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On the structure of two-dimensional local skew fields

A. B. Zheglov

Abstract. The concept of n-dimensional local skew field is a direct generalization of the concept of n-dimensional local field. We study 2-dimensional local skew fields and solve the classification problem for the those of characteristic 0 whose last residue field is contained in the centre, and suggest a condition under which there is a section of the residue map whose first residue skew field is commutative. Under this condition we solve the classification problem for all 2-dimensional local skew fields.

For skew fields of characteristic 0 whose last residue field is contained in the centre, we state a criterion for two elements to be conjugate.

The concept of n-dimensional local field is well known in algebraic geometry and algebraic number theory and has many applications (see [3], [4] and [5]). In [6] the study of n-dimensional local skew fields, that is, n-dimensional local fields without the commutativity condition, was recommended, and the problem was posed of classifying them and describing their conjugacy classes.

In this paper we consider these problems for two-dimensional local skew fields whose residue skew field is commutative. For such skew fields we give a condition (Theorem 1) under which the residue skew field can be embedded in it. The skew fields for which this condition holds are classified with respect to the isomorphisms preserving their local structure. We give an example of a skew field whose residue skew field cannot be embedded in it, and classify the two-dimensional skew fields of characteristic 0 whose last residue field is contained in the centre and the residue field can be embedded in the skew field.

This paper has the following structure. In the introduction we make definitions and prove some general theorems for arbitrary two-dimensional local skew fields, § 2 deals with the problem of embeddability of the residue field in the skew field and the classification of the skew fields for which this condition holds. In § 3 we classify the skew fields of characteristic 0 whose residue fields can be embedded in them and whose last residue fields are contained in their centres. The conjugacy theorems are proved in § 4. The main results of the first three sections are stated in Theorem 5 in § 3.

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§ 1. Introduction

Definition 1. Let K and k be arbitrary skew fields. We say that K *is an n-dimensional local skew field with last residue skew field k* if K has the following structure: either $n = 0$ (and $K = k$) or K has a discrete valuation $\nu\colon K^* \cup \{0\} \to \mathbb{Z}\cup\{\infty\}$, where $\nu\colon K^* \to \mathbb{Z}$ is a surjective homomorphism, $\nu(0) = \infty$ and $\nu(a+b) \geqslant \inf(\nu(a), \nu(b))$, its valuation ring $\mathcal{O} := \{x \in K\colon \nu(x) \geqslant 0\}$ is complete and separable in the topology defined by ν, and its residue skew field is an $(n-1)$-dimensional local skew field with last residue skew field k.

The following statements are well known in the valuation theory of division algebras (see, for example, [1]).

Lemma 1. *Let K be an arbitrary skew field with a discrete valuation. Then*

(i) *its valuation ring $\mathcal{O}$ is a topological group* (*with the topology defined by ν*) *and a metric space*,

(ii) *$\mathcal{O}$ is a local ring and a principal ideal domain.*

For every two-dimensional local skew field we have

$$K \supset \mathcal{O} \to \overline{K} \supset \overline{\mathcal{O}} \to k,$$

where $\overline{\mathcal{O}}$ is the valuation ring of $\overline{K}$. We have the filtrations

$$K \supset \mathcal{O} \supset \wp \supset \wp^2 \supset \cdots, \qquad \overline{K} \supset \overline{\mathcal{O}} \supset \overline{\wp} \supset \overline{\wp}^2 \supset \cdots,$$

where $\overline{\wp}$ is the maximal ideal in $\overline{\mathcal{O}}$. Let $\overline{\nu}$ be the discrete valuation of $\overline{K}$.

Definition 2. An *isomorphism of local skew fields K and K'* is defined to be a ring isomorphism of K and K' that preserves the above filtrations.

This is equivalent to the preservation of ν and $\overline{\nu}$.

Definition 3. A two-dimensional local skew field is said to be *splittable* if there is a homomorphism $\overline{K} \hookrightarrow \mathcal{O} \subset K$ that is a section of the map $\mathcal{O} \to \mathcal{O}/\wp = \overline{K}$.

By *local parameters* (or *variables*) *of* K we mean $z\in\mathcal{O}, \nu(z) = 1$, and $u\in\overline{\mathcal{O}} \subset \overline{K}$, $\overline{\nu}(u) = 1$,

Proposition 1. *Let K be splittable. Let z and u be some fixed parameters. Then K is isomorphic to $\overline{K}((z))$, which is defined to be the vector space of series with multiplication defined by the formula*

$$za = a^{\alpha}z + a^{\delta_1}z^2 + a^{\delta_2}z^3 + \cdots,$$

where $a \in \overline{K}$, α is an automorphism, and $\delta_i\colon \overline{K} \to \overline{K}$ are linear maps.

Proof. We first prove that any splittable skew field can be represented as $\overline{K}((z))$. Let $a \in K$ and $\nu(a) = j$. Then $\nu(az^{-j}) = 0$ and $\overline{az^{-j}} := az^{-j} \bmod \wp \in \overline{K}$. We can assume that this element of $\overline{K}$ belongs to $\mathcal{O}$, since there is a section. Then $\nu(az^{-j} - \overline{az^{-j}}) \geqslant 1$. Proceeding in this way, we establish that a can be written as $\sum_{i=j}^{\infty} a_i z^i$, $a_i \in \overline{K}$. The multiplication formula is obtained likewise: we put $a^{\alpha} = zaz^{-1} \bmod \wp$, where $a \in \overline{K}$. It is clear that α is an automorphism.

Since $\nu(zaz^{-1}) = 0$, zaz^{-1} can be written as $\sum_{i=0}^{\infty} a_i z^i$, where $a_i \in \overline{K}$. We have established that $a_0 = a^{\alpha}$. For the other coefficients we use the notation $a^{\delta_i} := a_i$, $i \geqslant 1$. It is easy to verify that the maps δ_i thus defined are linear.

Every δ_i satisfies a certain identity, for which we shall use the following notation.

Consider the ring of non-commutative polynomials in two variables $\mathbb{Z}\langle\alpha,\delta\rangle$. We define a map

$$\sigma\colon \mathbb{Z}\langle\alpha,\delta\rangle \to \mathbb{Z}\langle\alpha,\delta,\delta_i; i \geqslant 1\rangle$$

with values in the ring of non-commutative polynomials in α, δ, δ_i, $i \geqslant 1$, to be the map that replaces expressions of the type $\delta^i\alpha$ by δ_i in every word, beginning with the right-hand end, that is,

$$\sigma(\alpha^{a_1}\delta^{b_1}\dots\alpha^{a_n}\delta^{b_n}) = \alpha^{a_1}\delta_{b_1}\dots\delta_{b_{n-1}}\alpha^{a_n-1}\delta^{b_n},$$

where $a_1, b_n \geqslant 0$, $a_i, b_j \geqslant 1$, $i > 1$ and $j < n$ are positive integers.

We are interested in the values of this map on polynomials whose monomials end in α. In this case the values of σ belong to the ring $\mathbb{Z}\langle\alpha,\delta_i; i \geqslant 1\rangle$.

For example,

$$\sigma(\alpha^k) = \alpha^k, \qquad \sigma(\alpha^k\delta^l\alpha^i) = \alpha^k\delta_l\alpha^{i-1},$$

where k, l and i are positive integers, $i, l \geqslant 1$.

The polynomials S_i^k in $\mathbb{Z}\langle\alpha,\delta\rangle$ are defined to be the sums of the monomials belonging to the orbit of the monomial $\underbrace{\alpha\dots\alpha}_{i-k}\underbrace{\delta\dots\delta}_{k}$ under the action of the permutation group S_i:

$$S_i^k = \sum_{\tau\in S_i/G} \tau(\underbrace{\alpha\dots\alpha}_{i-k}\underbrace{\delta\dots\delta}_{k}),$$

where G is the stationary subgroup.

Lemma 2. *The polynomials S_i^k satisfy the recurrence relation*

$$S_{i+1}^{k+1} = \alpha S_i^{k+1} + \delta S_i^k,$$

with the initial conditions $S_i^i = \delta^i$, $S_i^0 = \alpha^i$, where $k, i \in \mathbb{Z}$, $i \geqslant k$, $i \geqslant 1$.

This lemma follows immediately from the definition.

We can now write down the identities for δ_i.

Proposition 2. *Let $i \geqslant 1$. Then*

$$\delta_i(ab) = \sum_{k=0}^{i} \sigma(\delta^{i-k}\alpha)(a)\sigma(S_i^k\alpha)(b), \qquad a, b \in \overline{K}.$$

Proof. Let $a, b \in \overline{K}$. We have

$$(ab)^{\alpha}z + (ab)^{\delta_1}z^2 + \dots = z(ab) = (a^{\alpha}z + a^{\delta_1}z^2 + \dots)b. \qquad (*)$$

Writing the right-hand side as a series with powers of z on the right and comparing the coefficients of powers of z on the left- and right-hand sides, we obtain general formulae for $\delta_i(ab)$. We claim that these formulae are those in the statement of the proposition.

Consider the formula for $\delta_i(ab)$ with an arbitrary $i \geqslant 1$ that follows from $(*)$. This formula gives the coefficient of z^{i+1}. Let x_k be the coefficients of z^{i+1} in the series for $z^{i+1-k}b$. Then the coefficient of z^{i+1} is equal to

$$\alpha(a)x_i + \sum_{k=0}^{i-1} \delta_k(a)x_k = \sum_{k=0}^{i} \sigma(\delta^{i-k}\alpha)(a)x_k.$$

Note that x_k are sums of monomials

$$\alpha^{a_1}\delta_{b_1}\dots\alpha^{a_n}\delta_{b_n}\alpha^{a_{n+1}}(b), \qquad a_k, b_k \in \mathbb{Z}, \quad a_k, b_k \geqslant 0$$

($\delta_0 = 1$). It is obvious that the coefficients of all different monomials are positive integers.

We claim that $x_k = \sigma(S_i^k\alpha)(b)$.

This will be proved if we show that all the summands in $\sigma(S_i^k\alpha)(b)$ occur in x_k, and x_k and $\sigma(S_i^k\alpha)(b)$ have the same number of summands. (By the number of summands we mean the sum of the coefficients of the monomials.)

Indeed, the monomials in $\sigma(S_i^k\alpha)(b)$ are different by definition and their coefficients are equal to 1. It is easy to see that the number of summands there is equal to $\binom{i}{k} = i!/(i-k)!k!$. On the other hand, the coefficients of different monomials in x_k are positive integers. Our assertion will be proved if we show that every monomial in $\sigma(S_i^k\alpha)(b)$ occurs as a summand in x_k and the number of summands in x_k is equal to $\binom{i}{k}$.

We first prove that every monomial in $\sigma(S_i^k\alpha)(b)$ occurs in x_k. By definition, every monomial in $\sigma(S_i^k\alpha)(b)$ has the form $\sigma(\tau(\underbrace{\alpha\dots\alpha}_{i-k}\underbrace{\delta\dots\delta}_{k})\alpha)(b)$, where $\tau \in S_i$. Therefore, it can be written as $\alpha^{a_1}\delta_{b_1}\dots\alpha^{a_n}\delta_{b_n}\alpha^{a_{n+1}}(b)$, where $a_j \geqslant 0$, $b_j \geqslant 1$, $\sum_{j=1}^{n} b_j = k$, and $\sum_{j=1}^{n+1} a_j = i-k+1-n$. If this monomial occurs in x_k, then it is equal to the coefficient of z^{i+1} in the series for $z^{i+1-k}b$. To show this, we write this series as

$$z^{i+1-k}b = z^{i+1-k-a_{n+1}}\alpha^{a_{n+1}}(b)z^{a_{n+1}} + \text{other terms}.$$

The explicitly written monomial can be written as

$$\begin{aligned} &z^{i+1-k-a_{n+1}}\alpha^{a_{n+1}}(b)z^{a_{n+1}} = \\ &\quad = z^{i+1-k-a_{n+1}-1}[\alpha^{a_{n+1}+1}(b)z + \dots + \delta_{b_n}\alpha^{a_{n+1}}(b)z^{b_n+1} + \dots]z^{a_{n+1}} \\ &\quad = z^{i+1-k-a_{n+1}-1}\alpha^{a_{n+1}+1}(b)z^{a_{n+1}+1} \\ &\qquad + z^{i+1-k-a_{n+1}-1}\delta_{b_n}\alpha^{a_{n+1}}(b)z^{b_n+1+a_{n+1}} + \dots . \end{aligned}$$

We write $z^{i+1-k-a_{n+1}-1}b_1$ in a similar way, where the role of b_1 is played by $\delta_{b_n}\alpha^{a_{n+1}}(b)$. This yields

$$z^{i+1-k-a_{n+1}-1-a_n-1}b_2 z^{b_n+1+a_{n+1}+a_n+b_{n-1}+1},$$

where $b_2 = \delta_{b_{n-1}} \alpha^{a_n} \delta_{b_n} \alpha^{a_{n+1}}(b)$. Proceeding by induction, we obtain the equality

$$\begin{aligned} &z^{i+1-k-\sum a_j - n} \alpha^{a_1} \delta_{b_1} \dots \alpha^{a_n} \delta_{b_n} \alpha^{a_{n+1}}(b) z^{\sum b_j + n + \sum a_j} \\ &\quad = \alpha^{a_1} \delta_{b_1} \dots \alpha^{a_n} \delta_{b_n} \alpha^{a_{n+1}}(b) z^{i+1}, \end{aligned}$$

which provides us with the desired monomial in x_k.

Hence, any summand in $\sigma(S_i^k \alpha)(b)$ occurs in x_k. We claim that there are precisely $\binom{i}{k}$ summands in x_k.

Let s_n^l be the number of summands in the coefficient of z^l in the series for $z^n a z^{-n}$, $a \in \overline{K}$. Then the number of summands in x_k is equal to s_{i+1-k}^k. The recurrence relation

$$s_n^d = \sum_{l=0}^{d} s_{n-1}^l$$

can be proved by induction as follows. If $n = 1$, then $s_1^d = 1$ for all $d \geqslant 0$. On the other hand, we have $s_0^l = 0$, $l > 0$, and $s_0^0 = 1$.

If n is any integer greater than 1, then

$$z^{n-1} a z^{-n+1} = y_0 + y_1 z + \cdots,$$

where $y_0 \in \overline{K}$. We have

$$z^n a z^{-n} = z y_0 z^{-1} + z y_1 z^{-1} z + \cdots = [y_0^\alpha + y_0^{\delta_1} z + \cdots] + [y_1^\alpha + y_1^{\delta_1} z + \cdots] z + \cdots.$$

Therefore, the coefficient of z^d is equal to

$$\sum_{j=1}^{d} \delta_j(y_{d-j}) + \alpha(y_d).$$

Since y_j contains precisely s_{n-1}^j summands, we have

$$s_n^d = \sum_{j=0}^{d} s_{n-1}^j.$$

Finally, we claim that $s_{i+1-k}^k = \binom{i}{k}$ if $k < i+1$. We prove this by induction. Let $i = 0$. Then $s_1^0 = 1 = \binom{i}{0}$. Assume that the formulae hold for $i-1$. Let $k < i+1$. Then

$$\begin{aligned} s_{i+1-k}^k &= \sum_{l=0}^{k} s_{i-k}^l = \binom{i}{k} + \binom{i-1}{k-1} + \cdots + \binom{i-k}{0} \\ &= \Bigl(\dots\Bigl(\Bigl(\Bigl(\binom{i-k}{0} + \binom{i-k+1}{1}\Bigr) + \binom{i-k+2}{2}\Bigr) + \binom{i-k+3}{3}\Bigr) + \cdots + \binom{i}{k}\Bigr) \\ &= \Bigl(\dots\Bigl(\Bigl(\Bigl(\binom{i-k+2}{1}\Bigr) + \binom{i-k+2}{2}\Bigr) + \binom{i-k+3}{3}\Bigr) + \cdots + \binom{i}{k}\Bigr) = \binom{i+1}{k}, \end{aligned}$$

which completes the proof of the proposition, since x_k contains precisely s_{i+1-k}^k summands.

Corollary 1. *Let* $\alpha = \mathrm{Id}$. *Then*

$$\delta_i(ab) = \delta_i(a)b + \sum_{k=1}^{i} \delta_{i-k}(a) \sum_{(j_1,\dots,j_l)} \binom{i-k+1l}{\delta}_{j_1} \dots \delta_{j_l}(b),$$

where $\delta_0 = \alpha$, $0 < l \leqslant \min\{i-k+1, k\}$, $j_m \geqslant 1$, $\sum j_m = k$, *and* $(j_1, \dots, j_l)$ *belongs to the orbit under the action of* S_l *of an integer vector whose coordinate sum is equal to* k.

The following definition will be used frequently below.

Definition 4. Let (α, β) be endomorphisms of a skew field L. A map $\delta\colon L \to L'$, where $L \subset L'$ are skew fields, is called an (α,β)-*derivation* if it is linear and the identity

$$\delta(ab) = \delta(a)b^{\alpha} + a^{\beta}\delta(b),$$

holds for $a, b \in L$. An $(\alpha, 1)$-derivation is called an α-*derivation*.

For example, δ_1 is an (α^2, α)-derivation.

If $\alpha = \mathrm{Id}$, then δ_1 is a derivation in the conventional sense. We have $\delta_2 = \delta_1^2 + \delta$, where δ is a derivation. The last assertion can be generalized as follows.

Corollary 2. *If* $\delta_1 = \dots = \delta_{k-1} = 0$, *then* δ_k *is an* (α^{k+1}, α)-*derivation*.

The following corollary will be used in § 3.

Corollary 3. *Let* $\overline{K}$ *be a field,* $\overline{K} = k((u))$, $k \subset Z(K)$, *and assume that the maps* δ_i, $i \geqslant 1$, *are continuous if* $\operatorname{char} k = 0$. *Then*

$$\delta_i\biggl(\sum_{j=N}^{\infty} x_j u^j\biggr) = \sum_{j=N}^{\infty} x_j \delta_i(u^j), \qquad x_j \in k.$$

So δ_i *is determined for any* i *by the values* $\delta_i(u)$ *and* $\delta_j(u)$ *with* $j < i$.

Proof. In the case when $\operatorname{char} k = p \neq 0$ the maps δ_i, $i \geqslant 1$, are continuous, since $\delta_i(a^{p^i}) = 0$ for any $a \in \overline{K}$. The topology on one-dimensional local fields of the same characteristic is defined unambiguously by the local structure. Therefore, the continuity of the above maps does not depend on the choice of parametrization. (For the relation between the local structure, parametrization and topology of local fields of any dimension, see [8].)

We claim that α is continuous. Since we are dealing with a one-dimensional local field, it is sufficient to show that α preserves the valuation. This will be proved (for an arbitrary characteristic) if we prove that $\overline{\nu}(\alpha(u')) = 1$ for any u', $\overline{\nu}(u') = 1$. Consider the automorphism α' defined by the formula

$$\alpha'(a) := \overline{z^{-1}az},$$

where $a \in \overline{K}$. (The notation is explained in Proposition 1.) It is obvious that $\alpha' = \alpha^{-1}$.

Let u' be an arbitrary parameter. Put $\kappa = \overline{\nu}(\alpha(u'))$. We claim that $|\kappa| \leqslant 1$ or $|\kappa| = p^q$, $q \in \mathbb{N}$. Assume the contrary. Then $\kappa = mp^q$, $(m,p) = 1$, $|m| \neq 1$. Hence, there are $c \in k$ and $a \in \overline{K}$ such that $\alpha(u') = ca^m$. Therefore,

$$u' = \alpha^{-1}(\alpha(u')) = c(\alpha^{-1}(a))^m,$$

whence

$$\overline{\nu}(u') = 1 = \overline{\nu}(c(\alpha^{-1}(a))^m) = m\overline{\nu}(\alpha^{-1}(a)).$$

We have arrived at a contradiction.

We claim that $\kappa \geqslant 0$. Assume the contrary: let $\kappa < 0$. Consider the element $u' + u'^2$ (or $u' + u'^3$, if $\operatorname{char} k = 2$). We have $\overline{\nu}(\alpha(u' + u'^2)) = 2\kappa < -1$. If $\operatorname{char} k \neq 2$, then this contradicts the assertion proved above that $|\overline{\nu}(\alpha(u'))| = p^q$ or $|\overline{\nu}(\alpha(u'))| \leqslant 1$ for any u'. If $\operatorname{char} k = 2$, then the same argument applies to $u' + u'^3$.

We prove likewise that $0 \leqslant \kappa' \leqslant 1$ or $\kappa' = p^l$ for $\kappa' := \overline{\nu}(\alpha^{-1}(u'))$.

To prove the inequality $\kappa \neq p^q$ by contradiction, we consider two cases.

(i) Let $\kappa' \leqslant 1$. There are $r \in k$ and $a_1 \in k((u))$ such that $\alpha(u') = c_2 u'^2 a_1^{p^q - 2}$. We have

$$1 = 2\overline{\nu}(\alpha^{-1}(u')) + (p^q - 2)\overline{\nu}(\alpha^{-1}(a_1)),$$

whence $(p^q - 2) \mid 1$. Therefore, $p = 3$, $q = 1$. Applying the same argument to $\alpha(u') = c_3 u'^5 a_1^{-2}$, we obtain that $\overline{\nu}(\alpha^{-1}(a_1)) = 2$, which contradicts the property of κ'.

(ii) Let $\kappa' = p^l$. We write $\alpha(u')$ as $\alpha(u') = cu'a^{p^q - 1}$ with some $c \in k$, $a \in k((u))$ and $\overline{\nu}(a) = 1$. We have

$$\overline{\nu}(u') = 1 = \overline{\nu}(\alpha^{-1}(u')) + (p^q - 1)\overline{\nu}.$$

We have a contradiction, since $\overline{\nu}(\alpha^{-1}(a)) \geqslant 0$ by the property of κ.

Hence, $\kappa = 0$ or $\kappa = 1$, and we have proved that $\overline{\nu}(\alpha(u')) = 0$ or $\overline{\nu}(\alpha(u')) = 1$ for any u'. Let $\kappa = 0$. Consider the element $x = u' + c_1 u'^2 + c_2(u'^3 + c_1 u'^4)$, where $c_1 = -w_0^{-1}$ if $\alpha(u') = w_0 + \cdots$, and c_2 is an element of k such that $\overline{\nu}(\alpha(x)) > 1$. (There is such an element, since $\overline{\nu}(\alpha(u' + c_1 u'^2)) > 0$.) We have a contradiction. Hence, $\kappa = 1$, which proves that α is a continuous automorphism.

Since $k((u))$ is a complete and separable topological space, it is sufficient to show that the series $\sum_{j=N}^{\infty} x_j \delta_i(u^j)$ converges, which can be done by induction on i. If $i = 0$, then $\overline{\nu}(\alpha(u^j)) = j$, and the series converges. If $i = 1$, then $\overline{\nu}(\delta_1(u^j)) = (j-1)\overline{\nu}(\delta_1(u))$, and the series converges since $k((u))$ is complete.

Finally, Proposition 2 implies that

$$\delta_i(u^j) = \delta_i(u^{j-1})y_0 + \sum_{k=0}^{i-1} \delta_k(u^{j-1})y_{i-k}$$

for $j > 1$, where $\overline{\nu}(y_k)$ does not depend on j. By the induction hypothesis,

$$\min\{\overline{\nu}(\delta_0(u^{j-1})y_i), \dots, \overline{\nu}(\delta_{i-1}(u^{j-1})y_1)\} > \min\{\overline{\nu}(\delta_0(u^{j-2})y_i), \dots, \overline{\nu}(\delta_{i-1}(u^{j-2})y_1)\}$$

and $\overline{\nu}(y_0) = 1$, whence

$$\min\{\overline{\nu}(\delta_i(u^{j-1})y_0), \overline{\nu}(\delta_0(u^{j-1})y_i), \dots, \overline{\nu}(\delta_{i-1}(u^{j-1})y_1)\} > \min\{\overline{\nu}(\delta_i(u^{j-2})y_0), \overline{\nu}(\delta_0(u^{j-2})y_i), \dots, \overline{\nu}(\delta_{i-1}(u^{j-2})y_1)\}.$$

Hence, the series converges.

§ 2. Splittable skew fields

For the rest of this paper we assume, unless otherwise stated, that $\overline{K}$ is a field.

For such local skew fields the canonical automorphism α of the residue skew field $\overline{K}$ can be defined in a natural way.

By the definition of local skew fields, we have two exact sequences

$$1 \to \mathcal{O}^* \to K^* \xrightarrow{\nu} \mathbb{Z} \to 1,$$

where $\mathcal{O}$ is the valuation ring, and

$$1 \to 1+\wp \to \mathcal{O}^* \to \overline{K}^* \to 1,$$

where $\wp$ is the maximal ideal.

Consider the map

$$\phi\colon K^* \to \operatorname{Int}(K), \qquad \phi(x)=\operatorname{ad}(x), \qquad \operatorname{ad}(x)(y)=x^{-1}yx,$$

where $\operatorname{Int}(K)$ is the group of interior automorphisms of K. Since the interior automorphisms preserve the valuations of elements, they can be restricted to $\mathcal{O}$. Moreover, they map $\wp$ into itself, which gives a map $\phi\colon K^* \to \operatorname{Aut}(\mathcal{O}/\wp)=\operatorname{Aut}(\overline{K})$. We claim that $\phi(\mathcal{O}^*)$ is the trivial automorphism of $\overline{K}$. Using the second exact sequence above, we obtain that the action of $\phi(1+\wp)$ on $\overline{K}$ is trivial, since $(1+\wp)^{-1}x(1+\wp)=x \bmod \wp$ for any $x\in\mathcal{O}$. We define an action of $\overline{K}$ on $\overline{K}$ by the formula $a^{-1}xa \bmod \wp$ (where a, x are representatives in $\mathcal{O}$), which can be written as $\bar{a}^{-1}\bar{x}\bar{a}=\bar{x}$ (the bar denotes the image in the residue skew field), since the residue skew field is assumed to be a field. Hence, this action of $\overline{K}$ on $\overline{K}$ is trivial.

Definition 5. The *canonical automorphism* α *of* $\overline{K}$ is defined by the formula $\alpha=\phi(z)$, where $z\in K^*$ and $\nu(z)=1$. It does not depend on the choice of z.

We now describe, up to isomorphism, the splittable two-dimensional local skew fields *with isomorphic last residue fields*. Let K and K' be two splittable skew fields, $K\cong\overline{K}((z))$ and $K'\cong\overline{K}'((z'))$. If $K\cong K'$, then the isomorphism $\varphi\colon K\to K'$ can be written as the composite of $\phi\colon K\to K'$, $\phi(u)=u'$, $\phi(z)=z'$, and some automorphism ψ of K. Any automorphism is defined by some change of variables

$$\begin{aligned} u &\mapsto u'=c_0+c_1z+c_2z^2+\cdots, && \overline{\nu}(c_0)=1,\\ z &\mapsto z'=a_0z+a_1z^2+\cdots, && a_0\neq 0, \end{aligned} \tag{$*$}$$

where $a_i, c_i\in\overline{K}$. (Let us recall that the local structure of skew fields is invariant under isomorphisms.)

It is obvious that every change of variables looks like this. It is also clear that any change can be decomposed into a sequence of changes $u\mapsto u'$, $z\mapsto z$; $u'\mapsto u'$, $z\mapsto z'=a_0'z+a_1'z^2+\cdots$ (or in the reverse order). The change $u\mapsto u'$ can be decomposed into a sequence of changes $u\mapsto u_1'=c_0$, $u_1'\mapsto u_2'=u_1'+c_1'z,\dots,u_i'\mapsto u_{i+1}'=u_i'+c_i'z^i,\dots$, and the change $z\mapsto z'$ can be decomposed into a sequence of changes $z\mapsto z_1'=a_0z$, $z_1'\mapsto z_2'=z_1'+a_1'z^2,\dots,z_i'\mapsto z_{i+1}'=z_i'+a_i'z^{i+1},\dots$.

Remark 1. It should be noted that any general change of variables $(*)$ defines a map $f\colon K\to K$. If the skew field is non-commutative, this map can fail to be an automorphism. Assume the contrary and consider the change $f(z)=z'$, $f(u)=u$, where z' is a new variable. Then

$$f(zu)=f(z)f(u)=z'u=u^{\alpha'}z'+u^{\delta_1'}z'^2+\cdots,$$
$$f(zu)=f(u^{\alpha}z+u^{\delta_1}z^2+\cdots)=u^{\alpha}z'+u^{\delta_1}z'^2+\cdots.$$

Hence, $\alpha=\alpha'$, $\delta_1=\delta_1'$, and so on, that is, $\delta_i'=\delta_i$ for all i.

Consider the skew field $\mathbb{Q}((u))((z))$ with the relation $zu=(u+u^2)z$, and consider the change $z\mapsto z'=z+z^2$. We have

$$\begin{aligned} z'u &= (z+z^2)u=(u+u^2)z+z(u+u^2)z\\ &=(u+u^2)z+[(u+u^2)z+(u+u^2)^2z]z\\ &=(u+u^2)z'+[u+2u^2+2u^3+u^4-u-u^2]z^2\\ &=(u+u^2)z'+[u^2+2u^3+u^4]z'^2+\cdots. \end{aligned}$$

Therefore, $\delta_1\neq\delta_1'$. This is a contradiction.

Proposition 3. *For any two-dimensional splittable local skew field K with $\alpha^n\neq\mathrm{Id}$ for all n there is a change of variables such that $z'a=a^{\alpha}z'$ for every a in $\overline{K}$.*

Proof. We shall prove this proposition by induction, making successive changes of variables in such a way that δ_1 and δ_2 vanish successively.

Lemma 3 (the Change Lemma). *Assume that the relation*

$$zaz^{-1}=a^{\alpha}+a^{\delta_j}z^j+a^{\delta_{j+1}}z^{j+1}+\cdots$$

holds in K with $j\in\mathbb{N}$ such that $\delta_1=\cdots=\delta_{j-1}=0$ and $\delta_j\neq 0$. Then

(i) *the relation*

$$z'az'^{-1}=a^{\alpha}+\cdots+a^{\delta_{q-1}}z'^{q-1}+(a^{\delta_q}+ba^{\alpha^{q+1}}-a^{\alpha}b)z'^q+\cdots$$

holds for the change $z\mapsto z'=z+bz^{q+1}$, $q\in\mathbb{N}$, that is, $a^{\delta_q'}=a^{\delta_q}+ba^{\alpha^{q+1}}-a^{\alpha}b$,

(ii) *if $\alpha^n=1$ for some n, then the relation*

$$\begin{aligned} z'az'^{-1}=a^{\alpha}+\cdots+a^{\delta_{q+j-1}}z'^{q+j-1}+\Bigg(&a^{\delta_{q+j}}+b(a^{\delta_j})^{\alpha^q}-a^{\delta_j}b^{\alpha^j}\\ &+b\sum_{k=1}^{q}((a^{\alpha^k})^{\delta_j})^{\alpha^{q-k}}-a^{\delta_j}\sum_{k=0}^{j-1}b^{\alpha^k}\Bigg)z'^{q+j}+\cdots \end{aligned}$$

holds for the change $z\mapsto z'=z+bz^{q+1}$, where $n\mid q$,

(iii) *the relation*

$$z'az'^{-1}=a^{\alpha}+a^{\delta_j}(b^{-1})^{\alpha}\dots(b^{-1})^{\alpha^j}z'^j+\cdots$$

holds for the change $z\mapsto z'=bz$, $b\neq 0$.

Corollary 4. *If* $\alpha = \mathrm{Id}$, *then*

$$z'az'^{-1} = a + \cdots + a^{\delta_{q+j-1}} z'^{q+j-1} + (a^{\delta_{q+j}} + (q-j)a^{\delta_j} b) z'^{q+j} + \cdots .$$

Proof of the lemma. (i) We have

$$\begin{aligned} z'az'^{-1} &= (1+bz^q)zaz^{-1}(1+bz^q)^{-1} \\ &= (zaz^{-1} + bz^q zaz^{-1})(1 - bz^q + bz^q bz^q - \cdots) \\ &= (zaz^{-1} - zaz^{-1}bz^q + \cdots + bz^q zaz^{-1} - \cdots) \\ &= (zaz^{-1} - [a^{\alpha} + a^{\delta_j} z^j + \cdots]bz^q + bz^q[a^{\alpha} + a^{\delta_j} z^j + \cdots] + \cdots) \\ &= (zaz^{-1} - [a^{\alpha} b + a^{\delta_j} b^{\alpha^j} z^j + \cdots]z^q + ba^{\alpha^{q+1}} z^q + \cdots) \\ &= (zaz^{-1} + (-a^{\alpha} b + ba^{\alpha^{q+1}})z^q + \cdots) \\ &= a^{\alpha} + \cdots + a^{\delta_{q-1}} z'^{q-1} + (a^{\delta_q} + ba^{\alpha^{q+1}} - a^{\alpha} b) z'^q + \cdots . \end{aligned}$$

(ii) We have

$$\begin{aligned} z'az'^{-1} &= (1+bz^q)zaz^{-1}(1+bz^q)^{-1} = (zaz^{-1} + bz^q zaz^{-1})(1+bz^q)^{-1} \\ &= (a^{\alpha} + a^{\delta_j} z^j + \cdots + a^{\delta_{q+j}} z^{q+j} + \cdots + bz^q(a^{\alpha} + a^{\delta_j} z^j + \cdots))(1+bz^q)^{-1} \\ &= \Bigg(a^{\alpha} + ba^{\alpha^{q+1}} z^q + a^{\delta_j} z^j + \cdots + a^{\delta_{q+j}} z^{q+j} + \cdots + b\sum_{k=1}^{q}((a^{\alpha^k})^{\delta_j})^{\alpha^{q-k}} z^{q+j} \\ &\quad + b(a^{\delta_j})^{\alpha^q} z^{q+j} + \cdots\Bigg)(1+bz^q)^{-1} \\ &= a^{\alpha} + \Bigg(a^{\delta_j} z^j + \cdots + a^{\delta_{q+j}} z^{q+j} + \cdots + b\sum_{k=1}^{q}((a^{\alpha^k})^{\delta_j})^{\alpha^{q-k}} z^{q+j} \\ &\quad + b(a^{\delta_j})^{\alpha^q} z^{q+j} + \cdots\Bigg)(1 - bz^q + bz^q bz^q - \cdots) \\ &= a^{\alpha} + a^{\delta_j} z^j + \cdots + a^{\delta_{q+j}} z^{q+j} + \cdots + b\sum_{k=1}^{q}((a^{\alpha^k})^{\delta_j})^{\alpha^{q-k}} z^{q+j} \\ &\quad + b(a^{\delta_j})^{\alpha^q} z^{q+j} + \cdots - a^{\delta_j} b^{\alpha^j} z^{q+j} + \cdots \\ &= a^{\alpha} + \cdots + a^{\delta_{q+j-1}} z'^{q+j-1} + \Bigg(a^{\delta_{q+j}} + b(a^{\delta_j})^{\alpha^q} - a^{\delta_j} b^{\alpha^j} \\ &\quad + b\sum_{k=1}^{q}((a^{\alpha^k})^{\delta_j})^{\alpha^{q-k}} - a^{\delta_j}\sum_{k=0}^{j-1} b^{\alpha^k}\Bigg) z'^{q+j}, \end{aligned}$$

since $z'^j = z^j + \sum_{k=0}^{j-1} b^{\alpha^k} z^{q+j} + \cdots$.

(iii) We have

$$\begin{aligned} z'az'^{-1} = bzaz^{-1}b^{-1} &= a^{\alpha} + ba^{\delta_j}(b^{-1})^{\alpha^j} z^j + \cdots \\ &= a^{\alpha} + a^{\delta_j}(b^{-1})^{\alpha} \ldots (b^{-1})^{\alpha^j} z'^j + \cdots . \end{aligned}$$

Lemma 4. *Every* (α,β)*-derivation* δ *of the field* $\overline{K}$, $\alpha\neq\beta$, *is interior, that is, there is a* $d\in\overline{K}$ *such that*

$$\delta(a)=da^{\alpha}-a^{\beta}d \qquad \forall\, a\in\overline{K}.$$

Proof. Let $d=\delta(a)/(a^{\alpha}-a^{\beta})$, where a is such that $a^{\alpha}\neq a^{\beta}$. (There is such an element, since $\alpha\neq\beta$.) Put $\delta_{\mathrm{int}}(x)=dx^{\alpha}-x^{\beta}d$. We claim that $\delta=\delta_{\mathrm{int}}$. Consider the (α,β)-derivation $\bar\delta=\delta-\delta_{\mathrm{int}}$. Let $b\in\overline{K}$. Then $\bar\delta(ab)=\bar\delta(ba)$. We have

$$\bar\delta(ab)=\bar\delta(a)b^{\alpha}+a^{\beta}\bar\delta(b)=a^{\beta}\bar\delta(b)$$

and

$$\bar\delta(ba)=\bar\delta(b)a^{\alpha}+b^{\beta}\bar\delta(a)=a^{\alpha}\bar\delta(b).$$

Therefore, $\bar\delta(b)=0$ for any b.

Proof of the proposition. Let

$$zaz^{-1}=a^{\alpha}+a^{\delta_1}z+a^{\delta_2}z^2+\cdots.$$

Proposition 2 and its corollary imply that δ_1 is an (α^2,α)-derivation. Since $\alpha^2\neq\alpha$, Lemma 4 implies that δ_1 is an interior derivation and $\delta_1(a)=da^{\alpha^2}-a^{\alpha}d$. By Lemma 3(i),

$$z_2az_2^{-1}=a^{\alpha}+a^{\delta_2'}z_2^2+\cdots$$

$(\delta_1'=0)$ for the change $z\mapsto z_2=z-d_1z^2$. By the corollary to Proposition 2, δ_2' is an (α^3,α)-derivation. Since $\alpha^3\neq\alpha$, this derivation is interior, which enables us to apply Lemma 3. In this case the change is given by the formula $z_2\mapsto z_3=z_2-d_2z_2^3$.

At the kth step of induction we have

$$z_kaz_k^{-1}=a^{\alpha}+a^{\delta_k'}z_k^k+\cdots$$

$(\delta_j'=0$ if $j<k)$. By the corollary to Proposition 2, δ_k' is an (α^{k+1},α)-derivation. Since $\alpha^{k+1}\neq\alpha$, this derivation is interior, which enables us to apply Lemma 3 again. Now the change is given by the formula $z_k\mapsto z_{k+1}=z_k-d_kz_k^{k+1}$.

So we have proved that the changes $\{z_n\}$: $z_{n+1}=z_n-d_nz_n^{n+1}$, annihilate the δ_i. It is obvious that the sequence $\{z_n\}$ converges in K. Since the topological space corresponding to the first valuation ν is complete and separable, this sequence has precisely one limit z, which is the desired element for which $\delta_i=0$, $i\geqslant 1$. The proposition is proved.

Theorem 1. *Let* K *be a two-dimensional local skew field whose first residue skew field is commutative, and assume that* $\alpha^n\neq 1$ *for all* $n\in\mathbb{N}$*. Then*

(i) $\operatorname{char} K=\operatorname{char}\overline{K}$,

(ii) *there is a homomorphism* $\overline{K}\hookrightarrow\mathcal{O}\subset K$ *that is a section of* $\mathcal{O}\mapsto\overline{K}$.

Proof. We shall give the proof in several steps.

Step 0. Let us prove that $\operatorname{char} K=\operatorname{char}\overline{K}$.

Assume the contrary. Then $\operatorname{char}\overline{K}=l>0$. Therefore, $\nu(l)=r>0$. For any $t\in K$ such that $\nu(t)=0$ we have $ltl^{-1} \bmod \wp=\bar t^{\alpha^r}$, where $\bar t$ is the image of t in $\overline{K}$. (Since $l=z^ru$, where u is the identity in $\mathcal{O}$, the last assertion can be proved in the same way as we proved that α does not depend on the choice of the local parameter z.) On the other hand, $lt=tl$, whence $ltl^{-1} \bmod \wp=\bar t$, which contradicts the inequality $\alpha^r\neq\mathrm{Id}$.

Step 1. It is clear the prime field can be embedded in $\mathcal{O}$.

Lemma 5. *There is a c in $\overline{K}$ such that $c^{\alpha^k} \neq c$ for all $k \in \mathbb{N}$.*

Proof. Since $\overline{K}$ is a one-dimensional local field, it has a filtration defined by the valuation $\overline{\nu}$. We claim that there is a sequence $\{c_{j_i}\}$, $j_i, i \in \mathbb{N}$, in $\overline{\mathcal{O}}$ such that

(i) $\overline{\nu}(c_{j_i}) > \overline{\nu}(c_{j_{i-1}})\ \ \forall\, i$,

(ii) if $k \in \mathbb{N}$: $k = 0 \bmod j_2 \dots j_l$ and $k \neq 0 \bmod j_2 \dots j_{l+1}$, then $\alpha^k(c_{j_1}) = c_{j_1}, \dots, \alpha^k(c_{j_{l-1}}) = c_{j_{l-1}}$, $\alpha^k(c_{j_l}) \neq c_{j_l}$ and

$$\overline{\nu}[(\alpha^k - \mathrm{Id})(c_{j_l})] < \overline{\nu}(c_{j_{l+1}}).$$

We construct this sequence by induction in the following way. Let c_{j_1} be such that $\alpha(c_{j_1}) \neq c_{j_1}$ and $\overline{\nu}(c_{j_1}) \geqslant 1$. (Let u be such that $\overline{\nu}(u) = 1$. If $\alpha(u) \neq u$, then $c_{j_1} := u$. If $\alpha(u) = u$, then we take any c_{j_1} such that $\alpha(c_{j_1}) \neq c_{j_1}$.) If $\overline{\nu}(c_{j_1}) = 0$, we put $\tilde{c}_{j_1} := c_{j_1} u$. Then $\overline{\nu}(\tilde{c}_{j_1}) = 1$ and $\alpha(c_{j_1} u) = \alpha(c_{j_1}) u \neq c_{j_1} u$. We put $j_1 := 1$.

Let j_2 be the least number such that $(\alpha^{j_1})^{j_2}(c_{j_1}) = c_{j_1}$, and let $k_1 = \max\{\overline{\nu}[(\alpha^{j_1})^m(c_{j_1}) - c_{j_1}],\ m \in \{1, \dots, j_2 - 1\}\}$. Let $\tilde{c}_{j_2}$ be such that $(\alpha^{j_1})^{j_2}(\tilde{c}_{j_2}) \neq \tilde{c}_{j_2}$. We put $c_{j_2} = \tilde{c}_{j_2} c_{j_1}^{k_1+1}$. Then $(\alpha^{j_1})^{j_2}(c_{j_2}) \neq c_{j_2}$ and $\overline{\nu}[(\alpha^{j_1})^m(c_{j_1}) - c_{j_1}] < \overline{\nu}(c_{j_2})$ for all $m < j_2$.

Proceeding by induction, we obtain a sequence for which (i) and (ii) hold.

Let $c = \sum_{i=1}^{\infty} c_{j_i}$. We claim that $\alpha^k(c) \neq c$ for all k. Let $k = 0 \bmod j_2 \dots j_l$ and $k \neq 0 \bmod j_2 \dots j_{l+1}$. By (ii),

$$\alpha^k(c) - c = \alpha^k(c_{j_l}) - c_{j_l} + \alpha^k\left(\sum_{i=l+1}^{\infty} c_{j_l}\right) - \sum_{i=l+1}^{\infty} c_{j_l}.$$

Since $\overline{\nu}(\alpha^k(c_{j_l}) - c_{j_l}) < \overline{\nu}(c_{j_{l+1}}) \leqslant \overline{\nu}\big(\alpha^k\big(\sum_{i=l+1}^{\infty} c_{j_l}\big) - \sum_{i=l+1}^{\infty} c_{j_l}\big)$, we have $\alpha^k(c) - c \neq 0$.

Let Π be a prime field. Let $\overline{F} = \Pi(c) \subset K$. We claim that this field can be embedded in $\mathcal{O}$.

Let $c' \in \mathcal{O}$ be any lift of c: $c' \bmod \wp = c$. It is obvious that c' commutes with any element of the prime field and that c is transcendental over the prime field. Otherwise its equation modulo $\wp$ would have infinitely many solutions (since $c^{\alpha^k} \neq c$ for all k), which is impossible. Therefore, $\Pi[c'] \cap \wp = 0$. Hence, the residue field $\overline{F}$ of this ring can be embedded in $\mathcal{O}$.

Let $\bar{L}$ be some maximal extension of $\overline{F}$ that can be embedded in $\mathcal{O}$, let $L \subset \mathcal{O}$ be its image under this embedding, and let $\bar{a} \in \overline{K}$, $\bar{a} \notin \bar{L}$. We have to prove that there is an $a \in \mathcal{O}$ such that $a \bmod \wp = \bar{a}$ and a commutes with any element of L.

Step 2. To prove that there is such an a, we consider an a such that $a \bmod \wp = \bar{a}$. Let x be an arbitrary element of L. We have $axa^{-1} \bmod \wp = x$, since $\overline{K}$ is a field. Let z be an element of K such that $\nu(z) = 1$. We can write axa^{-1} as

$$axa^{-1} = x + x^{\delta_1'} z,$$

where $x^{\delta_1'}$ is some element of $\mathcal{O}$. It is easy to verify that the map $\bar{\delta}_1'$ assigning to x the element $\overline{x^{\delta_1'}} \in \overline{K}$ is an α-derivation on L with values in $\overline{K}$:

$$\begin{aligned} a(x_1 + x_2)a^{-1} &= (x_1 + x_2) + (x_1 + x_2)^{\delta_1'} z, \\ a(x_1 + x_2)a^{-1} &= ax_1a^{-1} + ax_2a^{-1} = x_1 + x_1^{\delta_1'} z + x_2 + x_2^{\delta_1'} z \\ &= (x_1 + x_2) + (x_1^{\delta_1'} + x_2^{\delta_1'}) z. \end{aligned}$$

Hence, $\overline{(x_1+x_2)^{\delta_1'}} = \overline{x_1^{\delta_1'}} + \overline{x_2^{\delta_1'}}$. Further, we have

$$a(x_1x_2)a^{-1} = (ax_1a^{-1})(ax_2a^{-1}).$$

Therefore,

$$\begin{aligned} x_1x_2 + (x_1x_2)^{\delta_1'}z &= (x_1 + x_1^{\delta_1'}z)(x_2 + x_2^{\delta_1'}z) \\ &= x_1x_2 + x_1x_2^{\delta_1'}z + x_1^{\delta_1'}zx_2 + x_1^{\delta_1'}zx_2^{\delta_1'}z \\ &\equiv x_1x_2 + x_1x_2^{\delta_1'}z + x_1^{\delta_1'}x_2^{\alpha}z \bmod \wp^2 \\ &= x_1x_2 + (x_1^{\delta_1'}x_2^{\alpha} + x_1x_2^{\delta_1'})z \bmod \wp^2. \end{aligned}$$

Hence,

$$\overline{(x_1x_2)^{\delta_1'}} = \overline{x_1^{\delta_1'}}\,\overline{x_2^{\alpha}} + \overline{x_1}\,\overline{x_2^{\delta_1'}} = \overline{x_1^{\delta_1'}}x_2^{\alpha} + x_1\overline{x_2^{\delta_1'}}.$$

By Lemma 4, $\bar{\delta}_1'$ is an interior α-derivation. We put $\tilde{a}_1 := (1+a_1z)a$, where $a_1 \bmod \wp = -d$ if $\bar{\delta}_1'(x) = d(x^{\alpha}-x)$. It follows from the proof of the Change Lemma that

$$(1+a_1z)axa^{-1}(1+a_1z)^{-1} = x + (x^{\delta_1'} + a_1x^{\alpha} - xa_1)z + \text{terms belonging to } \wp^2.$$

Since $x^{\delta_1'} + a_1x^{\alpha} - xa_1 = 0 \bmod \wp$, we have $\tilde{a}_1x\tilde{a}_1^{-1} = x + x^{\delta_2'}z^2$. As before, we verify that $\bar{\delta}_2'\colon L \to \overline{K}$ is an α^2-derivation. Further, we obtain by induction (see the proof of Proposition 3) that there is an $\tilde{a}_i = (1+a_iz^i)\dots(1+a_1z)a$ such that

$$\tilde{a}_ix\tilde{a}_i^{-1} = x + x^{\delta_{i+1}'}z^{i+1},$$

and $\bar{\delta}_{i+1}'\colon L \to \overline{K}$ is an α^{i+1}-derivation. Applying Lemma 4, we obtain that $\bar{\delta}_{i+1}'$ is an interior derivation. Consider $\tilde{a}_{i+1} = (1+a_{i+1}z^{i+1})\tilde{a}_i$ with a suitable a_{i+1}. The calculation used in the proof of the Change Lemma completes the inductive step:

$$\tilde{a}_{i+1}x\tilde{a}_{i+1}^{-1} = x + x^{\delta_{i+2}'}z^{i+2} \qquad \forall\, x \in L.$$

It is obvious that $\tilde{a}_i$ converges in K. Since $\tilde{a}_i \bmod \wp = a$, its limit is the desired element.

Step 3. Assume, in the situation of Step 1, that $\bar{a}$ is transcendental over $\overline{K}$. Then it has a lift $a \in \mathcal{O}$ commuting with any element of L. In this case $L[a] \cap \wp = 0$, and the field of fractions $L(a)$ can be embedded in $\mathcal{O}$. So we have found a field that contains L. Hence, we can assume that $\overline{K}$ is algebraic over L.

Now let $\bar{a}$ be algebraic and separable over $\bar{L}$ and let a be a lift of $\bar{a}$ commuting with every element of L. By Hensel's lemma (see below), there is a lift $a' \in \mathcal{O}$ such that a' commutes with every element of L and a' is algebraic over L. Hence, $\bar{L}(a')$ can be embedded in $\mathcal{O}$.

Finally, let $\bar{a}$ be purely inseparable over $\bar{L}$. Let a be a lift of $\bar{a}$ commuting with every element of L. Then $a^{l^k} \bmod \wp = x \in L$, where $l = \operatorname{char} K$. Therefore, $a^{l^k} - x$ commutes with every element of L. If it is different from zero, then we put $\nu(a^{l^k} - x) = r$. The argument used in Step 0 shows that $(a^{l^k}-x)c(a^{l^k}-x)^{-1} \bmod \wp = c^{\alpha^r} \neq c$. This contradiction implies that $a^{l^k} - x = 0$. Therefore, $\bar{L}(a)$ can be embedded in $\mathcal{O}$, which completes the proof of the theorem.

Proposition 4 (Hensel's lemma[1]). *Let $\mathcal{O}$ be the complete valuation ring in K, let I be the valuation ideal, $\bigcap_{n=1}^{\infty} I^n = 0$, and let F be a subfield contained in $\mathcal{O}$. Let $A \in \mathcal{O}$ be such that $Al = lA$ for all $l \in F$. Let $f(X) \in F[X]$, $f'(A) \notin I$ and $f(A) \in I$. Then there is an $\hat{A} \in \mathcal{O}$ such that*

(a) *$\hat{A}$ commutes with A,*

(b) *$\hat{A} - A \in I$,*

(c) *$f(\hat{A}) = 0$,*

(d) *$\hat{A}l = l\hat{A}$ for all $l \in F$.*

Proof. If $\tilde{A}$ commutes with A, then

$$f(A + \tilde{A}) = f(A) + f'(A)\tilde{A} + P\tilde{A}^2$$

for some $P \in F[A, \tilde{A}]$. (Here we have used Taylor's formula.) Put $\tilde{A} = -(f'(A))^{-1} f(A)$. It is clear that $\tilde{A} \in I$ and $\tilde{A}$ commutes with A. Moreover, $\tilde{A}$ commutes with every element of F. Therefore, $f(A + \tilde{A}) = P\tilde{A}^2 \in I^2$ and $f'(A + \tilde{A}) = f'(A) + X\tilde{A} \notin I$, where $X \in F[A, \tilde{A}]$. In the same way we find an $\tilde{A}_2 = -(f'(A + \tilde{A}))^{-1} f(A + \tilde{A}) \in I^2$ commuting with A, $\tilde{A}$ and any element of F and such that

$$f(A + \tilde{A} + \tilde{A}_2) \in I^4.$$

Proceeding in this way, we construct the desired element $\hat{A} = A + \tilde{A} + \tilde{A}_2 + \cdots$ (this series converges since $\mathcal{O}$ is complete).

Remark 2. If $\alpha^n = \mathrm{Id}$, then the assertion of the theorem fails (see the Example in § 3).

Corollary 5. *Proposition* 3 *holds for every two-dimensional local skew field.*

Theorem 2. *Let K and K' be two-dimensional local skew fields such that $\alpha^n \neq \mathrm{Id}$, $\alpha'^n \neq \mathrm{Id}$ for all n, and the residue skew field $\overline{K}$ is commutative. Then*

(i) *K is isomorphic to $\overline{K}((z))$, $za = a^{\alpha} z$, $a \in \overline{K}$, where $\overline{K}$ is a one-dimensional local field with residue field k,*

(ii) *K is isomorphic to K' if and only if $k \cong k'$ and there is an isomorphism $f \colon \overline{K} \mapsto \overline{K}'$ such that $\alpha = f^{-1}\alpha' f$.*

The proof follows from Corollary 5 combined with the well-known classification of one-dimensional local fields with a given residue field (see, for example, [2]).

Definition 6. Let $\overline{K}$ be a one-dimensional local field with residue field k, let $\operatorname{char} \overline{K} = \operatorname{char} k$, and let α be an automorphism of $\overline{K}$ preserving the natural filtration of $\overline{K}$. We put $a_1 = \alpha(u)u^{-1} \bmod \wp \in k$ and define $i_\alpha \in \mathbb{N} \cup \infty$ as follows. We put $i_\alpha = 1$ *if a_1 is not a root of* 1 *in k. Otherwise* $i_\alpha = \overline{\nu}((\alpha^n - \mathrm{Id})(u))$, where $n \geqslant 1$: $a_1^n = 1$, $a_1^m \neq 1$ for all $m < n$.

[1]The idea of the proof of this lemma was suggested by N. I. Dubrovin.

Lemma 6. *Let k be an arbitrary field of characteristic* 0*. Every k-automorphism α of $k((u))$ with $\alpha(u) = \xi u + a_2u^2 + \cdots$, where $\xi^n = 1$ for $n \geqslant 1$ and $\xi^m \neq 1$ for $m < n$, is conjugate to the automorphism β defined by the formula $\beta(u) = \xi u + xu^{i_\alpha} + yu^{2i_\alpha - 1}$, where $x \in k^*$, $y \in k$, and x and y depend on α. Moreover, $i_\alpha = i_\beta$.*

Proof. We first show that α is conjugate to the automorphism β' defined by the formula $\beta'(u) = \xi u + xu^{in+1} + yu^{2in+1}$, where i is some positive integer, and then that $in + 1 = i_\alpha$.

Let us find an automorphism f: $f(u) = u + x_2u^2 + \cdots$ such that $\alpha f = f\beta'$:

$$\begin{aligned}\alpha f(u) &= (\xi u + a_2u^2 + \cdots) + x_2(\xi u + a_2u^2 + \cdots)^2 + \cdots = f\beta'(u)\\ &= \xi(u + x_2u^2 + \cdots) + x(u + x_2u^2 + \cdots)^{in+1} + y(u + x_2u^2 + \cdots)^{2in+1}.\end{aligned}$$

Let $i = 1$. The equations for the coefficients x_i of the powers of u can be written as follows.

The first equation (for u^2) is $x_2\xi^2 + a_2 = \xi x_2$.

The $(n-1)$th equation (for u^n) is $x_n\xi^n + \text{terms with } x_j,\ j < n + a_n = \xi x_n$.

Each of these equations has precisely one solution, since $\xi^j \neq \xi$ for all $j < n+1$.

The nth equation is $x_{n+1}\xi^{n+1} + \text{terms with } x_j,\ j < n + 1 + a_{n+1} = \xi x_{n+1} + x$.

Since x_{n+1} cancels here, we obtain an equation for x that has precisely one solution. If $x = 0$, then we put $i := i + 1$ and write the next equation. At some step we obtain that $x \neq 0$. In this way we determine the x_l: $n \nmid l - 1$. It is obvious that every other, say kth, equation (for x_k) has precisely one solution if $n \nmid k$.

Hence, $\alpha = f^{-1}\alpha' f$, where $\alpha'(u) = \xi u + xu^{in+1} + \cdots$, and $f^{-1} = \prod_{k=1, n\mid k}^{in} f_k$, where $f_k(u) = u + x_{k+1}u^{k+1}$ and the x_k are defined unambiguously. Therefore, we can replace α by α' and assume that $a_q = 0$ if $q < in+1$ (and $x_q = 0$ if $q < in+1$, $n \nmid q$). In a similar way we can write any equation for x_q, $q < in+1$, $n \mid q$, as

$$a_{q+in} + qxx_q = (in+1)xx_q.$$

(We assume here that x_k, $k < q$, are defined and $a_k = 0$, $k < q + in$.) This equation has precisely one solution x_q.

The equation for u^{2in+1} has the form

$$a_{2in+1} + (in+1)xx_{in+1} = (in+1)xx_{in+1} + y,$$

which enables us to find y. Since the x_q are defined unambiguously, x and y are invariant in the conjugacy class $f^{-1}\alpha f$, where $f(u) = u + \cdots$. The other equations (corresponding to the powers of u that are integer multiples of n) have the form

$$a_{q+in} + qxx_q = (in+1)xx_q + \text{terms with } x_k, \qquad k < q.$$

Since $q > in+1$, these equations can be solved for x_q.

We claim that $in + 1 = i_\alpha$. A simple induction yields

$$\beta'^n(u) = u + x\xi^{-in-1}\left(\sum_{j=0}^{n-1}(\xi^{in})^j\right)u^{in+1} + \cdots = u + x\xi^{-in-1}nu^{in+1} + \cdots.$$

Hence, $i_{\beta'} = in+1$. Since $f^{-1}\alpha f = \beta'$, we have $f^{-1}(\alpha^n - \mathrm{Id})f = \beta'^n - \mathrm{Id}$, whence $\overline{\nu}(f^{-1}(\alpha^n - \mathrm{Id})f(u)) = \overline{\nu}((\beta'^n - \mathrm{Id})(u)) = i_{\beta'}$.

Let $f(u) = u' = f_1 u + f_2 u^2 + \cdots$, $f_1 \neq 0$. We claim that $\overline{\nu} f^{-1}(\alpha^n - \mathrm{Id})(u') = i_\alpha$. It is sufficient to show that $\overline{\nu}(\alpha^n - \mathrm{Id})(u') = i_\alpha$. We have

$$\begin{aligned}(\alpha^n - \mathrm{Id})(u') &= [f_1(u + \bar{a}_{i_\alpha} u^{i_\alpha} + \cdots) + f_2(u + \bar{a}_{i_\alpha} u^{i_\alpha} + \cdots)^2 + \cdots] \\ &\qquad - [f_1 u + f_2 u^2 + \cdots] \\ &= [(f_1 u + f_1 \bar{a}_{i_\alpha} u^{i_\alpha} + \text{terms with } u^{>i_\alpha}) + (f_2 u^2 + \text{terms with } u^{>i_\alpha}) \\ &\qquad + (f_3 u^3 + \text{terms with } u^{>i_\alpha}) + \cdots] - [f_1 u + f_2 u^2 + \cdots] \\ &= f_1 \bar{a}_{i_\alpha} u^{i_\alpha} + \text{terms with } u^{>i_\alpha},\end{aligned}$$

which completes the proof of the lemma.

Proposition 5. *Let $\overline{K}$ be a one-dimensional local field with residue field k, assume that $\operatorname{char} \overline{K} = \operatorname{char} k$, k is algebraically closed and $\operatorname{char} k = 0$, and let α and β be automorphisms of $\overline{K}$. Then $\overline{K} = k((u))$ and $\alpha = f^{-1}\beta f$ (where f is an automorphism of $\overline{K}$) if and only if $(a_1, i_\alpha, y(\alpha)) = (b_1, i_\beta, y(\beta))$.*

Proof. We have only to prove the second assertion, since the first is obvious.

If $\alpha = f^{-1}\beta f$, then it is obvious that $a_1 = b_1$.

Assume that ξ in the hypotheses of Lemma 6 is not a root of 1. Since, in this case, each of the above equations has precisely one solution when $x = y = 0$, it is obvious that α is conjugate to β: $\beta(u) = \xi u$. This completes the proof of the proposition in the case when $i_\alpha = i_\beta = 1$.

Now assume that $i_\alpha = i_\beta \neq 1$, and let $a_1 = b_1$ be a root of 1.

Lemma 7. *In the notation of Lemma 6 let β and β' be k-automorphisms of $k((u))$: $\beta(u) = \xi u + x u^{in+1} + y u^{2in+1}$, $\beta'(u) = \xi u + \bar{x} u^{in+1} + \bar{y} u^{2in+1}$, where $\bar{x}/x \in (k^*)^{in}$ and $\bar{y} = (\bar{x}/x)^2 y$. Then β and β' are conjugate.*

Proof. Put $x_0 = (\bar{x}/x)^{(in)^{-1}}$. Let f be the automorphism defined by the formula $f(u) = x_0 u$. Then

$$f\beta(u) = \xi x_0 u + x(x_0 u)^{in+1} + y(x_0 u)^{2in+1} = x_0 \xi u + x_0 \bar{x} u^{in+1} + x_0 \bar{y} u^{2in+1} = \beta' f(u).$$

Combining Lemmas 6 and 7, we obtain that any two automorphisms α and β with equal triples $(a_1, i_\alpha, y(\alpha))$ and $(b_1, i_\beta, y(\beta))$ are conjugate when k is algebraically closed, and this completes the proof of the proposition.

Corollary 6. *Assume that the hypotheses of Proposition 5 hold, with the exception of the assumption that k be algebraically closed. Let $\alpha^n = \mathrm{Id}$. Then there is a parameter u' in $k((u))$ such that $\alpha(u') = a_1 u'$.*

The proof follows from Lemma 6.

Proposition 5 implies that the following theorem holds.

Theorem 3. *Let K and K' be two-dimensional local skew fields with last residue fields k and k' and canonical automorphisms α and α'. Assume that $\operatorname{char} K = \operatorname{char} k$, $\operatorname{char} K' = \operatorname{char} k'$, $\alpha^n \neq \mathrm{Id}$ and $\alpha'^n \neq \mathrm{Id}$ for all $n \in \mathbb{N}$, and that k and k' are algebraically closed fields of characteristic 0. Then K is isomorphic to K' if and only if $k \cong k'$ and $(a_1, i_\alpha, y(\alpha)) = (a_1', i_{\alpha'}, y(\alpha'))$.*

We shall now study the skew fields with $\alpha^n = 1$.

§ 3. Classification of splittable skew fields of characteristic 0

From now on we assume that $k \subset K$, $k \subset \overline{K}$, and that these embeddings are compatible with the local structure. We also assume that k is contained in the centre of K: $k \subset Z(K)$, the maps δ_i, $i \geqslant 1$, are continuous (see Corollary 3), and K is splittable.

These assumptions imply that $\operatorname{char} k = \operatorname{char} \overline{K} = \operatorname{char} K$ and $\overline{K} \simeq k((u))$. This follows from Cohn's theorem for one-dimensional local fields, since the residue field is embedded in $\overline{K}$. The Change Lemma and the proof of Corollary 3 imply that the continuity of δ_i, $i \geqslant 1$, does not depend on the choice of z (or u).

We first consider the case when $\alpha = \mathrm{Id}$.

Definition 7. We put

$$i = \nu((\phi_z - 1)(u)) \in \mathbb{N} \cup \infty,$$
$$r = \overline{\nu}[((\phi_z - 1)(u))z^{-i} \bmod \wp] \bmod i \in \mathbb{Z}/i\mathbb{Z},$$

where u and z are arbitrary local parameters of K and $\phi_z \colon K \to K$, $\phi_z(a) = ad(z)(a)$.

Proposition 6. *i and r do not depend on the choice of u or z.*

Proof. Since K is splittable, it can be represented as $K \cong k((u))((z))$. Let us fix this representation. Let u', z' be other parameters. Then

$$u' = (x_0 u + x_1 u^2 + \cdots) + c_1 z + c_2 z^2 + \cdots, \qquad x_i \in k, \quad c_i \in k((u)), \quad x_0 \neq 0,$$
$$z' = a_0 z + a_1 z^2 + \cdots, \qquad a_i \in k((u)), \quad a_0 \neq 0.$$

Put $z'' = a_0^{-1} z'$. It is clear that $\nu((\phi_{z''} - 1)(u)) = \nu((\phi_{z'} - 1)(u))$. On the other hand, Corollary 4 implies that $\nu((\phi_{z'} - 1)(u)) = \nu((\phi_z - 1)(u))$. Therefore, i does not depend on the choice of z.

We claim that $\nu((\phi_z - 1)(u')) = \nu((\phi_z - 1)(u))$. This is a consequence of the following lemma.

Lemma 8 (the Second Change Lemma). *Assume that the following relation holds in K:*

$$zuz^{-1} = u^{\alpha} + u^{\delta_j} z^j + \cdots,$$

where j is such that $\delta_1 = \cdots = \delta_{j-1} = 0$, $\delta_j \neq 0$ and $\alpha^n = 1$.

(i) *If $k < q$, then the relation*

$$zu'z^{-1} = u'^{\alpha} + u'^{\delta_1} z + \cdots + u'^{\delta_{q-1}} z^{q-1} + u'^{\delta_q} z^q + \cdots,$$

where $u'^{\delta_q} = u^{\delta_q} + b^{\alpha} - \partial/\partial u(u^{\alpha}) b$, $u'^{\delta_k} = u^{\delta_k}$, holds for the change $u \mapsto u' = u + bz^q$, $n \mid q$.

(ii) *If $\alpha(u) = \xi u$, $\xi \in k$, and $\xi^n = 1$ for some n, then the relation*

$$zu'z^{-1} = \xi u' + \cdots + (u^{\delta_q} + b^{\alpha} - \xi b) z^q + \cdots + u'^{\delta_{q+j-1}} z^{q+j-1} + u'^{\delta_{q+j}} z^{q+j} + \cdots,$$

where $u'^{\delta_{q+j}} = u^{\delta_{q+j}} + b^{\delta_j} - \partial/\partial u(u^{\delta_j})b$, *holds for the change* $u \mapsto u' = u + bz^q$, $n \mid q$.

(iii) *If* $\alpha = 1$, *then the relation*

$$zu'z^{-1} = u' + \left(u^{\delta_j}\frac{\partial}{\partial u}u'\right)z^j + \cdots$$

holds for the change $u \mapsto u' = x_0u + x_1u^2 + \cdots$, *where* $x_q \in k$ *and* $x_0 \neq 0$.

Proof. (i) We have

$$\begin{aligned} zu'z^{-1} &= z(u+bz^q)z^{-1} = u^\alpha + u^{\delta_1}z + \cdots + (b^\alpha + b^{\delta_1}z + \cdots)z^q \\ &= u^\alpha + u^{\delta_1}z + \cdots + (u^{\delta_q} + b^\alpha)z^q + \cdots \\ &= u'^\alpha + u'^{\delta_1} + \cdots + (u^{\delta_q} + b^\alpha - \partial/\partial u(u^\alpha)b)z^q + \cdots, \end{aligned}$$

since

$$u'^{\delta_1} = (u+bz^q)^{\delta_1} = x_0(u+bz^q) + x_1(u+bz^q)^2 + \cdots = u^{\delta_1} + \frac{\partial}{\partial u}(u^{\delta_1})bz^q + \cdots$$

if $u^{\delta_1} = x_0u + x_1u^2 + \cdots$.

(ii) We have

$$\begin{aligned} zu'z^{-1} &= z(u+bz^q)z^{-1} = \xi u + u^{\delta_j}z^j + \cdots + (b^\alpha + b^{\delta_j}z^j + \cdots)z^q \\ &= \xi u + u^{\delta_j}z^j + \cdots + (u^{\delta_q} + b^\alpha)z^q + u^{\delta_{q+1}}z^{q+1} + \cdots + u^{\delta_{q+j-1}}z^{q+j-1} \\ &\quad + (u^{\delta_{q+j}} + b^{\delta_j})z^{q+j} + \cdots = \xi u' + \cdots + (u^{\delta_q} + b^\alpha - \xi b)z^q \\ &\quad + u'^{\delta_{q+1}}z^{q+1} + \cdots + u'^{\delta_{q+j-1}}z^{q+j-1} + \left(u^{\delta_{q+j}} + b^{\delta_j} - \frac{\partial}{\partial u}(u^{\delta_j})b\right)z^{q+j}. \end{aligned}$$

(iii) We have

$$zu'z^{-1} = x_0(u+u^{\delta_j}z^j+\cdots)+x_1(u+u^{\delta_j}z^j+\cdots)^2+\cdots = u' + \left(u^{\delta_j}\frac{\partial}{\partial u}u'\right)z^j + \cdots.$$

Hence, i does not depend on the choice of u or z.

We claim that the same is true for r. By Lemma 8, the replacement of u by u' yields

$$zu'z^{-1} = u' + \left(u^{\delta_i}\frac{\partial}{\partial u}u'\right)z^i + \cdots.$$

Therefore, $\overline{\nu}[((\phi_z - 1)(u'))z^{-i}] = \overline{\nu}(u^{\delta_i}) = \overline{\nu}[((\phi_z - 1)(u'))z^{-i}]$.

The replacement of z by z' yields

$$z'uz'^{-1} = zuz^{-1} \bmod \wp^i,$$

whence

$$\begin{aligned} &\overline{\nu}[((\phi_{z'} - 1)(u))z'^{-i} \bmod \wp] = \overline{\nu}[((\phi_z - 1)(u))z'^{-i} \bmod \wp] = \\ &\quad = \overline{\nu}[((\phi_z - 1)(u))z^{-i} \bmod \wp] + \overline{\nu}(a_0^{-i}) = \overline{\nu}[((\phi_z - 1)(u))z^{-i} \bmod \wp] \bmod i. \end{aligned}$$

Definition 8. We put

$$a = \operatorname{res}_u \left\{ \frac{u^{\delta_{2i} - \frac{i+1}{2}\delta_i^2}}{(u^{\delta_i})^2}\, du \right\} \in k.$$

Proposition 7. $a = a(u^{\delta_{i+1}}, \dots, u^{\delta_{2i-1}})$, *that is, a depends only on* $u^{\delta_{i+1}}, \dots, u^{\delta_{2i-1}}$.

Proof. It is sufficient to show that a does not depend on the changes of variables preserving $\delta_{i+1}, \dots, \delta_{2i-1}$. We can assume that $\delta_{i+1} = 0, \dots, \delta_{2i-1} = 0$, since there is a change that transforms $\delta_{i+1}, \dots, \delta_{2i-1}$ into zero. (It is sufficient to make successive changes of the form $z \mapsto z' = z + bz^{j+1}$.) By Corollary 4, for every specified j there is a b such that $\delta'_j(u) = 0$.

We first show that any change $u \mapsto u' = u + c_1 z + \dots + c_i z^i$ is equivalent to $z \mapsto z' = z + a_1 z^2 + \cdots$, $u \mapsto u' = u + c'_i z^i + \cdots$, that is, the δ_j are the same for these changes. We proceed by induction.

Since the above change can be decomposed into a sequence of changes $u \mapsto u_1 = u + c_i z^i$, $u_1 \mapsto u_2 = u_1 + c_{i-1} z^{i-1}, \dots, u_{i-1} \mapsto u_i = u_{i-1} + c_1 z$, it is sufficient to show that for any j we can write $u_j \mapsto u_{j+1} = u_j + c_{i-j} z^{i-j}$ in the desired form.

The assertion is trivial for the first change. Consider the jth change. By Lemma 8, δ_{2i-j} is the first map that varies under this change. By Lemma 3(ii), there is a change under which δ_{2i-j} varies in the same way. To use the induction hypothesis and complete the proof, we have to verify that under the composition of the change $u_j \mapsto u_{j+1}$ with its inverse $z \mapsto z' = z + a_{i-j} z^{i-j+1}$ the map δ_{2i} is varied by $u \mapsto u' = u + bz^i$, that is, this change yields the same δ'_{2i} as the δ''_{2i} obtained by the composition of the above changes. Lemma 8 implies that this is the case if and only if

$$\operatorname{res}_{u_{j+1}} \frac{(\delta''_{2i} - \delta_{2i})(u_{j+1})}{(u_{j+1}^{\delta_i})^2}\, du_{j+1} = 0.$$

We can write $u_j \mapsto u_{j+1}$ as $u \mapsto u_{j+1} = u + c_{i-j} z^{i-j} + \dots + c_i z^i$. This change can be decomposed into $u \mapsto u' = u + c_{i-j} z^{i-j}$ and $u' \mapsto u'' = u' + c_{i-j+1} z^{i-j+1} + \dots + c_i z^i$. The latter change does not influence δ_{2i-j}. It is sufficient to prove that the residue for the composition of $u \mapsto u' = u + c_{i-j} z^{i-j}$ and $z \mapsto z' = z + a_{i-j} z^{i-j+1}$ is equal to zero.

Lemma 8 enables us to calculate a_{i-j}:

$$a_{i-j} = \frac{\partial}{\partial u}((ju^{\delta_i})^{-1} c_{i-j}) u^{\delta_i}.$$

Now we observe that the residue is equal to zero if $\overline{\nu}(c_{i-j})$ has a sufficiently large positive value. This follows from the proofs of Lemmas 3 and 8, which show that δ_{2i} varies by a polynomial in c_{i-j}, a_{i-j} and u^{δ_i}.

Let r be a sufficiently large positive value. Let $c_{i-j} = \sum_{h=N}^{r} x_h u^h + \sum_{h=r+1}^{\infty} x_h u^h$. Then the change $u \mapsto u' = u + c_{i-j} z^{i-j}$ can be decomposed into a finite sequence of changes $u \mapsto u'_1 = u + x_N u^N z^{i-j}, \dots, u'_{r-N-1} \mapsto u'_{r-N} = u'_{r-N-1} + \sum_{h=r+1}^{\infty} x_h u^h z^{i-j}$. Our assertion will be proved if we prove it for each individual change. Since we have already done this for the last change, it remains to do it for any change $u \mapsto u + xu^h z^j$, $x \in k$.

Consider the composition of this change with the change $z \mapsto z - (j-i)^{-1} \times (h-r)xu^{h-1}z^{j+1}$. We shall show that the residue is equal to zero. For convenience, we put $-(j-i)^{-1}(h-r)xu^{h-1} = b$, $xu^h = b'$.

We claim that δ_{i+qj}, $q \in \mathbb{N}$, are the only maps that vary under the composition of these changes, and then $\delta_{i+qj}(u)$ have the form $\mathrm{const} \cdot u^{r+q(h-1)}$. Indeed, the change $u \mapsto u' = u + b'z^j$ yields

$$\begin{aligned} zu'z^{-1} &= u + u^{\delta_i} z^i + u^{\delta_{2i}} z^{2i} + \cdots + (b' + b'^{\delta_i} + b'^{\delta_{2i}} z^{2i} + \cdots) z^j \\ &= u' + u^{\delta_i} z^i + b'^{\delta_i} z^{i+j} + u^{\delta_{2i}} z^{2i} + \cdots \\ &= u' + u'^{\delta_i} z^i + \left(\frac{\partial}{\partial u}(b') u^{\delta_i} - \frac{\partial}{\partial u}(u^{\delta_i}) b' \right) z^{i+j} \\ &\quad - \frac{1}{2!} \frac{\partial^2}{\partial u^2}(u^{\delta_i}) b'^2 z^{i+2j} - \cdots - \frac{1}{(e-1)!} \frac{\partial^{e-1}}{\partial u^{e-1}}(u^{\delta_i}) b'^{e-1} z^{i+(e-1)j} \\ &\quad + \left(u^{\delta_{2i}} - \frac{1}{e!} \frac{\partial^e}{\partial u^e}(u^{\delta_i}) b'^e \right) z^{2i} + \cdots , \end{aligned}$$

where $ej = i$ if $j \mid i$. If $j \nmid i$, then $u^{\delta_{2i}}$ does not vary.

Therefore,

$$u'^{\delta_{i+j}} = \frac{\partial}{\partial u}(b') u^{\delta_i} - \frac{\partial}{\partial u}(u^{\delta_i}) b'$$

and $\overline{\nu}(u'^{\delta_{i+j}}) = r + (h-1)$.

We proceed as follows:

$$u'^{\delta_{i+2j}} = -\frac{\partial}{\partial u}(u'^{\delta_{i+j}}) b' - \frac{1}{2!} \frac{\partial^2}{\partial u^2}, (u^{\delta_i}) b'^2$$

and $\overline{\nu}(u'^{\delta_{i+2j}}) = r + 2(h-1)$,

$$u'^{\delta_{i+qj}} = -\frac{\partial}{\partial u}(u'^{\delta_{i+(q-1)j}}) b' - \frac{1}{2!} \frac{\partial^2}{\partial u^2}(u'^{\delta_{i+(q-2)j}}) b'^2 - \cdots - \frac{1}{q!} \frac{\partial^q}{\partial u^q}, (u'^{\delta_i}) b'^q$$

and $\overline{\nu}(u'^{\delta_{i+qj}}) = r + q(h-1)$.

The change $z \mapsto z' = z + bz^{j+1}$ yields

$$\begin{aligned} z'u = (z + bz^{j+1})u &= uz + u^{\delta_i} z^{i+1} + u^{\delta_{i+j}} z^{i+j+1} + \cdots \\ &\quad \cdots + u^{\delta_{2i}} z^{2i+1} + \cdots + buz^{j+1} + (j+1) bu^{\delta_i} z^{i+j+1} + \cdots \\ &\quad \cdots + (j+1) bu^{\delta_{2i-j+1}} + \text{terms with } z^{>2i+1} \\ &= uz' + u^{\delta_i} z'^{i+1} + u^{\delta'_{i+j}} z'^{i+j+1} + \cdots + u^{\delta'_{2i}} z'^{2i+1} + \cdots \\ &= u(z + bz^{j+1}) + u^{\delta_i} (z + bz^{j+1})^{i+1} + u^{\delta'_{i+j}} (z + bz^{j+1})^{i+j+1} + \cdots \\ &\quad \cdots + u^{\delta'_{2i}} (z + bz^{j+1})^{2i+1} + \cdots . \end{aligned}$$

Therefore,

$$u^{\delta'_{i+j}} = u^{\delta_{i+j}} + b(j-i) u^{\delta_i}$$

and $\overline{\nu}(u^{\delta'_{i+j}}) = r + (h-1)$,

$$u^{\delta'_{i+2j}} = u^{\delta_{i+2j}} - \binom{i+12}{b}^2 u^{\delta_i} - \binom{i+j+11}{b} u^{\delta'_{i+j}} + (j+1)bu^{\delta_{i+j}}$$

and $\overline{\nu}(u^{\delta'_{i+2j}}) = r + 2(h-1)$,

$$\begin{aligned} u^{\delta'_{i+qj}} = u^{\delta_{i+qj}} &- \binom{i+1q}{b}^q u^{\delta_i} - \binom{i+j+1q-1}{b}^{q-1} u^{\delta'_{i+j}} - \cdots - \binom{i+(q-1)j+11}{b} u^{\delta'_{i+(q-1)j}} \\ &+ (j+1)bu^{\delta_{i+(q-1)j}} \end{aligned}$$

and $\overline{\nu}(u^{\delta'_{i+qj}}) = r + q(h-1)$.

If $j \nmid i$, then $u^{\delta_{2i}}$ does not vary, which completes the proof in this case. If $j \mid i$ and $e(h-1) - r \neq -1$, then the residue is equal to zero, which completes the proof in this case (note that $e(h-1) - r \neq -1$ if $(r-1, i) = 1$). We omit the direct calculations that prove the statement in the case when $e(h-1) - r = -1$.

So we have shown that the change $u \mapsto u' = u + c_1 z + \cdots + c_i z^i$ is equivalent to the change $z \mapsto z' = z + a_1 z^2 + \cdots,\ \ u \mapsto u' = u + c'_i z^i + \cdots$. By Lemma 8, a is invariant under the change $u \mapsto u' = u + c'_i z^i + \cdots$. By Lemma 3, the change $z \mapsto z' = z + a_1 z^2 + \cdots$ preserves the $\delta_{i+1}, \ldots, \delta_{2i-1}$ only if $a_1 = a_2 = \cdots = a_{i-1} = 0$, in which case a is invariant under this change. Hence, a is invariant under any change $z \mapsto z' = z + a_1 z^2 + \cdots,\ u \mapsto u' = u + c_1 z + \cdots$ preserving the $\delta_{i+1}, \ldots, \delta_{2i-1}$.

It remains to show that a is invariant under the changes $u \mapsto u' = x_0 u + x_1 u^2 + \cdots,\ \ x_j \in k$, and $z \mapsto z' = a_0 z,\ \ a_0 \neq 0 \in k((u))$. This is obvious for the former change. For the latter we have

$$\begin{aligned} u^{\delta'_{2i}} &= a_0^{-2i}[u^{\delta_{2i}} + ia_0(a_0^{-1})^{\delta_i} u^{\delta_i} - a_0^{-i}(a_0^{i-1} a_0^{\delta_i} + \cdots + a_0(a_0^{i-1})^{\delta_i})] \\ &= a_0^{-2i}[u^{\delta_{2i}} + i(i+1)/2 a_0^{-1} a_0^{\delta_i} u^{\delta_i}], \\ u^{(\delta'_i)^2} &= a_0^{-2i} u^{\delta_i^2} - ia_0^{-2i-1} a_0^{\delta_i} u^{\delta_i}. \end{aligned}$$

Therefore,

$$\frac{u^{\delta'_{2i}} - (i+1)/2u^{(\delta'_i)^2}}{(u^{\delta'_i})^2} = \frac{u^{\delta_{2i}} - (i+1)/2u^{\delta_i^2}}{(u^{\delta_i})^2} = a,$$

which completes the proof of the proposition.

Remark 3. If a two-dimensional local skew field is not splittable, then the following example shows that i, r and a are not invariant (see also Remark 2).

Example. [2] Consider the free associative algebra $\mathbb{Q}((u))\langle x_1, x_2\rangle$ and the ideal I generated by $[[x_1, x_2], x_1]$ and $[[x_1, x_2], x_2]$. It is easy to verify that the factor ring $S = \mathbb{Q}((u))\langle x_1, x_2\rangle / I$ is a $\mathbb{Q}$-algebra without divisors of zero in which $z = [x_1, x_2] + I$ is central and algebraically independent of $u_i = x_i + I \quad (i = 1, 2)$. Every element of this ring has the form

$$f_0 + f_1 z + f_2 z^2 + \cdots + f_m z^m,$$

[2] This example was kindly communicated to me by N. I. Dubrovin.

where $f_0, \dots, f_m$ are polynomials in u_1 and u_2 written in canonical form:

$$a + bu_1 + cu_2 + d_1u_1^2 + d_2u_1u_2 + d_3u_2^2 + \cdots ,$$

S is an Ore domain (see [7]), and the quotient skew field K has a discrete valuation such that $\nu(u_i) = 0$, $\nu(\mathbb{Q}) = 0$ and $\nu(z) = 1$. Completing with respect to this valuation, we obtain a two-dimensional local skew field. (For the second valuation we can take the valuation with respect to u). This skew field is not splittable.

Lemma 9. *Assume that the valuation ring of a two-dimensional local skew field K contains u_1 and u_2 such that $z = u_1u_2 - u_2u_1$ is the uniformization of the valuation ν, and the commutators $[u_i, m] = u_im - mu_i$ $(i = 1, 2)$ belong to $z^2\mathcal{O}$ for any $m \in z\mathcal{O} \setminus z^2\mathcal{O}$. Then the residue field cannot be embedded in K.*

Proof. Assume the contrary. Let $\pi\colon \overline{K} \mapsto K$ be an embedding. Consider $f \in \pi(u_1)$ and $g \in \pi(u_2)$. We have $m_1 = f - u_1$, $m_2 = g - u_2 \in z\mathcal{O}$, and

$$\begin{aligned} 0 &= [u_1 + m_1, u_2 + m_2] = [u_1, u_2] + [m_1, u_2] + [u_1, m_2] + [m_1, m_2] \\ &= z + [m_1, u_2] + [u_1, m_2] + [m_1, m_2]. \end{aligned}$$

The second and third summands in the last sum belong to $z^2\mathcal{O}$, and $[m_1, m_2] \in z^2\mathcal{O}$, since $m_1m_2, m_2m_1 \in z^2\mathcal{O}$. Therefore,

$$\infty = \nu(0) = \nu(z + [m_1, u_2] + [u_1, m_2] + [m_1, m_2]) = \nu(z) = 1.$$

This contradiction completes the proof.

In this skew field z is a central element, whence $i = \infty$, and r, a are not defined. However, the change $z \mapsto u_1z$ yields $i = 1$, $r = 0$, and $a = 0$.

Hence, these numbers depend on the choice of parameters in this non-splittable skew field.

Proposition 8. *Let K be a two-dimensional local skew field such that the assumptions stated at the beginning of this section hold. Let $\operatorname{char} k = 0$, $\alpha = 1$, and assume that i is greater than 1. Then K is isomorphic to a skew field of the form $k((u))((z))$, $zuz^{-1} = u + u^{\delta'_i}z^i + u^{\delta'_{2i}}z^{2i}$, where $\delta'_i(u) = cu^r$, r is the second invariant, $c \in k^*/(k^*)^e$ if $(r-1, i) = e > 1$ (otherwise $c = 1$), $\delta'_{2i}(u) = (a(0, \dots, 0) + r(i+1)/2)u^{-1}(\delta'_i(u))^2$, and the other $\delta'_j(u)$ are equal to zero.*

Proof. Let us make the change $z \mapsto z' = a_0z$. By Lemma 3(iii), we have $u^{\delta'_i} = a_0^{-i}u^{\delta_i}$. Hence, $u^{\delta'_i}$ can be written as

$$u^{\delta'_i} = c_0u^{\overline{\nu}(u^{\delta_i}) \bmod i}$$

with $c_0 \in k^*/(k^*)^i$. We know from Lemmas 8 and 3 that c_0 is influenced only by the change $z \mapsto z' = a_0z$, $u \mapsto u' = x_0u$, where $a_0, x_0 \in k$. This change transforms c_0 into $c = c_0a_0^{-i}x_0^{-r+1}$. Therefore, $c \in k^*/(k^*)^e$ if $(r-1, i) = e > 1$. If $(r-1, i) = 1$, then we can make c equal to 1 by the choice of a_0 and x_0.

We claim that there is a change $z \mapsto z' = z + a_1z^2 + \cdots$ transforming the δ_j, $2i > j > i$, into δ'_j: $\delta'_j(u) = 0$. To find this change, we make successive changes of the type $z \mapsto z' = z + bz^{j+1}$. By Corollary 4, for every j specified above there is a b such that $\delta'_j(u) = 0$.

Now we can make a change that transforms δ_{2i} into δ'_{2i}: $\delta'_{2i}(u') = (a + r(i+1)/2)u'^{-1}(u'^{\delta_i})^2$, using Lemma 8(ii). It is sufficient to show that there is a b such that

$$u^{\delta_{2i}} - (a + r(i+1)/2)u^{-1}(u^{\delta_i})^2 + b^{\delta_i} - (u^{\delta_i})'b = 0.$$

(Differentiation with respect to u is denoted by a prime.) By Corollary 2, δ_i is a derivation, which enables us to write the last equation as

$$u^{\delta_{2i}} - \left(a + \frac{r(i+1)}{2}\right)u^{-1}(u^{\delta_i})^2 + b'u^{\delta_i} - (u^{\delta_i})'b = 0.$$

This ordinary differential equation has a solution of the form $b = u^{\delta_j}\tilde{b}$ if $\tilde{b}' + u^{\delta_{2i}}(u^{\delta_i})^{-2} - (a + r(i+1)/2)u^{-1} = 0$, which is the case since $\operatorname{res}_u \frac{\delta_i^2(u)}{(\delta_i(u))^2}\,du = r$ (see the definition of a).

Repeating these arguments, we complete the proof.

Let us return to the case when $\alpha^n = \mathrm{Id}$ for some n.

Lemma 10. *Assume that the canonical automorphism α of the skew field $K \cong k((u))((z))$ is such that $\alpha^n = 1$ for some $n \in \mathbb{N}$, $n > 1$. Then there is a change $z \mapsto z' = z + a_1 z^2 + \cdots$ in K such that*

$$z'uz'^{-1} = u^\alpha + u^{\delta'_n}z'^n + u^{\delta'_{2n}}z'^{2n} + \cdots.$$

Here $\delta'_j = 0$ if $n \nmid j$.

Proof. Let

$$zuz^{-1} = u^\alpha + u^{\delta_1}z + u^{\delta_2}z^2 + \cdots.$$

By Corollary 2, δ_1 is an (α^2, α)-derivation. Since $n > 1$, we have $\alpha^2 \neq \alpha$. By Lemma 4, δ_1 is an interior derivation and $\delta_1(u) = du^{\alpha^2} - u^\alpha d$. By Lemma 3(i), the change $z \mapsto z' = z - dz^2$ yields

$$z'uz'^{-1} = u^\alpha + u^{\delta'_2}z'^2 + \cdots.$$

By Corollary 2, δ'_2 is an (α^3, α)-derivation. If $n \neq 2$, then it is interior, and we can apply Lemma 3. We prove by induction that there is a change such that

$$z'uz'^{-1} = u^\alpha + u^{\delta'_n}z'^n + u^{\delta'_{n+1}}z'^{n+1} + \cdots,$$

where δ'_n is an (α^{n+1}, α)-derivation. Hence, $\delta'_n\alpha^{-1}$ is a derivation.

Now we observe that δ'_{n+1} in the last formula is an (α^2, α)-derivation. Indeed, Proposition 2 implies that

$$\delta'_{n+1}(ab) = \sum_{k=0}^{n+1} \delta'_{n+1-k}(a)\sigma(S^k_{n+1}\alpha)(b), \qquad a, b \in \overline{K}.$$

We have $\delta'_j = 0$ if $j < n$. Therefore,

$$\delta'_{n+1}(ab) = \delta'_{n+1}(a)\alpha^{n+2}(b) + \delta'_n(a)\left(\sum_{k=0}^{n} \alpha^k \delta'_1 \alpha^{n-k}\right)(b) + \alpha(a)\delta'_{n+1}(b)$$
$$= \delta'_{n+1}(a)\alpha^2(b) + \alpha(a)\delta'_{n+1}(b).$$

By Lemma 4, δ'_{n+1} is an interior derivation. Using Lemma 3 for the change $z' \mapsto z'' = z' + bz'^{n+2}$ with a suitable b, we obtain that

$$z''uz''^{-1} = u^\alpha + u^{\delta'_n} z''^n + u^{\delta'_{n+2}} z''^{n+2} + \cdots ,$$

and $\delta'_{n+1} = 0$. Proceeding by induction, we assume that we have made a change $z \mapsto z'$ such that

$$z'uz'^{-1} = u^\alpha + u^{\delta'_n} z'^n + u^{\delta'_{2n}} z'^{2n} + \cdots + u^{\delta'_{k+1}} z'^{k+1} + \cdots .$$

If $n \nmid (k+1)$, then δ'_{k+1} is an (α^{k+2}, α)-derivation. Indeed,

$$\delta'_{k+1}(ab) = \sum_{l=0}^{k+1} \sigma(\delta'_{k+1-l}\alpha)(a)\sigma(S^l_{k+1}\alpha)(b)$$
$$= \delta'_{k+1}(a)\alpha^{k+2}(b) + \sum_{m=1}^{x} \delta'_{mn}(a)\sigma(S^{k+1-mn}_{k+1}\alpha)(b) + \alpha(a)\delta'_{k+1}(b),$$

where $x \in \mathbb{N}$: $xn \leqslant k+1$, $(x+1)n > k+1$, since $\delta'_j = 0$ if $j < k+1$ and $n \nmid j$.

Every monomial in $\sigma(S^{k+1-mn}_{k+1}\alpha)$ contains a δ'_j with $j < k+1$ and $n \nmid j$. This follows from the definition of S^l_{k+1}, since $n \nmid (k+1-mn)$. Therefore, $\sigma(S^{k+1-mn}_{k+1}\alpha)(b) = 0$ for all m and δ'_{k+1} is an (α^{k+2}, α)-derivation.

If $n \mid (k+1)$, then the same arguments show that δ'_{k+2} is an (α^{k+2}, α)-derivation. By Lemma 3, there is a change $z' \mapsto z'' = z' + bz'^{k+2}$ (or $z'' = z' + bz'^{k+3}$, if $n \mid (k+1)$) such that

$$z''uz''^{-1} = u^\alpha + u^{\delta'_n} z''^n + u^{\delta'_{2n}} z''^{2n} + \cdots + u^{\delta'_{k+2}} z''^{k+2} + \cdots$$
$$(\text{or} \quad u^{\delta'_{k+1}} z''^{k+1} + u^{\delta'_{k+3}} z''^{k+3} + \cdots \quad \text{if} \quad n \mid (k+1)).$$

Since at the lth step of induction we make the change $z_l \mapsto z_{l+1} = (1 + z_l^l)z_l$, the sequence $\{z_l\}_{l=1}^\infty$ converges in K, which completes the proof of the lemma.

Lemma 11. *There is a parameter u in K such that $\alpha(u) = \xi u$, where $\xi^n = 1$, and $\delta'_{jn}(u) = u\left(\sum_k y_{jk} u^{nk}\right) \in uk((u^n))$ for all j, where $y_{jk} \in k$. The other δ'_k are equal to zero.*

Proof. By Lemma 10, we can assume the relation stated there. We make successive changes of the type $u \mapsto u' = u + b_{jn} z^{jn}$. The proof of Lemma 8 implies that the maps δ_k, $n \nmid k$, are invariant under these changes. By Lemma 8(i) these changes yield $u'^{\delta'_{jn}} = u^{\delta'_{jn}} + b^\alpha - \partial/\partial u(u^\alpha)b$. By Corollary 6, we can assume that $\alpha(u) = \xi u$, where $\xi^n = 1$. Therefore, $u'^{\delta'_{jn}} = u^{\delta'_{jn}} + b^\alpha - \xi b$. Hence, b can be chosen in such a way that the assumption of the lemma holds.

The definitions of i_n, r_n and a_n are similar to those in the case when $\alpha = \mathrm{Id}$. Namely, the definition of a_n remains unchanged, whereas i_n and r_n are defined as follows.

Definition 9. We put

$$i_n = \nu((\phi_{z^n} - 1)(u)) \in \mathbb{N} \cup \infty,$$
$$r_n = \overline{\nu}[((\phi_{z^n} - 1)(u))z^{-i_n} \bmod \wp] \bmod i_n \in \mathbb{Z}/i_n\mathbb{Z},$$
$$a_n = \operatorname{res}_u \left\{ \frac{u^{\delta_{2i_n} - \frac{i_n+1}{2}\delta_{i_n}^2}}{(u^{\delta_{i_n}})^2}\, du \right\} \in k,$$

where u and z are arbitrary local parameters of K, $\phi_z \colon K \to K$, and $\phi_z(a) = \operatorname{ad}(z)(a)$.

The last two lemmas imply that if z is the parameter specified in Lemma 10, then $i_n \in n\mathbb{N}$ and $r_n = 1 \bmod n$. It is easy to see that i_n corresponds to the first non-zero δ_{i_n} in the assertion of Lemma 11. We prove the following statement in the same way as Proposition 7.

Proposition 9. $i_n = i_n(u^{\delta_j}, j \notin n\mathbb{N})$, $r_n = r_n(i_n)$, $a_n = a_n(u^{\delta_{i_n+1}}, \ldots, u^{\delta_{2i_n-1}})$.

Proposition 10. *Let K be a two-dimensional local skew field satisfying the assumptions stated at the beginning of this section. Let $\operatorname{char} k = 0$ and $\alpha^n = \mathrm{Id}$ for some n. Then K is isomorphic to the skew field $k((u))((z))$ with the relation $zuz^{-1} = \xi u + u^{\delta'_{i_n}} z^{i_n} + u^{\delta'_{2i_n}} z^{2i_n}$, where $\xi^n = 1$, $i_n = i_n(0, \ldots, 0)$, $\delta'_{i_n}(u) = cu^{r_n}$, $c \in k^*/(k^*)^e$ if $(r_n - 1, i) = e > 1$, otherwise $c = 1$, $\delta'_{2i_n}(u) = (a_n(0, \ldots, 0) + r_n(i_n + 1)/2)u^{-1}(\delta'_{i_n}(u))^2$.*

Proof. We can assume that the assumptions of Lemma 11 hold. The special form of $u^{\delta'_{jn}}$ enables us to prove this proposition by repeating the proof of Proposition 8.

The above results imply that the following theorem holds.

Theorem 4. *If the assumptions stated at the beginning of this section hold for two local skew fields K and K' of characteristic 0, then K and K' are isomorphic if and only if $k \cong k'$ and $(n, \xi, i_n, r_n, c, a_n)$ coincides with $(n', \xi', i'_n, r'_n, c', a'_n)$.*

Remark 4. If $n = 1$ and $i_n = \infty$, then K is a two-dimensional local field of the form $k((u))((z))$.

We summarize the results obtained above in definitive form.

Theorem 5. (I) *Let K be a two-dimensional local skew field whose first residue skew field is commutative. Then it is splittable if its canonical automorphism α is such that $\alpha^n \neq \mathrm{Id}$ for all n. If this condition does not hold, then K can be non-splittable.*

(II) *Let K and K' be two-dimensional local skew fields such that $\alpha^n \neq \mathrm{Id}$ and $\alpha'^n \neq \mathrm{Id}$ for all n, and assume that the residue skew field $\overline{K}$ is commutative. Then*

(a) *K is isomorphic to $\overline{K}((z))$, $za = a^\alpha z$, $a \in \overline{K}$, where $\overline{K}$ is a one-dimensional local field with residue field k,*

(b) *K is isomorphic to K' if and only if $k \cong k'$ and there is an isomorphism $f \colon \overline{K} \mapsto \overline{K}'$ such that $\alpha = f^{-1}\alpha' f$,*

(c) *if* $\operatorname{char} K = \operatorname{char} k$, $\operatorname{char} K' = \operatorname{char} k'$, *and* k *and* k' *are algebraically closed fields of characteristic* 0, *then* K *is isomorphic to* K' *if and only if* $k \cong k'$ *and* $(a_1, i_\alpha, y(\alpha)) = (a'_1, i_{\alpha'}, y(\alpha'))$.

(III) *Let* K *and* K' *be splittable two-dimensional local skew fields of characteristic* 0, *let* $k \subset Z(K)$, $k' \subset Z(K')$, *and let* $\alpha^n = \mathrm{Id}$, $\alpha'^{n'} = \mathrm{Id}$ *for some* $n, n' \geqslant 1$. *Then*

(a) K *is isomorphic to* $k((u))((z))$ *with the relation* $zuz^{-1} = \xi u + u^{\delta'_{i_n}} z^{i_n} + u^{\delta'_{2i_n}} z^{2i_n}$, *where* $\xi^n = 1$, $i_n = i_n(0, \dots, 0)$ *and* $\delta'_{i_n}(u) = cu^{r_n}$, *where* $c \in k^*/(k^*)^e$ *if* $(r_n - 1, i) = e > 1$, *otherwise* $c = 1$, $\delta'_{2i_n}(u) = (a_n(0, \dots, 0) + r_n(i_n + 1)/2)u^{-1}(\delta'_{i_n}(u))^2$ (i_n, r_n *and* a_n *are defined in Proposition* 9); *if* $n = 1$ *and* $i_n = \infty$, *then* K *is commutative*,

(b) K *is isomorphic to* K' *if and only if* $k \cong k'$ *and* $(n, \xi, i_n, r_n, c, a_n)$ *coincides with* $(n', \xi', i'_n, r'_n, c', a'_n)$.

§ 4. Classes of conjugate elements

Let K be a splittable local skew field of characteristic 0 whose first residue skew field is commutative and whose last residue skew field k is contained in its centre. We classified these skew fields in the preceding section. In this section we give necessary and sufficient conditions for two elements of K to be conjugate.

We fix a representation of K in the form $k((u))((z))$.

Definition 10. Let $\alpha = \mathrm{Id}$. A *residue* $\operatorname{res}_{i,r}$ *on* K is defined to be a map $\operatorname{res}_{i,r} \colon k((u))((z)) \mapsto k$,

$$\operatorname{res}_{i,r}(X) = \operatorname{res} \frac{x_i}{u^{\delta_i}}\, du,$$

where $X = \sum_l x_l z^l$.

Proposition 11. *Let* $\alpha = \mathrm{Id}$. *Let* $L, M \in K$, $\nu(L) = \nu(M) = -1$, $M = b_{-1}z^{-1} + b_0 + b_1 z + \cdots$, *and* $L = a_{-1}z^{-1} + a_0 + a_1 z + \cdots$. *The following conditions are equivalent*:

(i) *there is an* $S \in K$, $\nu(S) = 0$, *such that* $M = S^{-1}LS$,

(ii) $a_{-1} = b_{-1}$, $a_0 = b_0, \dots, a_{i-2} = b_{i-2}$,

$$\operatorname{res} \frac{a_{i-1} - b_{i-1}}{u^{\delta_i} a_{-1}}\, du \in \mathbb{Z}\,, \qquad u \frac{a_{i-1} - b_{i-1}}{u^{\delta_i} a_{-1}} \in k[[u]],$$

and $\operatorname{res}_{i,r}(M^j) = \operatorname{res}_{i,r}(L_j^j)$ *for all* $j \geqslant 1$, *where* $L_j = \widetilde{S}_j^{-1} L_{j-1} \widetilde{S}_j$, $L_0 := L$ *and* $\widetilde{S}_j = \widetilde{S}_j(M, L_{j-1})$.

Proof. K has the form $k((u))((z))$ with the relation $zuz^{-1} = u + u^{\delta_i} z^i + \cdots$. We have

$$SM = s_0 b_{-1} z^{-1} + (s_0 b_0 + s_1 b_{-1}) + \cdots + \left(\sum_{j=-1}^{i-2} b_j s_{i-2-j} \right) z^{i-2} + \\ + \left(\sum_{j=-1}^{i-1} b_j s_{i-1-j} \right) z^{i-1} + \cdots,$$

$$LS = s_0 a_{-1} z^{-1} + (s_0 a_0 + s_1 a_{-1}) + \cdots + \left(\sum_{j=-1}^{i-2} a_j s_{i-2-j} \right) z^{i-2} + \\ + \left(-a_{-1} s_0^{\delta_i} + \sum_{j=-1}^{i-1} a_j s_{i-1-j} \right) z^{i-1} + \cdots.$$

It follows that the conditions $a_{-1} = b_{-1},\ a_0 = b_0,\ \dots,\ a_{i-2} = b_{i-2}$ are necessary for M and L to be conjugate. Another necessary condition is that

$$\frac{s_0^{\delta_i}}{s_0} = \frac{a_{i-1} - b_{i-1}}{a_{-1}}.$$

Since δ_i is a derivation, we have

$$\frac{\frac{\partial}{\partial u} s_0}{s_0} = \frac{a_{i-1} - b_{i-1}}{u^{\delta_i} a_{-1}}.$$

Thus we obtain the second necessary condition:

$$\operatorname{res} \frac{a_{i-1} - b_{i-1}}{u^{\delta_i} a_{-1}} du \in \mathbb{Z} \quad \text{and} \quad u \frac{a_{i-1} - b_{i-1}}{u^{\delta_i} a_{-1}} \in k[[u]].$$

Conversely, if these two conditions hold, then there is an $s_0 \in k((u))$ such that the first $i+1$ summands in $L_1 = s_0^{-1} L s_0$ are the same as those in M. It is clear that L and M are conjugate if and only if L_1 and M are conjugate. The conjugating element $\tilde{S}$ has the form $1 + \cdots$ ($\widetilde{S}$ can be written as $(1+s_1 z)(1+s_2 z^2)\dots$). Note that

$$\begin{aligned}(1+s_j z^j)^{-1} & (x_{-1} z^{-1} + x_0 + x_1 z + \cdots)(1 + s_j z^j) \\ &= x_{-1} z^{-1} + x_0 + x_1 z + \cdots + x_{i+j-2} z^{i+j-2} \\ &\quad + (x_{i+j-1} + j x_{-1}^{\delta_i} s_j + x_{-1} s_j^{\delta_i}) z^{i+j-1} + \cdots\end{aligned}$$

for every $x_{-1} z^{-1} + x_0 + x_1 z + \cdots \in K$, since the proof of Lemma 3(ii) implies that

$$\begin{aligned}(1+s_j z^j)^{-1} & (x_{-1} + x_0 z + x_1 z^2 + \cdots)(1 + s_j z^j) \\ &= x_{-1} + x_0 z + \cdots + x_{i+j-2} z^{i+j-1} + (x_{i+j-1} + j x_{-1}^{\delta_i} s_j) z^{i+j} + \cdots)\end{aligned}$$

and

$$(1+s_jz^j)^{-1}z^{-1}(1+s_jz^j) = (1+s_jz^j)^{-1}(z^{-1}+s_jz^{j-1}-s_j^{\delta_i}z^{i+j-1}+\cdots)$$
$$= z^{-1}-s_j^{\delta_i}z^{i+j-1}+\cdots.$$

It follows that

$$(s_1a_{-1})^{\delta_i} = b_i - a_i$$

if $M = \widetilde{S}^{-1}L_1\widetilde{S}$, where a_i is the coefficient of L_1. This equation is soluble if and only if

$$\operatorname{res}\frac{b_i-a_i}{u^{\delta_i}}\,du = 0,$$

that is, $\operatorname{res}_{i,r}(M) = \operatorname{res}_{i,r}(L_1)$.

Conversely, if the residues are equal then there is an $s_1 \in k((u))$ such that the first $i+2$ summands in $L_2 = (1+s_1z)^{-1}L_1(1+s_1z)$ are the same as those in M.

Proceeding by induction, we obtain at the kth step that

$$ks_ka_{-1}^{\delta_i} + a_{-1}s_k^{\delta_i} = b_{i+k-1} - a_{i+k-1}$$

if $M = \overline{S}^{-1}L_k\overline{S}$. To solve this equation, we substitute $s_k = a_{-1}^{-k}s$ and obtain the equation

$$s' = a_{-1}^{k-1}\frac{b_{i+k-1}-a_{i+k-1}}{u^{\delta_i}}\,.$$

which is soluble if and only if

$$\operatorname{res}\frac{a_{-1}^{k-1}a_{i+k-1}}{u^{\delta_i}} = \operatorname{res}\frac{u^{-r}a_{-1}^{k-1}b_{i+k-1}}{u^{\delta_i}}\,.$$

The coefficient of z^i in M^k has the form

$$ka_{-1}^{k-1}b_{i+k-1} + f_M,$$

where f_M is a polynomial in $b_{i+k-2},\dots,b_{-1}$ and the values of δ_j at these points. The corresponding coefficient in L_k^k has the form

$$ka_{-1}^{k-1}a_{i+k-1} + f_{L_k},$$

and $f_{L_k} = f_M$, since $a_j = b_j$ for $j \leqslant i+k-2$. It follows that $\operatorname{res}_{i,r}L_k^k = \operatorname{res}_{i,r}M^k$ if and only if

$$\operatorname{res}\frac{a_{-1}^{k-1}a_{i+k-1}}{u^{\delta_i}} = \operatorname{res}\frac{a_{-1}^{k-1}b_{i+k-1}}{u^{\delta_i}},$$

which completes the proof of the proposition.

Definition 11. Let $\alpha \neq \mathrm{Id}$. We say that the *residue* $\operatorname{res}_\alpha$ *of* $X = \sum_l x_lz^l$ *is equal to zero* if

$$x_0 \in \operatorname{im}(\alpha - \mathrm{Id}).$$

We say that *two elements have the same residue* if the residue of their difference is equal to zero.

We define $\varphi\colon k((u)) \mapsto k((u))$ by the formula $\varphi(x) = x^{\alpha^{-1}}/x$.

Proposition 12. *Let* $\alpha \neq \mathrm{Id}$. *Let* $L, M \in K$, $\nu(L) = \nu(M) = -1$, $M = b_{-1}z^{-1} + b_0 + b_1 z + \cdots$ *and* $L = a_{-1}z^{-1} + a_0 + a_1 z + \cdots$ *. The following conditions are equivalent*:

(i) *there is an* $S \in K$, $\nu(S) = 0$, *such that* $M = S^{-1}LS$,

(ii) $b_{-1}/a_{-1} \in \operatorname{im}\varphi$, $\operatorname{res}_\alpha(M^j) = \operatorname{res}_\alpha(L_j^j)$ *for all* $j \geqslant 1$, *where* $L_j = \widetilde{S}_j^{-1} L_{j-1} \widetilde{S}_j$, $L_0 := L$, $\widetilde{S}_j = \widetilde{S}_j(M, L_{j-1})$.

The proof is similar to that of the preceding proposition. We have

$$SM = s_0 b_{-1} z^{-1} + (s_0 b_0 + s_1 b_{-1}^{\alpha}) + \cdots,$$
$$LS = a_{-1} s_0^{\alpha^{-1}} z^{-1} + (a_0 s_0 + a_{-1} s_1^{\alpha^{-1}}) + \cdots.$$

Therefore, $s_0 b_{-1} = a_{-1} s_0^{\alpha^{-1}}$, that is, $b_{-1}/a_{-1} \in \operatorname{im}\varphi$. If this condition holds, then we put $L_1 = s_0^{-1} L s_0$. The first coefficients in L_1 and M are equal.

Now we observe that

$$(1 + s_j)^{-1}(x_{-1}z^{-1} + x_0 + x_1 z + \cdots)(1 + s_j z^j)$$
$$= x_{-1}z^{-1} + \cdots + x_{j-2}z^{j-2} + (x_{j-1} + s_j x_{-1}^{\alpha^j} - x_{-1} s_j^{\alpha^{-1}}) z^{j-1} + \cdots$$

for any $x_{-1}z^{-1} + x_0 + x_1 z + \cdots \in K$, which follows from the calculation in the proof of Lemma 3(i).

The arguments used in the proof of the preceding proposition yield at the first step the following condition that is necessary for conjugacy:

$$s_1 a_{-1}^{\alpha} - a_{-1} s_1^{\alpha^{-1}} = \alpha(s_1^{\alpha^{-1}} a_{-1}) - (s_1^{\alpha^{-1}} a_{-1}) = b_0 - a_0.$$

This equation is soluble if and only if $(b_0 - a_0) \in \operatorname{im}(\alpha - \mathrm{Id})$, which is equivalent to the equality $\operatorname{res}_\alpha M = \operatorname{res}_\alpha L_1$.

At the jth step we have the condition

$$s_j a_{-1}^{\alpha^j} - a_{-1} s_j^{\alpha^{-1}} = a_{j-1} - b_{j-1}.$$

Hence,

$$(a_{-1}^{\alpha} a_{-1}^{\alpha^2} \ldots a_{-1}^{\alpha^{j-1}})(a_{j-1} - b_{j-1}) = (a_{-1}^{\alpha} a_{-1}^{\alpha^2} \ldots a_{-1}^{\alpha^j}) s_j - (a_{-1} \ldots a_{-1}^{\alpha^{j-1}}) s_j^{\alpha^{-1}}$$
$$= \alpha((a_{-1} \ldots a_{-1}^{\alpha^{j-1}}) s_j^{\alpha^{-1}}) - (a_{-1} \ldots a_{-1}^{\alpha^{j-1}}) s_j^{\alpha^{-1}}.$$

This equation is soluble if and only if $(a_{-1} \ldots a_{-1}^{\alpha^{j-1}})(a_{j-1} - b_{j-1}) \in \operatorname{im}(\alpha - \mathrm{Id})$, which is equivalent to the equality $\operatorname{res}_\alpha(M^j) = \operatorname{res}_\alpha(L_j^j)$, since the first $(j-1)$ coefficients in L_j are equal to the corresponding coefficients in M, and the coefficient of the 0th power of z in M^j is

$$a_{-1} \ldots a_{-1}^{\alpha^{-j+2}} b_{j-1}^{\alpha^{-j+1}} + b_{j-1} a_{-1}^{\alpha} \ldots a_{-1}^{\alpha^{j-1}} + \text{a sum of monomials with subscripts } < j - 1.$$

The corresponding coefficient in L_j^j is

$$a_{-1}\dots a_{-1}^{\alpha^{-j+2}}a_{j-1}^{\alpha^{-j+1}}+a_{j-1}a_{-1}^{\alpha}\dots a_{-1}^{\alpha^{j-1}}+\text{a sum of monomials with subscripts } < j-1.$$

Hence,

$$\begin{aligned}(a_{-1}\dots a_{-1}^{\alpha^{-j+2}}b_{j-1}^{\alpha^{-j+1}}-a_{-1}\dots a_{-1}^{\alpha^{-j+2}}a_{j-1}^{\alpha^{-j+1}}+b_{j-1}a_{-1}^{\alpha}\dots a_{-1}^{\alpha^{j-1}}\\ -a_{j-1}a_{-1}^{\alpha}\dots a_{-1}^{\alpha^{j-1}})=([a_{-1}\dots a_{-1}^{\alpha^{-j+2}}b_{j-1}^{\alpha^{-j+1}}-a_{-1}+\dots+a_{-1}^{\alpha^{-j+2}}a_{j-1}^{\alpha^{-j+1}}]\\ -\alpha[\dots]+\alpha[\dots]-\alpha^2[\dots]+\alpha^2[\dots]+\dots+\alpha^{j-1}[\dots]\\ +b_{j-1}a_{-1}^{\alpha}\dots a_{-1}^{\alpha^{j-1}}-a_{j-1}a_{-1}^{\alpha}\dots a_{-1}^{\alpha^{j-1}})=(2[a_{-1}^{\alpha}\dots a_{-1}^{\alpha^{j-1}}(a_{j-1}-b_{j-1})]).\end{aligned}$$

Remark 5. It was shown in [6] that $\mathrm{res}_{1,0}[X,Y]=0$ for the residue $\mathrm{res}_{1,0}$ in the skew field of pseudodifferential operators. (Here $[X,Y]$ is the commutator of two pseudodifferential operators.) There follow other examples of skew fields with this property.

Lemma 12. *Let K be a skew field such that $\alpha^n\neq \mathrm{Id}$ or $\alpha^n=\mathrm{Id}$ and $i_n=\infty$. Let $X,Y\in K$. Then $\mathrm{res}_\alpha[X,Y]=0$.*

Proof. It is sufficient to prove the assertion for $X=u^lz^k$ and $Y=u^mz^q$. If $k+q\neq 0$, then $\mathrm{res}_\alpha(XY)=\mathrm{res}_\alpha(YX)=0$. If $k+q=0$, then

$$XY-YX=u^l(u^m)^{\alpha^k}-u^m(u^l)^{\alpha^{-k}}=\alpha^k(u^m(u^l)^{\alpha^{-k}})-u^m(u^l)^{\alpha^{-k}}\in \mathrm{im}(\alpha-\mathrm{Id}).$$

In this case our propositions can be restated as follows.

Corollary 7. *Let K be a skew field such that $\alpha=\mathrm{Id}$, $i=1$, $r=0$ and $a=0$. (In this case K is the ring $k((u))((\partial^{-1}))$ of pseudodifferential operators.) Let $L,M\in K$, $\nu(L)=\nu(M)=-1$, $M=b_{-1}z^{-1}+b_0+b_1z+\cdots$ and $L=a_{-1}z^{-1}+a_0+a_1z+\cdots$. The following conditions are equivalent*:

(i) *there is an $S\in K$, $\nu(S)=0$, such that $M=S^{-1}LS$,*

(ii) $a_{-1}=b_{-1}$,

$$\mathrm{res}\,\frac{a_0-b_0}{a_{-1}}\,du\in\mathbb{Z}\,,\qquad \frac{u(a_0-b_0)}{a_{-1}}\in k[[u]],$$

and $\mathrm{res}_{1,0}(M^j)=\mathrm{res}_{1,0}(L^j)$ for all $j\geqslant 1$.

Corollary 8. *Assume that $\alpha^n\neq\mathrm{Id}$ for all $n\in\mathbb{N}$. Let $L,M\in K$, $\nu(L)=\nu(M)=-1$, $M=b_{-1}z^{-1}+b_0+b_1z+\cdots$ and $L=a_{-1}z^{-1}+a_0+a_1z+\cdots$. The following conditions are equivalent*:

(i) *there is an $S\in K$, $\nu(S)=0$, such that $M=S^{-1}LS$,*

(ii) *$b_{-1}/a_{-1}\in\mathrm{im}\,\varphi$ and $\mathrm{res}_\alpha(M^j)=\mathrm{res}_\alpha(L^j)$ for all $j\geqslant 1$.*

The following examples show that the identity $\mathrm{res}_{\dots}([X,Y])=0$ does not hold in other cases.

Example 1. Let K be a skew field with $\alpha=1$, $a(0,\dots,0)\neq 0$, $r\neq 1$. We assume that K has the form specified in Theorem 5. Let $M=z^{-1}$ and $L=z^{-1}+z^i\in k((z))\subset K$. If $\mathrm{res}_{i,r}([X,Y])=0$, then M and L are conjugate by Proposition 11.

Let $S = 1 + s_1 z + \cdots$. We have

$$SM = z^{-1} + s_1 + s_2 z + \cdots = LS = (z^{-1} + z^i)(1 + s_1 z + \cdots)$$
$$= (z^{-1} + s_1 + s_2 z + \cdots) + (z^i - s_1^{\delta_i} z^i) + (s_1 z^{i+1} - s_2^{\delta_i} z^{i+1}) + \cdots$$
$$\cdots + (s_i z^{2i} + s_1^{\delta_i^2 - \delta_{2i}} z^{2i} - s_{i+1}^{\delta_i} z^{2i}) + \cdots .$$

Hence, $1 - s_1^{\delta_i} = 0$. Since $r \neq 1$, this equation is soluble, and $s_1 = (1-r)^{-1} c^{-1} u^{1-r}$. Solving the subsequent equations, we obtain $s_2, s_3, \ldots$. Each of these elements consists of a single monomial whose valuation is different from $r - 1$.

Further, we have $s_i z^{2i} + s_1^{\delta_i^2 - \delta_{2i}} z^{2i} - s_{i+1}^{\delta_i} z^{2i} = 0$. If $a(0, \ldots, 0) \neq 0$, then, by Theorem 5, $s_1^{\delta_i^2 - \delta_{2i}}$ contains a monomial whose valuation is equal to $r - 1$. Therefore, the equation cannot be solved for s_{i+1}, and M is not conjugate to L. This contradiction completes the proof of the assertion.

Example 2. Let K be a skew field with $\alpha = 1$, $a(0, \ldots, 0) = 0$. In this case $i > 1$, since $r = 0$ for $i = 1$, and we obtain the ring of pseudodifferential operators. We assume that K has the form specified in Theorem 5. Then $zuz^{-1} = u + cu^r z^i + r(i+1)/2c^2 u^{2r-1} z^{2i}$. Therefore, $\delta_{2i} = \delta_i^2$. We have

$$z^{-1} x z = x - x^{\delta_i} z^i + \text{terms with } z^{>2i}$$

for any $x \in k((u))$. We put $X = u^{-r-1} z^{-i}$ and $Y = u^2$. Then

$$XY = u^{1-r} z^{-i} + \cdots + C u^{r-1} z^i + \cdots , \qquad C \in \mathbb{Q}, \quad C \neq 0.$$

Hence, $\operatorname{res}_{i,r}([X, Y]) \neq 0$.

An example with $a(0, \ldots, 0) \neq 0$, $r = 1$ can be obtained likewise.

Example 3. Let K be a skew field with $\alpha^n = 1$, $i_n \neq \infty$. We put $X = u^{-r_n} z^{-i_n}$ and $Y = u$. Then

$$XY = \xi^{-i_n} u^{1-r_n} z^{-i_n} + C + \cdots ,$$

where $C = -i_n \xi^{-i_n+1} c \neq 0$. Hence, $\operatorname{res}_\alpha([X, Y]) \neq 0$.

Let K be the ring $k((u))((\partial_u^{-1}))$ of pseudodifferential operators. We have shown that this is the only slew field such that $\operatorname{res}_{1,0}([X, Y]) = 0$. Let us deduce a criterion for two elements of this skew field to be conjugate.

Let $n \in \mathbb{N}$. Consider the skew field $K' = k((t))((\partial_t^{-1}))$ with $t^n = u$. Then $\partial_t = nt^{n-1} \partial_u$ and $K \subset K'$.

Lemma 13. *Let $L = l_{-m} \partial_t^m + \cdots + l_0 + l_1 \partial_t^{-1} + \cdots$ be an arbitrary element of K'. L belongs to K if and only if $l_i \in t^i k((t^n))$.*

Proof. Assume that $L \in K$. Then $L = b_{-m} \partial_u^m + \cdots$, where $b_i \in k((u)) = k((t^n))$. Let $j \in \mathbb{N}$. We have $\partial_u^j = (n^{-1} t^{1-n} \partial_t)^j$, $\partial_u^{-j} = (\partial_t^{-1} n t^{n-1})^j$.

We first prove the assertion of the lemma for l_{-i} $(i > 0)$. For $i = 1$ we have $\partial_u^i = n^{-1} t^{1-n} \partial_t$ and $b_{-1} \partial_u = l_{-1} n^{-1} t^{1-n} \partial_t$. The assertion of the lemma holds since $t^{1-n} \in tk((t^n))$.

For an arbitrary i we have

$$\begin{aligned}\partial_u^i &= \frac{\partial_t}{\partial_t t}(n^{-1}t^{1-n})(n^{-1}t^{1-n}\partial_t)^{i-1} + (n^{-1}t^{1-n})^2\partial_t^2(n^{-1}t^{1-n}\partial_t)^{i-2} \\ &= (1-n)(n^{-1}t^{-n})(n^{-1}t^{1-n}\partial_t)^{i-1} + (n^{-1}t^{1-n})^2\partial_t^2(n^{-1}t^{1-n}\partial_t)^{i-2}.\end{aligned}$$

Since the coefficients in the expression for L in K belong to $k((t^n))$, it is sufficient to show that the lemma holds for ∂_u^i.

We prove by induction that the assumption of the lemma holds for all the coefficients in $(n^{-1}t^{1-n}\partial_t)^{i-1}$. The same is true for $(n^{-1}t^{1-n}\partial_t)^{i-2}$. Let $(n^{-1}t^{1-n}\partial_t)^{i-2} = \sum_{k=0}^{i-2}\tilde{l}_k\partial_t^k$. (Let us note that there are no negative powers of ∂_t in the expansion of ∂_u^i, $i > 0$, and the minimal power of ∂_t occurring in it is 1.) We have

$$(n^{-1}t^{1-n})^2\partial_t^2\left(\sum_{k=0}^{i-2}\tilde{l}_k\partial_t^k\right) = (n^{-1}t^{1-n})^2\left(\sum_{k=0}^{i-2}\tilde{l}_k\partial_t^{k+2} + \sum_{k=0}^{i-2}\tilde{l}'_k\partial_t^{k+1} + \sum_{k=0}^{i-2}\tilde{l}''_k\partial_t^k\right).$$

Therefore, $(n^{-1}t^{1-n})^2\,\tilde{l}_k \in t^{k+2}k((t^n))$, $(n^{-1}t^{1-n})^2\,\tilde{l}'_k \in t^{k+1}k((t^n))$, $(n^{-1}t^{1-n})^2\,\tilde{l}''_k \in t^k k((t^n))$.

For $i = 0$ we have $l_0 = b_0 \in k((t^n))$.

Let us prove that the assertion of the lemma holds for ∂^{-i}, $i > 0$. For $i = 1$ we have

$$\partial_u^{-1} = n\sum_{k=0}^{n-1}(t^{n-1})^{(k)}\partial_t^{-1-k}\binom{k-1}{\cdot}$$

Assume that $\partial_u^{-k} = \sum_{j=0}^{\infty}\tilde{l}_j\partial_t^{-k-j}$, $\tilde{l}_j \in t^{-k-j}k((t^n))$ for $k < i$:

$$\begin{aligned}\partial_u^{-i} = (\partial_t^{-1}nt^{n-1})^i &= \left(n\sum_{k=0}^{n-1}\binom{k-1}{(}t^{n-1})^{(k)}\partial_t^{-1-k}\right)(\partial_t^{-1}nt^{n-1})^{i-1} = \\ &= \left(n\sum_{k=0}^{n-1}\binom{k-1}{(}t^{n-1})^{(k)}\partial_t^{-1-k}\right)\left(\sum_{j=0}^{\infty}\tilde{l}_j\partial_t^{-i+1-j}\right).\end{aligned}$$

We have

$$\partial_t^{-1-k}\tilde{l}_j = \sum_{p=0}^{\infty}\binom{p-1-k}{\tilde{l}_j}^{(p)}\partial_t^{-1-k-p}$$

for every $k \in \{0,\dots,n-1\}$. This yields the following conditions on the coefficients for fixed k and j: the coefficient of $\partial_t^{-1-k-p-i+1-j}$, $p \geqslant 0$, belongs to $t^{-1-k-i+1-j-p}k((t^n))$.

Conversely, assume that the assumptions of the lemma on the coefficients hold. We have obtained that $\partial_u^i = \sum_{j\geqslant 0}c_j\partial_t^{i-j}$ and $c_j \in t^{i-j}k((t^n))$ for any $i \in \mathbb{Z}$.

Consider the highest monomial in L:

$$l_{-m}\partial_t^m = l_{-m}c_0^{-1}\partial_u^m - l_{-m}\left(\sum_{j\geqslant 1}c_jc_0^{-1}\partial_t^{m-j}\right).$$

We have $l_{-m}c_0^{-1} \in k((t^n))$ and $l_{-m}c_jc_0^{-1} \in t^{m-j}k((t^n))$. Hence, $L = l_{-m}c_0^{-1}\partial_u^m + L_1$, where $\nu(L_1) > \nu(L)$, and the the assumptions of the lemma hold for the coefficients in L_1. We complete the proof by induction.

Lemma 14. *Let $L, M \in K \subset K'$ and $\nu(L) = \nu(M) = -n$. Let $M = SLS^{-1}$, where $S \in K'$. Then $S \in K$ if and only if*

$$\operatorname{res} \frac{l_{\nu(L)+1} - m_{\nu(M)+1}}{l_{\nu(L)}} = 0 \quad \text{and} \quad t\frac{l_{\nu(L)+1} - m_{\nu(M)+1}}{l_{\nu(L)}} \in k[[t]].$$

The proof is similar to that of Proposition 11.

Theorem 6. *Let $L, M \in K = k((u))((\partial_u^{-1})), \nu(L) = \nu(M) < 0, M = m_{\nu(M)}\partial_t^{-\nu(M)} + \cdots$ and $L = l_{\nu(L)}\partial_t^{-\nu(L)} + \cdots$. The following conditions are equivalent*:

(i) *there is an $S \in K$, $\nu(S) = 0$, such that $M = S^{-1}LS$,*

(ii) $\nu(L) = \nu(M)$, $m_{\nu(M)} = l_{\nu(L)}$,

$$\operatorname{res} \frac{l_{\nu(L)+1} - m_{\nu(M)+1}}{l_{\nu(L)}} = 0 \quad \text{and} \quad t\frac{l_{\nu(L)+1} - m_{\nu(M)+1}}{l_{\nu(L)}} \in k[[t]],$$

and $\operatorname{res}(M^{j/(-\nu(M))}) = \operatorname{res}(L^{j/(-\nu(L))})$ *in K' for all $j \geqslant 1$.*

The proof follows immediately from Corollary 7, Lemmas 13, 14 and the fact that L (and M) has precisely one nth root in K'.

Theorem 7. *Assume that $L, M \in K = k((u))((\partial_u^{-1}))$ and $\nu(L) = \nu(M) = 0$.*

(i) *If $l_0 = m_0 \neq \text{const}$ and $l_1 = m_1$, then $M = SLS^{-1}$.*

(ii) *If $l_0 = m_0 = \text{const}$, then $M = SLS^{-1}$ if and only if $(M - m_0)^{-1} = S(L - l_0)^{-1}S^{-1}$ (see Theorem 6).*

The proof is obvious.

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