z)$ of a C-singular point z of the function* $\operatorname{minmax}(f)$ *of a family f in general position satisfies the inequalities*

$$l(z) - 1 \leqslant \eta(z) \leqslant m + l(z) - (\max(1, s_1(z) - k_1) + \ldots + \max(1, s_{l(z)} - k_1),$$

where $k_1 = k - m$, $l(z)$ is the number of distinct points (say, $y^1, \ldots, y^{l(z)}$) of the fibre $p_2^{-1}(z)$ at which $\max(f)$ *attains the value* $\operatorname{minmax}(f(z))$, *and $s_i(z)$ is the number of distinct points of the fibre $p_1^{-1}(y^i)$ at which f attains this value.*

3. Sketch of the proofs of Theorems 1 and 3.
Let f be a family in general position, $y \in Y$, $\max(f(y)) = 0$. Let $X(s)$ be the inverse image of the diagonal under the immersion p_1^s: $X^s \to Y^s$, where $(\cdot)^s$ is the s-fold direct product. We denote by $r(y)$ the rank of the composition of the embedding $X(s) \hookrightarrow X^s$ and of the map f^s: $X^s \to \mathbf{R}^s$ at the multipoint $(x^1, \ldots, x^s)$, where $s = s(y)$ and $f(x^i) = \max(f)(y)$, $p_1(x^i) = y$.

Lemma. *$r(y) = s(y)$ for a family f in general position and for every point $y \in Y$, except for a set $Y(f)$ of dimension zero. If $y \in Y(f)$ then $r(y) = s(y) - 1$, and the points x^i $(i = 1, \ldots, s(y))$ are Morse points of the function $f \mid p_1^{-1}(y)$.*

Introducing suitable coordinate systems in neighbourhoods of the points x^i we find that the germ $[\max(f)]_y$ is topologically equivalent to the germ $[F]_0$ of the function F: $\mathbf{R}^k \to \mathbf{R}$, where $F(v) = \max\{\max\{f_i(u, v) \mid u \in \mathbf{R}^{n_i}, \ \|u\| \leqslant \varepsilon\} \mid i = 1, \ldots, s\}$, $\varepsilon > 0$, and $0 \in \mathbf{R}^{n_i}$ is a maximum point of the smooth function $f_i \mid \mathbf{R}^{n_i} \times \{0\}$. In the case $r_y = s-1$ it follows from the implicit function theorem that $[F]_0 = \max(\varphi_1, \ldots, \varphi_s)$, where $\varphi_i \in \mathfrak{m}(k)$.

Lemma. *If the map* $\mathbf{R}^{n_i} \times \mathbf{R}^k \to \mathbf{R}^s$ *with the components* $f_1, \ldots, f_s$ *has rank* s *at* 0, *then the germ* $[F]_0$ *is topologically equivalent to the germ of a linear function.*

Lemma. *If* f *is a family in general position and if the germ* $(\mathbf{R}^k, 0) \to (\mathbf{R}^s, 0)$ *with the components* $\varphi_1, \ldots, \varphi_s$ *has rank* $s-1$ *at* 0, *then the germ* $[F]_0$ *is topologically equivalent to the direct sum of the germ* $\max(\gamma_1, \ldots, \gamma_s)$, *where* $\gamma_i \in \mathfrak{m}(s-1)$, *and the germ of a non-degenerate quadratic form from* $\mathfrak{m}(k-s+1)$. *Here the set of germs* $\gamma_1, \ldots, \gamma_s$ *satisfies the conditions of the following lemma.*

Lemma. *If any* p *out of the* q *vectors* $\operatorname{grad} \gamma_i(0)$ $(i = 1, \ldots, q;\ q > p)$, *where* $\gamma_i \in \mathfrak{m}(p)$, *are linearly independent, then the germ* $\max(\gamma_1, \ldots, \gamma_q)$ *is topologically equivalent either to the germ* y_i *or to the germ* $y_1^2 + \ldots + y_p^2$, *where* $y_1, \ldots, y_p$ *are coordinates in* $\mathbf{R}^p$.

4. Sketch of the proofs of Theorems 2 and 4.

Let f be a family in general position, $z \in Z$, $\operatorname{minmax}(f)(z) = 0$. Suppose that for $i = 1, \ldots,$; $j = 1, \ldots, s_i$, where $l = l(z)$, $s_i = s_i(z)$, the points x^{ij} are such that $x^{ij} \in p_1^{-1}(y^i)$ and $f(x^{ij}) = \operatorname{minmax}(f)(z)$. We denote by $r_i(z)$ the rank of the restriction of the composition of the embedding $X(s_i) \hookrightarrow X^{s_i}$ and f^{s_i}: $X^{s_i} \to \mathbf{R}^{s_i}$ to the inverse image of the fibre $p_2^{-1}(z)$ under the map $(s_i) \to Y$ at the multipoint $(x^{i1}, \ldots, x^{is_i})$. Let $X(s_1, \ldots, s_l)$ be the inverse image of the diagonal in the composite map $X(s_1) \times \ldots \times X(s_l) \to Y^l \to Z^l$. We denote by $r(z)$ the rank of the composition of the embedding $X(s_1, \ldots, s_l) \hookrightarrow X^s$ and the map f^s: $X^s \to \mathbf{R}^s$, where $s = s_1 + \ldots + s_l$, at the multipoint $(x^{11}, \ldots, x^{ls_l})$.

Lemma. $r(z) = s(z)$ *for a family* f *in general position and for every* $z \in Z$, *except for a set* $Z(f)$ *of dimension zero. If* $z \in Z(f)$, *then* $r(z) = s(z) - 1$, $r_i(z) = \min(s_i(z) - 1, k - m)$, *and the points* x^{ij} $(j = 1, \ldots, s_i(z))$ *are Morse points of the function* $f \mid p_1^{-1}(y^i)$, $(i = 1, \ldots, l(z))$.

Lemma. *For a family* f *in general position the germ* $[\operatorname{minmax}(f)]_z$ *in the case* $z \in Z \setminus Z(f)$ *is topologically equivalent to the germ of a linear function* $\mathbf{R}^m \to \mathbf{R}$, *and in the case* $z \in Z(f)$ *to the direct sum of the germ* $\min(\alpha_1, \ldots, \alpha_l) \bigoplus_{i=1}^{l} \max(\beta_1^i, \ldots, \beta_{q_i}^i)$, *where* $l = l(z)$, $\alpha_j \in \mathfrak{m}(l-1)$, $\beta_j^i \in \mathfrak{m}(s_i - r_i - 1)$, *and the germ of a non-degenerate quadratic form from* $\mathfrak{m}(m + r_1 + \ldots + r_l - s + 1)$. *Here the sets* $\{\alpha_j\}$ *and* $\{\beta_1^i, \ldots, \beta_{q_i}^i\}$ *satisfy the conditions of the last lemma in* 3 *above.*

5. In the proof of the last lemma we use Theorem 5.

***Theorem* 5.** *Let* g: $\mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^r$ *be a semi-quasihomogeneous function of degree* d *with indices* $(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)$ *and quasihomogeneous part* φ, *and let* $0 \in \mathbf{R}^p$ *be a point of a local minimum of the functions* $g \mid \mathbf{R}^p \times \{0\}$ *and* $\varphi \mid \mathbf{R}^p \times \{0\}$. *Then for small* $\varepsilon > 0$ *the germ* $\hat{g}$: $(\mathbf{R}^q, 0) \to (\mathbf{R}, 0)$, *where* $\hat{g}(y) = \min\{g(x, y) \mid \|x\| \leqslant \varepsilon\}$, *is equal to* $\hat{\varphi} + \psi$, *where* $\hat{\varphi}(y) = \min\{\varphi(x, y) \mid \|x\| \leqslant \varepsilon\}$ *is a quasihomogeneous germ of degree* d *with indices* $(\beta_1, \ldots, \beta_q)$, *while on compact sets* $\psi(t^{\beta_1} y_1, \ldots, t^{\beta_q} y_q)/t^q \rightrightarrows 0$ *as* $t \to +0$.

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