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The Cauchy problem for a quasilinear parabolic equation with gradient absorption

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Abstract. The qualitative properties of solutions to the Cauchy problem for a degenerate parabolic equation containing a nonlinear operator of Baouendi-Grushin type and with gradient absorption whose density depends on time, as well as the space variables, are investigated. Bounds for the diameter of the support of the solution which are sharp with respect to time are obtained, together with its maximum. A condition which determines whether or not the phenomenon of decay to zero of the total mass of the solution occurs is discovered.

Bibliography: 35 titles.

Keywords: operator of Baouendi-Grushin type, quasilinear parabolic equation, gradient absorption, decay of the total mass of a solution, estimate for the support of the solution.

§ 1. Introduction

We investigate the solution of the Cauchy problem for a degenerate quasilinear parabolic equation of the following form:

$$\frac{\partial u}{\partial t} = \operatorname{div}_L\bigl(|D_L u|^{\lambda-1} D_L u\bigr) - a(\rho(z)) f(t) |D_L u^\nu|^q, \qquad (z,t) \in S_T = \mathbb{R}^{N+M} \times (0,T), \tag{1.1}$$

$$u(z,0) = u_0(z) \in L_1(\mathbb{R}^{M+N}), \qquad u_0(z) \geqslant 0 \text{ a.e.}, \qquad z = (x,y), \quad x \in \mathbb{R}^N, \quad y \in \mathbb{R}^M. \tag{1.2}$$

Here $\lambda > 1$, $1 < q < \lambda + 1$, $\nu q > \lambda$, and $x = (x_1, \dots, x_N)$ and $y = (y_1, \dots, y_M)$ are arbitrary points in the Euclidean spaces $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively, where $N \geqslant 1$, $M \geqslant 1$. Let $f(t)$ and $a(\rho(z))$ be nonnegative measurable functions. Furthermore,

$$a(s) \text{ is a continuous nondecreasing function such that } s^q/a(s) \text{ is also nondecreasing for } s > 0. \tag{H_1}$$

The function $\rho(z) = (|x|^{2(\alpha+1)} + (\alpha+1)^2 |y|^2)^{1/2(\alpha+1)}$ is an analogue of the distance function for points in Euclidean space; we describe it accurately in § 2; finally,

$$f(t) \text{ is a continuous nondecreasing function such that for each } t > 0 \text{ there exists a number } \mu, \; 0 < \mu < (\nu q - \lambda)/\lambda, \text{ such that } t^\mu/f(t) \text{ is also nondecreasing.} \tag{H_2}$$

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The symbol $D_L u$ denotes the vector

$$D_L u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_N}, |x|^\alpha \frac{\partial u}{\partial y_1}, |x|^\alpha \frac{\partial u}{\partial y_2}, \dots, |x|^\alpha \frac{\partial u}{\partial y_M}\right).$$

The corresponding vector field is often called the Baouendi-Grushin vector field (see [1] and [2]). By analogy we call the operator defined by means of these vector fields an operator of Baouendi-Grushin type. In what follows we assume that $\alpha > 0$ since the embedding theorems on which we rely have been proved in this case. Further,

$$|D_L u| = \sqrt{\sum_{i=1}^{N}\left(\frac{\partial u}{\partial x_i}\right)^2 + |x|^{2\alpha}\sum_{j=1}^{M}\left(\frac{\partial u}{\partial y_j}\right)^2},$$

$$\operatorname{div}_L \vec{F}(x,y) = \sum_{i=1}^{N}\frac{\partial F_i}{\partial x_i} + |x|^\alpha \sum_{j=1}^{M}\frac{\partial F_{j+N}}{\partial y_j}.$$

From the standpoint of physics the term $-a(\rho(z))f(t)|D_L u^\nu|^q$ produces a damping effect, and in what follows we shall call it the *damping* term in the equation.

We define the action $L_{\lambda,\alpha}[u]$ of a Baouendi-Grushin type operator on an arbitrary function by

$$L_{\lambda,\alpha}[u] := \operatorname{div}_L(|D_L u|^{\lambda-1} D_L u).$$

Equations involving a Baouendi-Grushin type operator are of independent interest. If we set $\alpha = 0$ in the undamped equation (1.1), it describes a process with slow diffusion (see the survey in [3]). Operators of type $L_{1,\alpha} = \Delta_x + |x|^{2\alpha}\Delta_y$, where Δ_x and Δ_y are the Laplace operators with respect to the x- and y-variables, respectively, were first investigated in [4] and [5]. In [6] and [7] qualitative properties of the solution of the equation $L_{\lambda,\alpha}[u] = f$ were discussed; this is an undamped elliptic analogue of (1.1) (see also [1] and the references listed there). In [8], questions of solvability for the Cauchy problem in the absence of damping were investigated, and sharp bounds for the maximum of the solution and for the propagation rate of its support were obtained. In [9] and [10] it was proved that the solution is locally Hölder continuous.

There is interest in problem, (1.2) because the presence of a damping term in the form of gradient absorption can significantly change the properties of the solution of (1.1) in certain circumstances. The resulting effect is that the mass of the solution decays with time. In the case $\alpha = 0$, $a(s) = f(t) = 1$ the properties of solutions of (1.1) have been investigated by many authors; see, for example, [11]–[17] for the standard Laplace operator and [18]–[22] for the λ-Laplace operator.

In [23] the Cauchy problem was considered for the equation

$$\begin{gathered}\frac{\partial u}{\partial t} = \operatorname{div}\bigl(u^{m-1}|Du|^{\lambda-1}Du\bigr) - \varepsilon|Du^\nu|^q + \delta u^p,\\ u(x,0) = u_0(x) \in L_1(\mathbb{R}^N),\end{gathered} \tag{1.3}$$

which has a double nonlinearity and a source, where $(x,t) \in S_T = \mathbb{R}^N \times (0,T)$, $T > 0$, and $p > 1$. It was assumed that the initial function is nonnegative and belongs to $L_1(\mathbb{R}^N)$, and also that $\varepsilon, \delta \geqslant 0$, $m + \lambda - 2 > 0$, $\lambda > 0$, $1 < q < \lambda + 1$

and $\nu q > m + \lambda - 1$. It is easy to see that when $\delta = 0$, $m = 1$ this equation is a special case of (1.1); it coincides with (1.1) for $\alpha = 0$, $a(s) = \varepsilon$ and $f(t) = 1$. The authors of [23] discovered a critical condition on q, which ensures that the total mass of the solution decays to zero. Results of this kind had been established previously by many authors (for example, in [11], [13], [15], [17]) for semilinear linear equations. Although they have an independent value, results for an equation containing a damping term were used in [23] to investigate the solvability of the Cauchy problem for an equation with both a source and damping.

Our aim here is to find a condition on the parameters of the problem which determines whether the total mass of the solution decays to zero or not. Moreover, our approaches allow us to find the rate of decay of the total mass. Upper bounds for the maximum of the solution and for the 'radius' of the support of the solution to the Cauchy problem (1.1), (1.2) are obtained. Apparently, our results are sharp, but it would be of interest to find similar lower bounds. We also find the limiting exponent q^* and describe the dependence of the exponent characterizing the order of decrease of the mass of the solution in the special case when nonlinear weights of the form $a(s) = s^\gamma$, $f(t) = t^\beta$ multiply the damping term.

Now we define a weak solution of problem (1.1), (1.2). Assume that conditions (H_1) and (H_2) hold.

Definition 1. A nonnegative function $u(z,t) \in L_{\infty,\mathrm{loc}}(S_T)$ is called a *weak solution of equation* (1.1) in $S_T = \mathbb{R}^{N+M} \times (0,T)$ if for each t, $T > t > 0$, and $R > 0$,

$$u \in C\big((0,T), L_{2,\mathrm{loc}}(\mathbb{R}^{N+M})\big),$$
$$|D_L u|^{\lambda+1}, a(\rho(z))f(t)|D_L u^\nu|^q \in L_{1,\mathrm{loc}}(S_T)$$

and u satisfies (1.1) in the sense of the integral identity

$$\iint_{B_R\times(t,T)} -u\eta_\tau + \big(|D_L u|^{\lambda-1} D_L u\big) D_L \eta + \eta a(\rho(z)) f(\tau)|D_L u^\nu|^q \, dx\, dy\, d\tau = 0, \quad (1.4)$$

where $\eta(x,y,t)$ is an arbitrary smooth function with support in $B_R \times (t,T)$.

Definition 2. A weak solution of equation (1.1) $u(z,t) \in L_{\infty,\mathrm{loc}}(S_T)$ is called a *weak solution of problem* (1.1), (1.2) if

$$\lim_{t\to 0} \int_{\mathbb{R}^{N+M}} u\eta(z)\, dz = \int_{\mathbb{R}^{N+M}} u_0 \eta(z)\, dz \qquad \forall\, \eta(z) \in C_0^\infty(\mathbb{R}^{N+M}). \qquad (1.5)$$

Before we state the main results of this paper, we introduce the requisite notation. Throughout,

$$K = Q(\lambda-1) + \lambda + 1, \qquad K_{1+\theta} = Q(\lambda-1) + (\lambda+1)(1+\theta),$$
$$H = (\lambda+1)(\nu q - 1) - q(\lambda-1) > 0, \qquad Q = N + M(\alpha+1),$$
$$B_R = \{z \in \mathbb{R}^{N+M} \colon \rho(z,0) \leqslant R\},$$

where we have already defined $\rho(z) = (|x|^{2(\alpha+1)} + (\alpha+1)^2|y|^2)^{1/2(\alpha+1)}$. The parameter K is similar to the Barenblatt exponent and is crucial in describing the qualitative properties of solutions of degenerate equations for $\alpha = 0$ (see [3]), Q is the homogeneous dimension of a Carnot-Carathéodory space (see [1] for details). Finally, B_R is a natural extension of the notion of an open ball in a Carnot-Carathéodory space. We will give a more thorough description in § 2. Throughout, we let C, C_i, C_i' denote various positive constants which only depend on the parameters λ, N, M, α, ν and q.

By the *radius* of the support of the solution we mean the quantity

$$Z(t) = \inf\bigl\{\rho : u(\,\cdot\,,t) = 0 \text{ for almost all } (x,y) \in \mathbb{R}^{N+M} \setminus B_\rho\bigr\}.$$

Let $\varphi(s)$ be the inverse function of $a(s)^{\lambda-1}s^H$. The function

$$\omega(t) \equiv \frac{\varphi\bigl(t^{\nu q-\lambda}/f(t)^{\lambda-1}\bigr)}{t^{1/K}} \tag{1.6}$$

will also be of importance to us.

Now we state our main results. We require the following asymptotic assumptions about $\omega(t)$:

$$\lim_{t\to\infty} \omega(t) = 0, \tag{1.7}$$

$$\exists\, C_1 > 0: \quad C_1 \leqslant \lim_{t\to\infty} \omega(t), \tag{1.8}$$

for sufficiently large t

$$\exists\, C_2, \epsilon > 0: \quad C_2 t^\epsilon \leqslant \omega(t). \tag{1.9}$$

If, for example, we take a certain special weight, (1.8) becomes a condition on the limiting exponent q^*.

Theorem 1. *Assume that conditions* (H_1) *and* (H_2) *hold. Then a weak solution* u *of* (1.1), (1.2) *exists, and if* $\omega(t)$ *is the function defined in* (1.6) *and* $\operatorname{supp} u_0 \subset B_{R_0}$, $R_0 < \infty$, *then for sufficiently large* t

$$Z(t) \leqslant C_3 \omega(t) t^{\frac{1}{K}}. \tag{1.10}$$

Remark 1. Whether the solution of (1.1), (1.2) is unique remains an open question. The difficulties in establishing uniqueness are connected with the presence of gradient absorption; they could be overcome, for instance, by means of uniform estimates for the gradient of the solution, but this is a nontrivial problem in its own right.

Remark 2. If (1.7) holds, then it follows from (1.10) that for sufficiently large t the estimate (1.10) is stronger than the estimate for the support of the solution of the same equation without absorption, that is, with $a(s) = f(t) = 0$ (see Proposition 7 below or [8]).

Theorem 2. *Let* u *be a weak solution of* (1.1), (1.2) *and assume that* (H_1) *and* (H_2) *hold, and that* $\operatorname{supp} u_0 \subset B_{R_0}$, $R_0 < \infty$. *Assume that* $0 < \alpha < M(\nu q - 1)/q$. *Then for sufficiently large* t

$$\int_{\mathbb{R}^{M+N}} u\,dz \leqslant C_4 \omega^{\frac{K}{\lambda-1}}(t), \tag{1.11}$$

$$\|u\|_{\infty,\mathbb{R}^{N+M}} \leqslant C_5 \omega^{\frac{\lambda+1}{\lambda-1}}(t) t^{-\frac{Q}{K}}. \tag{1.12}$$

Remark 3. When (1.7) holds, inequality (1.11) ensures that the total mass of the solution $\|u\|_{1,\mathbb{R}^{N+M}}$ decays to zero.

We also consider two cases when we can find an estimate for the mass of the solution without assuming that the initial function has compact support.

Theorem 3. *Let u be a weak solution of (1.1), (1.2) and assume that (H_1) and (H_2) hold. Then for sufficiently large t*

$$\int_{\mathbb{R}^{N+M}} u(t)\,dz \leqslant \int_{\rho>\omega(t)t^{\frac{1}{K}}} u_0\,dz + C_6\omega^{\frac{K}{\lambda-1}}(t), \tag{1.13}$$

provided that $\nu = 1$, $\lambda > 1$ and $0 < \alpha < M(q-1)/q$, On the other hand, if $\lambda = 1$ and $0 < \alpha < M(\nu q-1)/q$, then

$$\int_{\mathbb{R}^{N+M}} u(t)\,dz \leqslant \int_{\rho>\sqrt{t}} u_0\,dz + C_6' \frac{t^{\frac{Q(\nu q-1)+q-2}{2(\nu q-1)}}}{a(\sqrt{t})^{\frac{1}{\nu q-1}} f(t)^{\frac{1}{\nu q-1}}}. \tag{1.14}$$

Remark 4. If (1.7) holds, then (1.13) ensures that the total mass of the solution $\|u\|_{1,\mathbb{R}^{N+M}}$ decays to zero.

For example, it follows from (1.14) that when

$$q < \frac{Q+2}{\nu Q+1},$$

the total mass of the solution tends to zero as $t \to \infty$. We also observe that for $\alpha = 0$, $\nu = 1$, and $a = f = 1$ the critical exponent has the value

$$q = \frac{M+N+2}{M+N+1}$$

(see, for example, [12]).

Theorem 4. *Let u be a weak solution of (1.1), (1.2) and assume that conditions (H_1), (H_2) hold and that $\operatorname{supp} u_0 \subset B_{R_0}$, $R_0 < \infty$. If (1.8) is fulfilled, then for sufficiently large t, $t \geqslant t_0 = \|u_0\|_{1,\mathbb{R}^{N+M}}^{\lambda-1}/R_0^K$,*

$$\int_{\mathbb{R}^{N+M}} u\,dz \leqslant C_7 \left(\int_{t_0}^{t} \frac{a(\tau^{\frac{1}{K}}) f(\tau)}{\tau^{\frac{Q(\nu q-1)+q}{K}}}\,d\tau \right)^{-\frac{1}{\nu q-1}}. \tag{1.15}$$

In particular, it follows from (1.15) that if there exist $C > 0$ and $\varrho \in (0,1)$ such that $C\varrho \leqslant \omega(t) \leqslant C$, then for sufficiently large t, $t \geqslant t_0 = \|u_0\|_{1,\mathbb{R}^{N+M}}^{\lambda-1}/R_0^K$,

$$\int_{\mathbb{R}^{N+M}} u\,dz \leqslant C_8 \left(\ln\left(\frac{t}{t_0}\right)\right)^{-\frac{1}{\nu q-1}}. \tag{1.16}$$

Theorem 5. *Let u be a weak solution of (1.1), (1.2) and assume that (H_1), (H_2) hold and that $\operatorname{supp} u_0 \subset B_{R_0}$, $R_0 < \infty$. If (1.9) is fulfilled, then for sufficiently large t*

$$\int_{\mathbb{R}^{N+M}} u(t)\,dz \geqslant C_9 > 0, \tag{1.17}$$

where C_9 is a positive constant depending only on the parameters of the problem and $\|u_0\|_1$.

Example. Let u be a weak solution of (1.1), (1.2), where $\operatorname{supp} u_0 \subset B_{R_0}$, $R_0 < \infty$. Let $a(s) = s^\gamma$, $\gamma < q$, $f(t) = t^\beta$, $\beta < \frac{\nu q - \lambda}{\lambda}$,

$$q^* = \frac{Q + \gamma + K(1+\beta)}{\nu Q + 1}, \qquad A = \frac{(q^* - q)(\nu Q + 1)}{H + \gamma(\lambda - 1)}. \tag{1.18}$$

Then $\omega(t) = t^{-\frac{A(\lambda-1)}{K}}$. Note that q^* is the critical exponent because for $q < q^*$ (1.7) holds, and the estimates (1.10), (1.11), (1.12) hold in the following form:

$$Z(t) \leqslant C_{10} t^{\frac{1}{K} - \frac{A(\lambda-1)}{K}}, \tag{1.19}$$

$$\int_{\mathbb{R}^{N+M}} u(t)\,dz \leqslant C_{11} t^{-A}, \tag{1.20}$$

$$\|u(t)\|_{\infty,\mathbb{R}^{N+M}} \leqslant C_{12} t^{-\frac{Q}{K} - \frac{A(\lambda+1)}{K}}. \tag{1.21}$$

Note that in the case $q = q^*$ inequality (1.8) becomes an equality, and then the bound (1.16) is unchanged. Thus, the effect that the mass of the solution decays to zero occurs for $q \leqslant q^*$.

For $q^* < q < \lambda + 1$, (1.9) holds, and the total mass is decreasing, but is bounded below by a positive constant, as the estimate in Theorem 5 shows.

The critical exponent we have found coincides with the ones in [23] (for $\alpha = 0$, $a(s)f(t) \equiv \varepsilon > 0$) and [24] (for $\alpha = 0$, $f(t) \equiv 1$).

Section 2 is devoted to auxiliary results, which we state without proof. In §§ 3, 4, 5, 6 and 7 we are concerned with the proofs of Theorems 1, 2, 3, 4 and 5, respectively.

We approach the proofs of Theorems 1–5 using methods developed in [23], [24] and [8].

In proving estimates for the size of the support of the solution we have used ideas from [25]–[27] and also relied strongly on [23], [24] and [8], where these methods were further developed.

§ 2. Auxiliary statements

Above we used the notation

$$D_L u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_N}, |x|^\alpha \frac{\partial u}{\partial y_1}, |x|^\alpha \frac{\partial u}{\partial y_2}, \dots, |x|^\alpha \frac{\partial u}{\partial y_M}\right).$$

The corresponding vector field is called the Baouendi-Grushin vector field. We introduce some further notation:

$$\frac{\partial u}{\partial x_1} = X_1, \quad \frac{\partial u}{\partial x_2} = X_2, \quad \dots, \quad \frac{\partial u}{\partial x_N} = X_N,$$
$$|x|^\alpha \frac{\partial u}{\partial y_1} = Y_1, \quad |x|^\alpha \frac{\partial u}{\partial y_2} = Y_2, \quad \dots, \quad |x|^\alpha \frac{\partial u}{\partial y_M} = Y_M.$$

It is known that if α is an even positive integer, then the C^∞-vector fields $X_1, X_2, \dots, X_N$, $Y_1, \dots, Y_M$ satisfy Hörmander's rank condition

$$\operatorname{rank} \operatorname{Lie}[X_1, X_2, \dots, X_N, Y_1, \dots, Y_M] = M + N.$$

For a fixed set of C^∞-vector fields we can define the Carnot-Carathéodory metric space (see [1], [28]–[30]) with distance defined by

$$\rho(z,z^0)=(|x-x^0|^{2(\alpha+1)}+(\alpha+1)^2|y-y^0|^2)^{1/2(\alpha+1)},\quad z=(x,y),\ z^0=(x^0,y^0).$$

This function satisfies the standard conditions of homogeneity and symmetry. We also have a triangle inequality, but with a constant C greater than 1:

$$\forall\, z^1,z^2,z^3\in\mathbb{R}^{N+M}\qquad \rho(z^1,z^2)\leqslant C\big(\rho(z^1,z^3)+\rho(z^3,z^2)\big).$$

We define an anisotropic dilation, which can be applied to $L_{\lambda,\alpha}$, as follows:

$$\delta_\mu(z)=(\mu x,\mu^{\alpha+1}y),\qquad \mu>0,\quad z=(x,y)\in\mathbb{R}^{N+M}.$$

The distance introduced above is said to be homogeneous because for all $z\in\mathbb{R}^{N+M}$

$$\rho(\delta_\mu(z))=\mu\rho(z).$$

Obviously, making a change of variables in the Lebesgue integral we obtain

$$d\circ\delta_\mu(x,y)=\mu^Q\,dx\,dy,\qquad Q=N+(\alpha+1)M,$$

where Q is the homogeneous dimension of the Carnot-Carathéodory space.

Let $\Omega\subset\mathbb{R}^{N+M}$ be an arbitrary bounded open subset of R^{N+M} with sufficiently smooth boundary. For $1\leqslant p<\infty$ we define the space $\mathfrak{L}_{1,p}(\Omega)$ as the closure of $C^\infty(\bar\Omega)$ with respect to the norm

$$\|f\|_{\mathfrak{L}^{1,p}(\Omega)}=\left(\int_\Omega(|D_Lf|^p+|f|^p)\,dx\,dy\right)^{\frac1p},$$

which is equivalent to the norm

$$\left(\int_\Omega|D_Lf|^p\,dx\,dy\right)^{\frac1p}+\left(\int_\Omega|f|^r\,dx\,dy\right)^{\frac1r}$$

for $0<r\leqslant p$.

We set $\mathring{\mathfrak{L}}_{1,p}(\Omega)$ to be the subspace of $\mathfrak{L}_{1,p}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm of $\mathfrak{L}_{1,p}(\Omega)$. The spaces $\mathfrak{L}_{1,p}(\Omega)$ and $\mathring{\mathfrak{L}}_{1,p}(\Omega)$ belong to the class of Carnot-Carathéodory spaces. Sobolev-type embedding theorems and multiplicative inequalities of the Gagliardo-Nirenberg type have been fairly well investigated for these. The reader interested in the theory of Carnot-Carathéodory spaces is referred to the survey [1], where further references are available.

Proposition 1 (a Gagliardo-Nirenberg type inequality; see [1]). *For each function $f\in\mathfrak{L}_{1,\lambda+1}(\mathbb{R}^{N+M})$,*

$$\int_{\mathbb{R}^{N+M}}|f|^{\lambda+1}\,dz\leqslant C_{14}\left(\int_{\mathbb{R}^{N+M}}|D_Lf|^{\lambda+1}dz\right)^{\beta_1}\left(\int_{\mathbb{R}^{N+M}}|f|^{\beta_2}\,dz\right)^{\frac{(1-\beta_1)(\lambda+1)}{\beta_2}}. \tag{2.1}$$

Here $\beta_1=\frac{Q(\lambda-1)}{K_{1+\theta}}$, $\beta_2>0$.

Proposition 2 (a Poincaré-type inequality; see [1]). *If* $\omega \in \mathring{\mathfrak{L}}_{1,q}(\mathbb{R}^{N+M})$ *then*

$$\int_\Omega |\omega|^q \, dz \leqslant C_{15} \operatorname{mes}\{\operatorname{supp}\omega\}^{\frac{q}{Q}} \int_\Omega |D_L\omega|^q \, dz. \tag{2.2}$$

The next result is not a consequence of [2], but can easily be proved by similar methods.

Proposition 3 (a Hardy-type inequality). *Let $a(s)$ be a continuous monotonically increasing function such that $s^q/a(s)$ is also monotonically increasing. Then for each function $\omega \in \mathring{\mathfrak{L}}_{1,q}(\Omega)$*

$$\int_\Omega a(\rho(z)) \frac{|D_L\rho|^q}{\rho^q} |\omega|^q \, dz \leqslant \left(\frac{q}{Q}\right)^q \int_\Omega a(\rho)|D_L\omega|^q \, dz. \tag{2.3}$$

For each $\omega \in \mathfrak{L}_{1,q}(\mathbb{R}^{N+M})$

$$\int_{\mathbb{R}^{N+M}} a(\rho(z)) \frac{|D_L\rho|^q}{\rho^q} |\omega|^q \, dz \leqslant \left(\frac{q}{Q}\right)^q \int_{\mathbb{R}^{N+M}} a(\rho)|D_L\omega|^q \, dz. \tag{2.4}$$

Proposition 4 (a change of variables). *Let*

$$\rho(z,0) = \left(|x|^{2(\alpha+1)} + (\alpha+1)^2|y|^2\right)^{\frac{1}{2(\alpha+1)}}.$$

Consider the change of the variables (x, y) to (ρ, φ) using the formulae

$$\begin{cases} x_1 = \rho \cos^{\frac{1}{\alpha+1}} \varphi_{M+N-1} \cos^{\frac{1}{\alpha+1}} \varphi_{M+N-2} \cos^{\frac{1}{\alpha+1}} \varphi_2 \cos^{\frac{1}{\alpha+1}} \varphi_1, \\ x_2 = \rho \cos^{\frac{1}{\alpha+1}} \varphi_{M+N-1} \cos^{\frac{1}{\alpha+1}} \varphi_{M+N-2} \cos^{\frac{1}{\alpha+1}} \varphi_2 \sin^{\frac{1}{\alpha+1}} \varphi_1, \\ \dots \\ x_M = \rho \cos^{\frac{1}{\alpha+1}} \varphi_{M+N-1} \cos^{\frac{1}{\alpha+1}} \varphi_{M+N-2} \cos^{\frac{1}{\alpha+1}} \varphi_M \sin^{\frac{1}{\alpha+1}} \varphi_{M-1}, \\ y_1 = \dfrac{\rho^{\alpha+1}}{\alpha+1} \cos\varphi_{M+N-1} \cos\varphi_{M+N-2} \cos\varphi_{M+1} \sin\varphi_M, \\ y_2 = \dfrac{\rho^{\alpha+1}}{\alpha+1} \cos\varphi_{M+N-1} \cos\varphi_{M+N-2} \cos\varphi_{M+2} \sin\varphi_{M+1}, \\ \dots \\ y_N = \dfrac{\rho^{\alpha+1}}{\alpha+1} \sin\varphi_{M+N-1}. \end{cases} \tag{2.5}$$

This transformation has Jacobian

$$J(\rho,\varphi) = \frac{\partial(x,y)}{\partial(\rho,\varphi)} = \frac{\cos^{\frac{1}{\alpha+1}} \varphi_2 \cos^{\frac{2}{\alpha+1}} \varphi_3 \cdots \cos^{\frac{M-2}{\alpha+1}} \varphi_{M-1} \cos^{\frac{M-1}{\alpha+1}} \varphi_M \cos^{\frac{M}{\alpha+1}} \varphi_{M+1}}{(\cos\varphi_1 \cdots \cos\varphi_M)^{\frac{\alpha}{\alpha+1}}} \times \frac{\cos^{\frac{M}{\alpha+1}+1} \varphi_{M+2} \cdots \cos^{\frac{M}{\alpha+1}+N-2} \varphi_{M+N-1}}{(\sin\varphi_1 \cdots \sin\varphi_{M-1})^{\frac{\alpha}{\alpha+1}}}. \tag{2.6}$$

In what follows we require the following iterative lemma.

Proposition 5 (see [31], Lemma 5.6 for $b > 1$). *Let y_h, $h = 0, 1, 2, \dots$, be a sequence of nonnegative numbers satisfying the recurrence relation*

$$y_{h+1} \leqslant C_{16} b^h y_h^{1+\varepsilon}, \qquad h = 0, 1, 2, \dots, \tag{2.7}$$

with some positive constants C_{16}, $\varepsilon > 0$ and $b > 1$. If

$$y_0 \leqslant C_{16}^{-\frac{1}{\varepsilon}} b^{-\frac{1}{\varepsilon^2}}, \tag{2.8}$$

then $y_h \to 0$ as $h \to \infty$.

Proposition 6 (see [32], Ch. I, Lemma 4.3). *Let y_h, $h = 0, 1, 2, \dots$, be a sequence of nonnegative numbers satisfying the recurrence relation*

$$y_h \leqslant C_{17} b^h y_{h+1}^{1-\varepsilon}, \qquad h = 0, 1, 2, \dots, \tag{2.9}$$

with some positive constants C_{17}, $\varepsilon > 0$ and $b < 1$. If

$$y_h \leqslant C_{18} \text{ for each } h, \tag{2.10}$$

then $y_0^{\varepsilon} \leqslant C_{19}$.

For a locally integrable function $f(z)$ we set

$$|||f|||_r = \sup_{\rho \geqslant r} \biggl(\rho^{-\frac{K}{\lambda - 1}} \int_{B_\rho} |f| \, dx \, dy \biggr).$$

Since the solution of (1.1), (1.2) is a subsolution of the problem without damping (with $a(\rho) = f(t) = 0$), we have the following result (see [8]).

Proposition 7. *Let $u_0 \in L_{1,\mathrm{loc}}(R^{N+M})$ and*

$$|||u_0|||_r < \infty$$

for some fixed $r > 0$. Then there exist positive constants C_{20}, C_{21}, C_{22} and C_{23}, which depend only on the parameters λ, N, M and α, such that the (generalized) solution of problem (1.1), (1.2), for all $0 < t < T_r$ and $R > r$, where $T_r = C_{20} |||u_0|||_r^{-\frac{1}{\lambda - 1}}$, satisfies the estimates

$$|||u(t)|||_r \leqslant C_{21} |||u_0|||_r, \tag{2.11}$$

$$\|u(t)\|_{L_\infty(B_R)} \leqslant C_{22} t^{-\frac{Q}{K}} R^{\frac{\lambda+1}{\lambda-1}} |||u_0|||_r^{\frac{\lambda+1}{K}}. \tag{2.12}$$

Let $u_0 \in L_1(R^{N+M})$ and $\operatorname{supp} u_0 \subset B_{R_0}$, $R_0 < \infty$. Then for each $t > 0$

$$Z(t) = \inf\bigl\{\rho : u(t) = 0 \text{ for a.e. } (x, y) \in R^{N+M} \setminus B_\rho\bigr\} \leqslant 4R_0 + C_{23} t^{\frac{1}{K}} \|u_0\|_1^{\frac{\lambda - 1}{K}}. \tag{2.13}$$

Proposition 8 (see [8], Lemma 3.3). *Under the assumptions of Proposition 7, for all $t \in (0, T)$ and $R \geqslant r$*

$$\|u(t)\|_{\infty, B_R \times (t/2, t)} \leqslant \frac{C_{24}}{t^{\frac{Q}{K}}} \biggl(\fint_{t/4}^{t} \int_{B_{2R}} u \, dx \biggr)^{\frac{\lambda+1}{K}}. \tag{2.14}$$

§ 3. The proof of Theorem 1

We start by proving that problem (1.1), (1.2) is solvable. We look at the sequence of problems

$$\frac{\partial u_n}{\partial t} = \operatorname{div}_L\bigl(|D_L u_n|^{\lambda-1} D_L u_n\bigr) - \min\bigl\{a(\rho(z)) f(t) |D_L u_n^{\nu}|^q, n\bigr\}, \tag{3.1}$$

$$(z,t) \in S_{T,n} = B_n \times (0,T), \qquad B_n \equiv \{z \in \mathbb{R}^{N+M} \colon \rho(z,0) \leqslant n\},$$

$$u_n(z,t) = 0 \quad \text{on } \partial B_n \times (0,T), \tag{3.2}$$

$$u_n(z,0) = u_{0n}(z) \in C_0^{\infty}(B_n), \quad u_{0n}(z) \geqslant 0, \quad \text{where } \|u_{0n}\|_{1,\mathbb{R}^{N+M}} \leqslant \|u_0\|_{1,\mathbb{R}^{N+M}}, \tag{3.3}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^{N+M}} u_{0n}\eta(z)\,dz = \int_{\mathbb{R}^{N+M}} u_0\eta(z)\,dz \qquad \forall\, \eta(z) \in C_0^{\infty}(\mathbb{R}^{N+M}).$$

Using Lions' monotonicity and compactness method [33], for each fixed n we can prove that a solution u_n exists in the class $C(0,T;L_2(B_n)) \cap L_{\lambda+1}(0,T;\mathfrak{L}_{1,\lambda+1}(B_n))$. Now we show that the solutions u_n of problem (3.1)–(3.3) converge to the solution of the original Cauchy problem. To do this, we must pass to the limit as $n \to \infty$ in the integral identity for u_n. Note that, as the equation contains a damping term, the limiting procedure is nontrivial. Below we use the scheme set out in [34] and [35]. To do this we need the gradients of the solutions to converge almost everywhere. We shall prove that the gradients converge strongly in $L_{\lambda+1}(0,T;\mathfrak{L}_{1,\lambda+1,\mathrm{loc}}(\mathbb{R}^{N+M}))$. We pick an arbitrary compact subset $\mathfrak{K}$ of S_T. There exists n_0 such that $\mathfrak{K} \subset (0,T) \times B_{n_0}$. First of all we claim that for all u_n

$$\int_{B_{n_0}} u_n(t)\,dz \leqslant \int_{B_{n_0}} u_{0n}\,dz.$$

We multiply (3.1) by the function $u_n/(u_n+\varepsilon)$ with compact support, where $\varepsilon > 0$. Integrating by parts we obtain

$$\int_{B_{n_0}} \int_0^{u_n(t)} \frac{s}{s+\varepsilon}\,ds\,dz + \varepsilon \int_0^t \int_{B_{n_0}} \frac{|D_L u_n|^{\lambda+1}}{(u_n+\varepsilon)^2}\,dz\,d\tau + \int_0^t \int_{B_{n_0}} \frac{a(z) f(\tau) |D_L u_n^{\nu}|^q u_n}{u_n+\varepsilon}\,dz\,d\tau \leqslant \int_{B_{n_0}} \int_0^{u_{n0}} \frac{s}{s+\varepsilon}\,ds\,dz.$$

Passing to the limit as $\varepsilon \to 0$ we obtain the inequality

$$\int_{B_{n_0}} u_n(t)\,dz + \int_0^t \int_{B_{n_0}} a(z) f(\tau) |D_L u_n^{\nu}|^q\,dz\,d\tau \leqslant \int_{B_{n_0}} u_{0n}\,dz, \tag{3.4}$$

which will also be useful in what follows. As follows from Proposition 8 and (3.4), the u_n are uniformly bounded:

$$\|u_n(\,\cdot\,,t)\|_{\infty,B_n} \leqslant C t^{-\frac{Q}{K}} \|u_0\|_{1,\mathbb{R}^{N+M}}^{\frac{\lambda+1}{K}} \qquad \forall\, t > 0. \tag{3.5}$$

Obviously, the addition of a positive damping term to the equation does not take the solution outside the Hölder class. Since the Hölder constants depends on $\|u_n\|_\infty$, which is independent of n in view of (3.5), the sequence $\{u_n\}$ is locally Hölder continuous on the compact set $\mathfrak{K}$ for all the n simultaneously. Hence we can extract a subsequence converging uniformly to u which we also denote by $\{u_n\}$, $n \geqslant n_0$. Furthermore, we have another asymptotic estimate for $\{u_n\}$, $n \geqslant n_0$:

$$\sup_{0<t<T} \int_{B_{n_0}} u_n^2(t)\,dz + \iint_{S_{T,n_0}} |D_L u_n|^{\lambda+1}\,dz\,d\tau \\ + \iint_{S_{T,n_0}} \min\bigl\{a(\rho(z))f(t)|D_L u_n^{\nu}|^q, n\bigr\} u_n\,dz\,d\tau \leqslant \int_{B_{n_0}} u_{0n}^2\,dz. \tag{3.6}$$

Hence

$$\iint_{\mathfrak{K}} |D_L u_n|^{\lambda+1}\,dz\,d\tau \leqslant C(\mathfrak{K}, u_0). \tag{3.7}$$

Using the uniform convergence of $\{u_n\}$, we shall prove that

$$\{D_L u_n\} \in L_{\lambda+1}(0, T; \mathfrak{L}_{1,\lambda+1}(\mathfrak{K}))$$

is a Cauchy sequence. We take two arbitrary elements of $\{u_n\}$, u_j and u_k, $k, j > n_0$. Then $u_j - u_k$ satisfies

$$\frac{\partial(u_j - u_k)}{\partial t} - \bigl(\operatorname{div}_L(|D_L u_j|^{\lambda-1} D_L u_j) - \operatorname{div}_L(|D_L u_k|^{\lambda-1} D_L u_k)\bigr) \\ + \min\bigl\{a(\rho(z))f(t)|D_L u_j^{\nu}|^q, j\bigr\} - \min\bigl\{a(\rho(z))f(t)|D_L u_k^{\nu}|^q, k\bigr\} = 0. \tag{3.8}$$

Multiply this equation by $(u_j - u_k)\xi^2$; here $\xi(z,\tau)$ is defined in the cylinder S_{T,n_0} and $\xi(z,\tau) = \zeta(z)\eta(\tau)$, where ζ is a smooth cutoff function associated with the ball B_{n_0}, that is,

$$0 \leqslant \xi \leqslant 1, \qquad \xi = 1 \quad \text{in } B_{n_0/2}, \quad \xi = 0 \quad \text{outside } B_{n_0},$$

and

$$|D_L \zeta(z)| \leqslant \frac{C}{n_0}.$$

Further, $\eta(\tau)$ is a smooth function in the interval $(0, T)$ such that

$$0 \leqslant \eta \leqslant 1, \qquad \eta(\tau) = 0 \quad \text{for } 0 \leqslant \tau \leqslant \frac{t}{2}, \quad \eta(\tau) = 1 \quad \text{for } t \leqslant \tau \leqslant T,$$

and

$$0 \leqslant \eta_\tau \leqslant \frac{C}{t}.$$

As the operator

$$A(u) = -\operatorname{div}_L(|D_L u|^{\lambda-1} D_L u)$$

is strongly monotone, from Hölder's inequality we obtain

$$\begin{aligned}
\iint_{S_{T,n_0}} |D_L u_j - D_L u_k|^{\lambda+1} \xi^2 \, dz \, d\tau &\leqslant C \iint_{S_{T,n_0}} |u_j - u_k|^2 \xi |\xi_\tau| \, dz \, d\tau \\
&\quad + C \iint_{S_{T,n_0}} \big(|D_L u_j|^\lambda + |D_L u_k|^\lambda\big) \xi |D_L \xi| \, |u_j - u_k| \, dz \, d\tau \\
&\quad + C \iint_{S_{T,n_0}} \big|a(\rho(z)) f(t) |D_L u_j^\nu|^q + a(\rho(z)) f(t) |D_L u_k^\nu|^q\big| \, |u_j - u_k| \xi^2 \, dz \, d\tau \\
&\leqslant C \iint_{S_{T,n_0}} |u_j - u_k|^2 \xi |\xi_\tau| \, dz \, d\tau \\
&\quad + C \Big(\iint_{S_{T,n_0}} |D_L u_j|^{\lambda+1} \, dz \, d\tau \Big)^{\frac{\lambda}{\lambda+1}} \Big(\iint_{S_{T,n_0}} |u_j - u_k|^{\lambda+1} \xi^{\lambda+1} |D_L \xi|^{\lambda+1} \, dz \, d\tau \Big)^{\frac{1}{\lambda+1}} \\
&\quad + C \Big(\iint_{S_{T,n_0}} |D_L u_k|^{\lambda+1} \, dz \, d\tau \Big)^{\frac{\lambda}{\lambda+1}} \Big(\iint_{S_{T,n_0}} |u_j - u_k|^{\lambda+1} \xi^{\lambda+1} |D_L \xi|^{\lambda+1} \, dz \, d\tau \Big)^{\frac{1}{\lambda+1}} \\
&\quad + \Big(\int_{B_{n_0}} u_{0j} \, dz + \int_{B_{n_0}} u_{0k} \, dz \Big) \sup_{B_{n_0} \times (t/2, T)} |u_j - u_k|^2 \to 0 \quad \text{as } j, k \to \infty.
\end{aligned} \tag{3.9}$$

In passing to the limit, alongside (3.4) and (3.7), we have used the uniform convergence of $\{u_n\}$. Now we can pick a subsequence for which

$$\begin{aligned}
D_L u_n \to D_L u \quad &\text{a.e. on each compact set } \mathfrak{K}, \\
D_L u_n \to D_L u \quad &\text{strongly in } L_{\lambda+1}(0, T; \mathfrak{L}_{1,\lambda+1}(B_{n_0})).
\end{aligned}$$

We can pass to the limit in the integral identity for u_n. Thus we can take it as established that u_n converges uniformly to the solution of (1.1) on each compact subset of S_T. Passing to the limit as $n \to \infty$ we see that estimates (3.4) and (3.5) also hold for u. To prove that the weak solution of the equation is a weak solution of the Cauchy problem (1.1), (1.2), we must track the behaviour of u as $t \to 0$. Obviously, (3.5) does not guarantee uniform boundedness as $t \to 0$, so we cannot use the uniform convergence of $\{u_n\}$ as we did above. We start with an auxiliary estimate. We multiply (1.1) by $t^\beta u^\theta \xi^{\lambda+1}(z)$; we shall choose $R, \beta, \theta > 0$ in what follows. As a cutoff function we take a function in $C_0^1(\mathbb{R}^{N+M})$ such that

$$\xi(z) = \begin{cases} 0 & \text{for } \rho(z) > R, \\ 1 & \text{for } \rho(z) < \dfrac{R}{2}, \end{cases} \qquad |D_L \xi| \leqslant C \frac{2}{R}. \tag{3.10}$$

Integrating over $B_R \times (0,t)$ in the standard way we obtain

$$\begin{aligned}\frac{1}{\theta+1}\int_{B_R} t^\beta \xi^{\lambda+1}(z)u^{\theta+1}(z,t)\,dz + \theta\iint_{S_{t,R}} \tau^\beta\xi^{\lambda+1}u^{\theta-1}|D_L u|^{\lambda+1}\,dz\,d\tau \\ + (\lambda+1)\iint_{S_{t,R}} \tau^\beta\xi^\lambda u^\theta |D_L u|^{\lambda-1}D_L u D_L\xi\,dz\,d\tau \\ + \iint_{S_{t,R}} a(\rho(z))f(\tau)|D_L u^\nu|^q\tau^\beta\xi^{\lambda+1}u^\theta\,dz\,d\tau \\ = \frac{\beta}{\theta+1}\iint_{S_{t,R}} \tau^{\beta-1}\xi^{\lambda+1}(z)u^{\theta+1}(z,t)\,dz\,d\tau. \qquad (3.11)\end{aligned}$$

Using Young's inequality and (3.5) we find an estimate

$$\begin{aligned}\frac{1}{\theta+1}\int_{B_R} t^\beta \xi^{\lambda+1}(z)u^{\theta+1}(z,t)\,dz + \theta\iint_{S_{t,R}} \tau^\beta\xi^{\lambda+1}u^{\theta-1}|D_L u|^{\lambda+1}\,dz\,d\tau \\ + \iint_{S_{t,R}} a(\rho(z))f(\tau)|D_L u^\nu|^q t^\beta\xi^{\lambda+1}u^\theta\,dz\,d\tau \\ \leqslant C\iint_{S_{t,R}} \tau^{\beta-1}u^{\theta+1}(1+\tau R^{-(\lambda+1)}u^{\lambda-1})\,dz\,d\tau \leqslant C\iint_{S_{t,R}} \tau^{\beta-1}u^{\theta+1}\,dz\,d\tau. \qquad (3.12)\end{aligned}$$

Thus for $\theta = (\lambda-1)/\lambda$, $0<\beta<1/\lambda$ and fixed R we have obtained

$$\iint_{S_{t,R}} \tau^\beta u^{-\frac{1}{\lambda}}|D_L u|^{\lambda+1}\,dz\,d\tau \leqslant C\iint_{S_{t,2R}} \tau^{\beta-1}u^{\frac{\lambda-1}{\lambda}+1}\,dz\,d\tau. \qquad (3.13)$$

To estimate the last integral we can use Proposition 8 and inequality (3.5):

$$\begin{aligned}\iint_{S_{t,R}} \tau^{\beta-1}u^{\frac{\lambda-1}{\lambda}+1}\,dz\,d\tau \leqslant \left(\sup_{0<\tau<t}\int_{B_R} u\,dz\right)\int_0^t \tau^{\beta-1}\|u\|_{\infty,B_R}^{\frac{\lambda-1}{\lambda}}\,d\tau \\ \leqslant Ct^{\beta-\frac{Q(\lambda-1)}{K\lambda}}\left(\sup_{0<\tau<t}\int_{B_R} u\,dz\right)^{1+\frac{(\lambda+1)(\lambda-1)}{K\lambda}} \leqslant Ct^{\beta-\frac{Q(\lambda-1)}{K\lambda}}\|u_0\|_1^{1+\frac{(\lambda+1)(\lambda-1)}{K\lambda}}. \qquad (3.14)\end{aligned}$$

Now we can prove directly that u solves the original problem. For an arbitrary compact subset $\mathfrak{K}$ of $\mathbb{R}^{N+M}$ we can find $R>0$ such that $\mathfrak{K}\subset B_{R/2}$. Multiplying (1.1) by the cutoff function ξ defined in (3.10) and integrating over $B_R\times(0,t)$ in the standard way, we obtain

$$\begin{aligned}\int_{B_R} u(t)\xi\,dz + \iint_{S_{t,R}} |D_L u|^{\lambda-1}D_L u D_L\xi\,dz\,d\tau \\ + \iint_{S_{t,R}} a(\rho(z))f(\tau)|D_L u^\nu|^q\xi\,dz\,d\tau = \int_{B_R} u_0\xi\,dz, \\ \left|\int_{B_R} u(t)\xi\,dz - \int_{B_R} u_0\xi\,dz\right| \\ \leqslant \frac{C}{R}\iint S_{t,R}|D_L u|^\lambda\,dz\,d\tau + \iint_{S_{t,R}} a(\rho(z))f(\tau)|D_L u^\nu|^q\xi\,dz\,d\tau \equiv \frac{C}{R}I_1 + I_2. \qquad (3.15)\end{aligned}$$

We find estimates for these two integrals:

$$\begin{aligned}
I_1 &= \iint_{S_{t,R}} |D_L u|^\lambda \, dz\, d\tau \\
&\leqslant \left(\iint_{S_{t,R}} \tau^\beta u^{-\frac{1}{\lambda}} |D_L u|^{\lambda+1} \, dz\, d\tau \right)^{\frac{\lambda}{\lambda+1}} \left(\iint_{S_{t,R}} \tau^{-\beta\lambda} u \, dz\, d\tau \right)^{\frac{1}{\lambda+1}} \\
&\leqslant \left(C t^{\beta - \frac{Q(\lambda-1)}{K\lambda}} \left(\sup_{0<\tau<t} \int_{B_{2R}} u\, dz \right)^{1+\frac{(\lambda+1)(\lambda-1)}{K\lambda}} \right)^{\frac{\lambda}{\lambda+1}} \left(t^{1-\beta\lambda} \left(\sup_{0<\tau<t} \int_{B_{2R}} u \, dz \right) \right)^{\frac{1}{\lambda+1}} \\
&\leqslant C t^{\frac{1}{K}} \left(\sup_{0<\tau<t} \int_{B_{2R}} u\, dz \right)^{1+\frac{\lambda-1}{K}} \leqslant \frac{C}{R} t^{\frac{1}{K}} \|u_0\|_1^{1+\frac{\lambda-1}{K}} \to 0 \quad \text{as } t \to 0.
\end{aligned}$$

The integral I_2 is bounded above (see (3.5)) and converges to zero with t. Hence (1.2) is proved.

We now turn to the proof of (1.10).

Lemma 3.1. *Let u be a nonnegative solution of equation (1.1) such that $\operatorname{supp} u_0 \subset B_{R_0}$, $R_0 < \infty$; then the following inequality holds for any $0 \leqslant t_1 \leqslant t_2$ and $R > 2R_0$, $R_i = R(1-2^{-i-1})$, $\tilde R_i = (R_i + R_{i+1})/2$, $R_i < \tilde R_i$, $U_i = \{\rho(z) > R_i\}$, $\tilde U_i = \{\rho(z) > \tilde R_i\}$:*

$$\begin{aligned}
\sup_{t_1<\tau<t_2} \int_{\tilde U_i} u^{\theta+1}\, dz + \int_{t_1}^{t_2} \int_{\tilde U_i} u^{\theta-1} |D_L u|^{\lambda+1} \, dz\, d\tau \\
+ \int_{t_1}^{t_2} \int_{\tilde U_i} a(\rho(z)) f(\tau) u^\theta |D_L u^\nu|^q \, dz\, d\tau \leqslant C \frac{2^{i(\lambda+1)}}{R^{\lambda+1}} \int_{t_1}^{t_2} \int_{U_i \setminus \tilde U_i} u^{\lambda+\theta} \, dz\, d\tau.
\end{aligned} \tag{3.16}$$

Proof. We multiply the equation by $u^\theta \xi_i^{\lambda+1}(z)$; we shall choose $\theta > 0$ in what follows. We take a cutoff function in the space $C^1(\mathbb{R}^{N+M})$ such that

$$\xi_i(z) = \begin{cases} 0 & \text{for } \rho(z) < R_i, \\ 1 & \text{for } \rho(z) > \tilde R_i, \end{cases} \qquad |D_L \xi_i| \leqslant C \frac{2^i}{R}.$$

Integrating over $\tilde U_i \times (0, t_2)$ in the standard way and using Young's inequality we obtain the required result. Note that the inequality does not contain the term $\sup \int_{\tilde U_i} u_0^{\theta+1}\, dz$ because we have integrated over a domain disjoint from the support of u_0.

Lemma 3.2. *Let u be a nonnegative solution of (1.1) such that $\operatorname{supp} u_0 \subset B_{R_0}$, $R_0 < \infty$; then for any $0 \leqslant t$ and $R > 2R_0$, $R_i = R(1-2^{-i-1})$, $\tilde R_i = (R_i + R_{i+1})/2$, $R_i < \tilde R_i$, $U_i = \{\rho(z) > R_i\}$, $\tilde U_i = \{\rho(z) > \tilde R_i\}$,*

$$y_{i+1} \leqslant C \frac{2^{i(\lambda+1)}}{R^{\lambda+1}} t^{\frac{(1+\theta)(\lambda+1)}{K_{1+\theta}}} y_i^{1+\frac{(\lambda-1)(\lambda+1)}{K_{1+\theta}}}, \tag{3.17}$$

where

$$y_{i+1} \equiv \sup_{0<\tau<t} \int_{\tilde{U}_{i+1}} u^{\theta+1}\, dz + \int_0^t \int_{\tilde{U}_{i+1}} u^{\theta-1} |D_L u|^{\lambda+1}\, dz\, d\tau \\ + \int_0^t \int_{\tilde{U}_{i+1}} a(\rho(z)) f(\tau) u^\theta |D_L u^\nu|^q\, dz\, d\tau + \frac{2^{i(\lambda+1)}}{R^{\lambda+1}} \int_0^t \int_{U_{i+1}\setminus \tilde{U}_{i+1}} u^{\lambda+\theta}\, dz\, d\tau.$$

Proof. Let

$$\bar{\xi}_{i+1}(z) = \begin{cases} 0 & \text{for } \rho(z) < \tilde{R}_i, \\ 1 & \text{for } \rho(z) > R_{i+1}, \end{cases} \qquad |D_L \bar{\xi}_i| \leqslant C\frac{2^i}{R}.$$

In (2.1) we set $f = v_{i+1} = (u\bar{\xi}_{i+1}(z))^\omega$, $\omega = (\lambda+\theta)/(\lambda+1)$, $\beta_1 = Q(\lambda-1)/K_{1+\theta}$, $\beta_2 = (\theta+1)/\omega > 1$, and integrate (2.1) from 0 to t with respect to $d\tau$:

$$\int_0^t \int_{U_{i+1}\setminus\tilde{U}_{i+1}} u^{\lambda+\theta}\, dz\, d\tau = \int_0^t \int_{U_{i+1}\setminus\tilde{U}_{i+1}} v_{i+1}^{\lambda+1}\, dz\, d\tau \\ \leqslant C \int_0^t \left(\int_{\mathbb{R}^{N+M}} |D_L v_{i+1}|^{\lambda+1}\, dz\right)^{\beta_1} \left(\int_{\tilde{U}_i} v_{i+1}^{\beta_2}\, dz\right)^{\frac{(1-\beta_1)(\lambda+1)}{\beta_2}} d\tau \\ \leqslant C t^{1-\beta_1} \left(\int_0^t \int_{\mathbb{R}^{N+M}} |D_L v_{i+1}|^{\lambda+1}\, dz\, d\tau\right)^{\beta_1} \left(\sup_{0<\tau<t} \int_{\tilde{U}_i} v_{i+1}^{\beta_2}\, dz\right)^{\frac{(1-\beta_1)(\lambda+1)}{\beta_2}} \\ \leqslant C t^{1-\beta_1} y_i^{\beta_1 + \frac{\lambda+1}{\beta_2}(1-\beta_1)} = C t^{1-\beta_1} y_i^{1+(1-\beta_1)(\frac{\lambda+1}{\beta_2}-1)}.$$

Using Lemma 3.1 we obtain the required result.

Now we use the damping term of the equation to derive an auxiliary recursive inequality.

Lemma 3.3. *Under the assumptions of Lemma 3.2 let*

$$Z(t) = \inf\{\rho : u(\,\cdot\,, t) = 0 \textit{ for almost all } z \in \mathbb{R}^{N+M} \setminus B_\rho\}.$$

Then

$$y_{i+1} \leqslant C \frac{2^{i(\lambda+1)}}{R^{\lambda+1}} R^{Q\frac{\nu q-\lambda}{\nu q+\theta} - q\frac{\lambda+\theta}{\nu q+\theta}} \frac{t^{\frac{\nu q-\lambda}{\nu q+\theta}}}{f(t)^{\frac{\lambda+\theta}{\nu q+\theta}}} \left(\frac{Z(t)^{2q}}{a(Z(t))}\right)^{\frac{\lambda+\theta}{\nu q+\theta}} y_i^{\frac{\lambda+\theta}{\nu q+\theta}}. \tag{3.18}$$

Proof. Let $\omega = u^{\frac{\nu q+\theta}{q}}$. It is clear from Proposition 7 that $Z(t)$ is finite for each t. Hence $\omega \in \mathring{\mathfrak{L}}_{1,p}(B_{Z(t)})$ for each $t > 0$. Using the notation and results from Lemmas 3.1 and 3.2 alongside (2.2), Hölder's inequality and the properties of the weight functions $a(\rho)$ and $f(t)$ we obtain

$$\begin{aligned}
y_{i+1} &\leqslant C\frac{2^{i(\lambda+1)}}{R^{\lambda+1}}\int_0^t\int_{U_{i+1}\setminus\tilde{U}_{i+1}} u^{\lambda+\theta}\,dz\,d\tau \\
&\leqslant C\frac{2^{i(\lambda+1)}}{R^{\lambda+1}}\int_0^t\Big(\int_{B_{Z(t)}\setminus B_{R_{i+1}}}\omega^q\,dz\Big)^{\frac{\lambda+\theta}{\nu q+\theta}}\operatorname{mes}\{U_{i+1}\setminus\tilde{U}_{i+1}\}^{\frac{\nu q-\lambda}{\nu q+\theta}}\,d\tau \\
&\leqslant C\frac{2^{i(\lambda+1)}}{R^{\lambda+1}}R_{i+1}^{-q\frac{\lambda+\theta}{\nu q+\theta}} \\
&\quad\times\int_0^t\Big(\frac{Z(t)^{2q}}{a(Z(t))f(\tau)}\int_{B_{Z(t)}\setminus B_{R_{i+1}}} a(\rho(z))f(\tau)|D_L\omega|^q\,dz\Big)^{\frac{\lambda+\theta}{\nu q+\theta}}\tilde{R}_i^{Q\frac{\nu q-\lambda}{\nu q+\theta}}\,d\tau \\
&\leqslant C\frac{2^{i(\lambda+1)}}{R^{\lambda+1}}R^{Q\frac{\nu q-\lambda}{\nu q+\theta}-q\frac{\lambda+\theta}{\nu q+\theta}}\Big(\frac{Z(t)^{2q}}{a(Z(t))}\Big)^{\frac{\lambda+\theta}{\nu q+\theta}}y_i^{\frac{\lambda+\theta}{\nu q+\theta}}\Big(\int_0^t\Big[\frac{1}{f(\tau)}\Big]^{\frac{\lambda+\theta}{\nu q-\lambda}}\,d\tau\Big)^{\frac{\nu q-\lambda}{\nu q+\theta}} \\
&\leqslant C\frac{2^{i(\lambda+1)}}{R^{\lambda+1}}R^{Q\frac{\nu q-\lambda}{\nu q+\theta}-q\frac{\lambda+\theta}{\nu q+\theta}}\frac{t^{\frac{\nu q-\lambda}{\nu q+\theta}}}{f(t)^{\frac{\lambda+\theta}{\nu q+\theta}}}\Big(\frac{Z(t)^{2q}}{a(Z(t))}\Big)^{\frac{\lambda+\theta}{\nu q+\theta}}y_i^{\frac{\lambda+\theta}{\nu q+\theta}}.
\end{aligned}$$

The choice of θ is important here. We take θ small enough to ensure that

$$-\mu\frac{\lambda+\theta}{\nu q-\lambda}+1>0.$$

Lemma 3.4. *If $\{y_i\}$ is a sequence such that*

$$y_{i+1}\leqslant C2^{i(\lambda+1)}Ay_i^{\frac{1}{a}},\quad a<1,\qquad y_{i+1}\leqslant C2^{i(\lambda+1)}By_i^{\frac{1}{b}},\quad b>1, \tag{3.19}$$

and if

$$(y_0B^b)^{\frac{1-a}{b}}A^a\leqslant C^{-\frac{1}{(1-a)/b}}2^{-\frac{(\lambda+1)(a+b)}{((1-a)/b)^2}}\equiv C_0 \tag{3.20}$$

is sufficiently small, where $C_0<1$ is fixed, then $y_i\to 0$ as $i\to\infty$.

Proof. By hypothesis

$$y_{i+1}^a\leqslant C2^{i(\lambda+1)a}A^ay_i,\qquad y_{i+1}^b\leqslant C2^{i(\lambda+1)b}B^by_i,$$

so that

$$\frac{y_{i+1}^a}{A^a}+\frac{y_{i+1}^b}{B^b}\leqslant C2^{i(\lambda+1)(a+b)}y_i.$$

Using Young's inequality with exponents

$$\frac{1}{p}=\frac{b}{b+1-a},\qquad\frac{1}{p'}=\frac{1-a}{b+1-a},$$

we obtain

$$\frac{y_{i+1}^{\frac{b}{b+1-a}}}{A^{\frac{ab}{b+1-a}}B^{\frac{b(1-a)}{b+1-a}}}=\frac{y_{i+1}^{a\frac{1}{p}}y_{i+1}^{b\frac{1}{p'}}}{A^{a\frac{1}{p}}B^{b\frac{1}{p'}}}\leqslant\frac{y_{i+1}^a}{A^a}+\frac{y_{i+1}^b}{B^b}.$$

Let $b/(b+1-a)=\varepsilon_1<1$; then

$$\frac{y_{i+1}^{\varepsilon_1}}{A^{a\varepsilon_1}B^{b(1-\varepsilon_1)}}\leqslant C2^{i(\lambda+1)(a+b)}y_i,$$

and if (3.20) holds, then we have convergence to zero by Proposition 5 on iterations.

Let us introduce some auxiliary notation:

$$A \equiv R^{-(\lambda+1)} t^{\frac{(1+\theta)(\lambda+1)}{K_{1+\theta}}}, \qquad a = \frac{1}{1+(\lambda-1)(\lambda+1)/K_{1+\theta}} < 1,$$
$$B \equiv R^{Q\frac{\nu q-\lambda}{\nu q+\theta}-(\lambda+1)-q\frac{\lambda+\theta}{\nu q+\theta}} \frac{t^{\frac{\nu q-\lambda}{\nu q+\theta}}}{f(t)^{\frac{\lambda+\theta}{\nu q+\theta}}} \left(\frac{Z(t)^{2q}}{a(Z(t))}\right)^{\frac{\lambda+\theta}{\nu q+\theta}}, \qquad b = \frac{\nu q+\theta}{\lambda+\theta} > 1. \tag{3.21}$$

Lemma 3.5. *Under the hypotheses of Lemmas 3.1 and 3.2 condition (3.20) holds if*

$$\left(\frac{Z(t)^{2q}}{a(Z(t))}\right)^{\lambda-1} R^{-2q(\lambda-1)-H} \frac{t^{\nu q-\lambda}}{f(t)^{\lambda-1}} \leqslant \delta, \tag{3.22}$$

where δ is a sufficiently small fixed number.

Proof. It follows from Lemmas 3.3 and 3.2 that (3.19) holds. In Lemma 3.1, for an arbitrary $\hat{R}$, $\hat{R} \geqslant 2R \geqslant 4R_0$, we set

$$\tilde{R}_i = \hat{R}_i, \quad R_i = \hat{R}_{i+1}, \quad R_i < \tilde{R}_i, \qquad \text{where } \hat{R}_i = \frac{\hat{R}}{4}(1+2^{-i}), \quad t_1 = 0, \quad t_2 = t.$$

Then Lemma 3.1 yields

$$\begin{aligned} \sup_{0<\tau<t} \int_{\rho\geqslant\hat{R}_i} u^{\theta+1}\,dz &+ \int_0^t \int_{\rho\geqslant\hat{R}_i} u^{\theta-1}|D_L u|^{\lambda+1}\,dz\,d\tau \\ &+ \int_0^t \int_{\rho\geqslant\hat{R}_i} a(\rho(z)) f(\tau) u^\theta |D_L u^\nu|^q\,dz\,d\tau \\ &\leqslant C\frac{2^{i(\lambda+1)}}{R^{\lambda+1}} \int_0^t \int_{\hat{R}_{i+1}\leqslant\rho\leqslant\hat{R}_i} u^{\lambda+\theta}\,dz\,d\tau. \end{aligned} \tag{3.23}$$

We shall repeat part of the proof of Lemma 3.3. Let $\omega = u^{\frac{\nu q+\theta}{q}} \in \mathring{\mathfrak{L}}_{1,p}(B_{Z(t)})$ for all $t > 0$. We find an estimate for the right-hand side of (3.23) using Hölder's inequality and Poincaré's inequality (2.2):

$$\begin{aligned} \int_0^t \int_{\hat{R}_{i+1}\leqslant\rho\leqslant\hat{R}_i} u^{\lambda+\theta}\,dz\,d\tau &= \int_0^t \int_{\hat{R}_{i+1}\leqslant\rho\leqslant\hat{R}_i} \omega^{\frac{(\lambda+\theta)q}{\nu q+\theta}}\,dz\,d\tau \\ &\leqslant C\int_0^t \left(\frac{Z(t)^{2q}}{a(Z(t))f(\tau)} \int_{\hat{R}_{i+1}\leqslant\rho\leqslant Z(t)} a(\rho(z)) f(\tau)|D_L\omega|^q\,dz\right)^{\frac{\lambda+\theta}{\nu q+\theta}} \hat{R}_{i+1}^{-q\frac{\lambda+\theta}{\nu q+\theta}} \hat{R}_i^{Q\frac{\nu q-\lambda}{\nu q+\theta}}\,d\tau \\ &\leqslant C\hat{R}^{Q\frac{\nu q-\lambda}{\nu q+\theta}-q\frac{\lambda+\theta}{\nu q+\theta}} \left(\frac{Z(t)^{2q}}{a(Z(t))}\right)^{\frac{\lambda+\theta}{\nu q+\theta}} \\ &\quad\times \left(\int_0^t \int_{\hat{R}_{i+1}\leqslant\rho} a(\rho(z)) f(\tau)|D_L\omega|^q\,dz\,d\tau\right)^{\frac{\lambda+\theta}{\nu q+\theta}} \frac{t^{\frac{\nu q-\lambda}{\nu q+\theta}}}{f(t)^{\frac{\lambda+\theta}{\nu q+\theta}}}. \end{aligned}$$

Here we again need a condition on θ. We take θ sufficiently small so that

$$-\mu\frac{\lambda+\theta}{\nu q-\lambda} + 1 > 0.$$

Let

$$y^{(i)}(\hat{R}) \equiv \sup_{0<\tau<t} \int_{\rho\geqslant \hat{R}_i} u^{\theta+1}\, dz + \int_0^t \int_{\rho\geqslant \hat{R}_i} u^{\theta-1}|D_L u|^{\lambda+1}\, dz\, d\tau$$
$$+ \int_0^t \int_{\rho\geqslant \hat{R}_i} a(\rho(z)) f(\tau) u^{\theta}|D_L u^{\nu}|^{q}\, dz\, d\tau + \frac{2^{i(\lambda+1)}}{R^{\lambda+1}} \int_0^t \int_{\hat{R}_{i+1}\leqslant\rho\leqslant\hat{R}_i} u^{\lambda+\theta}\, dz\, d\tau,$$
$$\varepsilon = \frac{\nu q-\lambda}{\nu q+\theta}.$$

Similarly to Lemma 3.3 we obtain

$$y^{(i)}(\hat{R}) \leqslant C\frac{2^{i(\lambda+1)}}{R^{\lambda+1}} R^{Q\frac{\nu q-\lambda}{\nu q+\theta}-q\frac{\lambda+\theta}{\nu q+\theta}} \frac{t^{\frac{\nu q-\lambda}{\nu q+\theta}}}{f(t)^{\frac{\lambda+\theta}{\nu q+\theta}}} \left(\frac{Z(t)^{2q}}{a(Z(t))}\right)^{\frac{\lambda+\theta}{\nu q+\theta}} y^{(i+1)}(\hat{R})^{\frac{\lambda+\theta}{\nu q+\theta}}$$
$$= C2^{i(\lambda+1)} B[y^{(i+1)}(\hat{R})]^{1-\varepsilon}.$$

With (2.11)–(2.13) and (3.23) at our disposal we can say that for $\tau\in(0,t)$

$$y^{(i)}(\hat{R}) \leqslant C_1'(\hat{R}, t, \|u_0\|_{1,\mathbb{R}^{N+M}}) \quad \text{for all } i,$$

so that

$$[y^{(0)}(\hat{R})]^{\varepsilon} \leqslant CB$$

by Proposition 6 on iterations. Using the inequality obtained, we estimate y_0 in (3.20):

$$y_0 \equiv \sup_{0<\tau<t} \int_{\tilde{U}_0} u^{\theta+1}\, dz + \int_0^t \int_{\tilde{U}_0} u^{\theta-1}|D_L u|^{\lambda+1}\, dz\, d\tau$$
$$+ \int_0^t \int_{\tilde{U}_0} a(\rho(z)) f(\tau) u^{\theta}|D_L u^{\nu}|^{q}\, dz\, d\tau + \frac{2^{i(\lambda+1)}}{R^{\lambda+1}} \int_0^t \int_{U_0\setminus\tilde{U}_0} u^{\lambda+\theta}\, dz\, d\tau$$
$$\leqslant y^{(0)}\left(\frac{5R}{4}\right) + y^{(0)}(R) \leqslant CB^{\frac{1}{\varepsilon}} = CB^{\frac{\nu q+\theta}{\nu q-\lambda}}.$$

Here we select radii so that $\tilde{R}_0 = 5R/8$, $R_0 = R/2$, $\hat{R}_0 = \hat{R}/2$ and $\hat{R}_1 = 3\hat{R}/8$. To prove the lemma we must find a condition ensuring (3.20) holds. In view of the estimate obtained, we must prove that

$$(B^{\frac{\nu q+\theta}{\nu q-\lambda}} B^{b})^{\frac{1-a}{b}} A^{a} \leqslant \delta.$$

Using the notation (3.21) again and raising the inequality to the power

$$\frac{[(\nu q-\lambda)(K_{1+\theta}+\lambda^2-1)]}{[(\lambda+1)(\lambda+\theta)]},$$

we obtain a condition in the following form:

$$\left(\frac{Z(t)^{2q}}{a(Z(t))}\right)^{\lambda-1} R^{(\lambda+1)(1-\nu q)-q(\lambda-1)} \frac{t^{\nu q-\lambda}}{f(t)^{\lambda-1}} \leqslant \delta. \tag{3.24}$$

This completes the proof because $(\lambda+1)(1-\nu q) = -q(\lambda-1) - H$.

Taking sufficiently large R so as to ensure equality in (3.24), Proposition 5 on iterations yields the convergence $y_i \to 0$ as $i \to \infty$. Hence $u = 0$ a.e. outside B_R, $Z(t) \leqslant R$, so it follows from (3.24) that

$$\frac{t^{\nu q-\lambda}}{f(t)^{\lambda-1}} \leqslant \delta[a(R)]^{\lambda-1} R^H,$$

and we have

$$Z(t) \leqslant \varphi\biggl(\frac{t^{\nu q-\lambda}}{f(t)^{\lambda-1}}\biggr).$$

It is clear from definition (1.6) that (1.10) holds, which completes the proof of Theorem 1.

§ 4. The proof of Theorem 2

We shall prove (1.11). We multiply both sides of (1.1) by u^θ, $\theta > 0$, and integrate over the ball $B_{Z(t)}$. For $0 < \tau < t$ we have

$$\begin{aligned}\frac{1}{1+\theta}\,\frac{d}{d\tau}\int_{B_{Z(t)}} u^{\theta+1}\,dz &\leqslant -\int_{B_{Z(t)}} a(\rho(z))f(\tau)u^\theta|D_L u^\nu|^q\,dz\\ &\leqslant -C\int_{B_{Z(t)}} a(\rho(z))f(\tau)|D_L u^{\frac{\nu q+\theta}{q}}|^q\,dz.\end{aligned}$$

We look at two integrals. The first is as follows:

$$\int_{B_{Z(t)}} u^{\theta+1}\,dz \leqslant \biggl(\int_{B_{Z(t)}} u^{\nu q+\theta}|D_L\rho|^q\,dz\biggr)^{\frac{1+\theta}{\nu q+\theta}} \biggl(\int_{B_{Z(t)}} |D_L\rho|^{-q\frac{1+\theta}{\nu q-1}}\,dz\biggr)^{\frac{\nu q-1}{\nu q+\theta}}.$$

We use the change of variables in Proposition 4. Then

$$B_R \equiv \left\{\begin{matrix} 0 \leqslant \rho \leqslant R\\ 0 \leqslant \varphi_1 \leqslant 2\pi\\ 0 \leqslant \varphi_2 < \pi\\ \dots\\ 0 \leqslant \varphi_{M+N-1} < \pi\end{matrix}\right\}, \tag{4.1}$$

and setting

$$\Psi \equiv \left\{\begin{matrix} 0 \leqslant \varphi_1 \leqslant 2\pi\\ 0 \leqslant \varphi_2 < \pi\\ \dots\\ 0 \leqslant \varphi_{M+N-1} < \pi\end{matrix}\right\} \tag{4.2}$$

we obtain an estimate for the second factor:

$$\begin{aligned}&\int_{B_R} |D_L\rho|^{-q\frac{1+\theta}{\nu q-1}}\,dz = \int_{B_R}\biggl(\frac{\rho^\alpha}{|x|^\alpha}\biggr)^{q\frac{1+\theta}{\nu q-1}} J(\rho,\varphi)\,d\rho\,d\varphi_1\cdots d\varphi_{M+N-1}\\ &\quad= \int \frac{J(\rho,\varphi)\,d\rho\,d\varphi}{\bigl((\cos\varphi_{M+N-1}\cdots\cos\varphi_1)^2+\cdots+(\cos\varphi_{M+N-1}\cdots\sin\varphi_{M-1})^2\bigr)^{\frac{\alpha q(\theta+1)}{2(\alpha+1)(\nu q-1)}}}\\ &\quad= \int_0^R \rho^{Q-1}\,d\rho\int\cdots\int_\Psi \Phi(\varphi)\,d\varphi_1\cdots d\varphi_{M+N-1} \leqslant CR^Q,\end{aligned}$$

where

$$\Phi(\varphi)=\frac{\cos^{\frac{1}{\alpha+1}}\varphi_2\cos^{\frac{2}{\alpha+1}}\varphi_3\cdots\cos^{\frac{M-2}{\alpha+1}}\varphi_{M-1}\cos^{\frac{M-1}{\alpha+1}}\varphi_M\cos^{\frac{M}{\alpha+1}}\varphi_{M+1}}{(\cos\varphi_{M+N-1}\cdots\varphi_{M+1})^{\frac{\alpha q(1+\theta)}{(\alpha+1)(\nu q-1)}}(\sin\varphi_1\cdots\sin\varphi_{M-1})^{\frac{\alpha}{\alpha+1}}}$$
$$\times\frac{\cos^{\frac{M}{\alpha+1}+1}\varphi_{M+2}\cdots\cos^{\frac{M}{\alpha+1}+N-2}\varphi_{M+N-1}}{\cos\varphi_M^{\frac{\alpha q(1+\theta)}{(\alpha+1)(\nu q-1)}+\frac{\alpha}{\alpha+1}}(\cos\varphi_1\cdots\cos\varphi_{M-1})^{\frac{\alpha}{\alpha+1}}}.$$

Now the integral with respect to $d\varphi$ can have singularities only for $\varphi_j=\varphi_{01j}$ because $\sin\varphi_{01j}=0$, $j=M-1,\dots,M+N-1$, and $\varphi_j=\varphi_{02j}$ because $\cos\varphi_{02j}=0$, $j=M,\dots,M+N-1$. Now we look at these two cases. In a neighbourhood of φ_{01j} the integrand $\Phi(\varphi)$ is equivalent to

$$\frac{1}{\sin(\varphi_j-\varphi_{01j})^{\frac{\alpha}{\alpha+1}}}\sim\frac{1}{(\varphi_j-\varphi_{01j})^{\frac{\alpha}{\alpha+1}}},$$

that is, it has a removable singularity. In a neighbourhood of φ_{02M} the integrand $\Phi(\varphi)$ is equivalent to $(\varphi_M-\varphi_{02M})^{\frac{M-1}{\alpha+1}-\frac{\alpha q(\theta+1)}{(\alpha+1)(\nu q-1)}-\frac{\alpha}{\alpha+1}}$, and it has a removable singularity if

$$M-\frac{\alpha q}{\nu q-1}>0,\qquad 0<\theta\leqslant\frac{M(\nu q-1)-\alpha q}{\alpha q}.$$

In a neighbourhood of φ_{02j}, $j=M+1,\dots,M+N-1$, the integrand satisfies

$$\Phi(\varphi)\sim(\varphi_j-\varphi_{02j})^{\frac{M}{\alpha+1}+(j-M-1)-\frac{\alpha q(\theta+1)}{(\alpha+1)(\nu q-1)}},$$

and the singularity is removable if

$$M-\frac{\alpha q}{\nu q-1}>0\qquad\text{and}\quad 0<\theta\leqslant\frac{M(\nu q-1)-\alpha q}{\alpha q}.$$

In a neighbourhood of the point φ_{02j}, $j=1,\dots,M-1$, the integrand satisfies $\Phi(\varphi)\sim(\varphi_j-\varphi_{02j})^{\frac{j-1}{\alpha+1}-\frac{\alpha}{\alpha+1}}$, and the singularity is removable. Hence the integral with respect to $d\varphi$ converges. Now we have

$$\int_{B_{Z(t)}}u^{\theta+1}\,dz\leqslant C\biggl(\int_{B_{Z(t)}}u^{\nu q+\theta}|D_L\rho|^q\,dz\biggr)^{\frac{1+\theta}{\nu q+\theta}}Z(t)^{Q\frac{\nu q-1}{\nu q+\theta}}.$$

We look at the second integral:

$$\begin{aligned}-C\int_{B_{Z(t)}}a(\rho(z))f(\tau)|D_Lu^{\frac{\nu q+\theta}{q}}|^q\,dz&\leqslant-C\int_{B_{Z(t)}}\frac{a(\rho(z))}{\rho^q}f(\tau)|D_L\rho|^q\,|u^{\frac{\nu q+\theta}{q}}|^q\,dz\\
&\leqslant-CZ(t)^{-q}a(Z(t))f(\tau)\int_{B_{Z(t)}}|D_L\rho|^q\,|u^{\frac{\nu q+\theta}{q}}|^q\,dz\\
&=-CZ(t)^{-q}a(Z(t))f(\tau)\int_{B_{Z(t)}}|D_L\rho|^q u^{\nu q+\theta}\,dz.\end{aligned}$$

Here we have used the Hardy-type inequality (2.3). Hence

$$\frac{d}{d\tau}\biggl(\int_{B_{Z(t)}} u^{\theta+1}\,dz\biggr) \leqslant -CZ(t)^{-q}a(Z(t))f(\tau)Z(t)^{-\frac{Q(\nu q-1)}{1+\theta}}\biggl(\int_{B_{Z(t)}} u^{\theta+1}\,dz\biggr)^{\frac{\nu q+\theta}{1+\theta}}. \tag{4.3}$$

Solving this differential inequality we obtain

$$\int_{B_{Z(t)}} u^{\theta+1}\,dz \leqslant CZ(t)^{\frac{q(1+\theta)}{\nu q-1}+Q}a(Z(t))^{-\frac{1+\theta}{\nu q-1}}t^{-\frac{1+\theta}{\nu q-1}}f(t)^{-\frac{1+\theta}{\nu q-1}}.$$

Now we estimate the mass of the solution:

$$\begin{aligned}\int_{\mathbb{R}^{N+M}} u\,dz = \int_{B_{Z(t)}} u\,dz &\leqslant \biggl(\int_{B_{Z(t)}} u^{\theta+1}\,dz\biggr)^{\frac{1}{1+\theta}} \operatorname{mes}^{\frac{\theta}{1+\theta}}\{B_{Z(t)}\}\\ &\leqslant C\frac{Z(t)^{Q+\frac{q}{\nu q-1}}}{a(Z(t))^{\frac{1}{\nu q-1}}t^{\frac{1}{\nu q-1}}f(t)^{\frac{1}{\nu q-1}}}.\end{aligned} \tag{4.4}$$

It is clear from definition (1.6) that

$$\omega(t)t^{\frac{1}{K}} = \varphi\biggl(\frac{t^{\nu q-\lambda}}{f(t)^{\lambda-1}}\biggr), \qquad a(\omega(t)t^{\frac{1}{K}})^{\lambda-1}(\omega(t)t^{\frac{1}{K}})^H = \frac{t^{\nu q-\lambda}}{f(t)^{\lambda-1}},$$

$$a(\omega(t)t^{\frac{1}{K}})f(t) = \frac{t^{\frac{\nu q-\lambda}{\lambda-1}-\frac{H}{K(\lambda-1)}}}{\omega^{\frac{H}{\lambda-1}}(t)}. \tag{4.5}$$

Using (4.4) we easily obtain (1.11) from (4.5). This gives us the corresponding estimate for the L_∞-norm of the solution. Following Proposition 8 we can show that for $t > t_0 = R_0^{-K}\|u_0\|_{1,\mathbb{R}^{N+M}}^{\lambda-1}$,

$$\|u\|_{\infty,\mathbb{R}^{N+M}} \leqslant C\|u\|_{1,\mathbb{R}^{N+M}}^{\frac{\lambda+1}{K}}t^{-\frac{Q}{K}} \leqslant C\|u\|_{\infty,\mathbb{R}^{N+M}}^{\frac{\lambda+1}{K}}Z(t)^{\frac{Q(\lambda+1)}{K}}t^{-\frac{Q}{K}},$$

$$\|u\|_{\infty,\mathbb{R}^{N+M}}^{1-\frac{\lambda+1}{K}} \leqslant CZ(t)^{\frac{\lambda+1}{\lambda-1}}t^{-\frac{1}{\lambda-1}}.$$

The proof of Theorem 2 is complete.

§ 5. The proof of Theorem 3

In this theorem we look at the cases $\lambda = 1$ and $\nu = 1$, when we can estimate the mass of the solution without referring to the compact support of the initial function. We have

$$\int_{\mathbb{R}^{N+M}} u\,dz = \int_{B_R} u\,dz + \int_{\mathbb{R}^{N+M}\setminus B_R} u\,dz := E_1(t) + E_2(t).$$

Here R is a fixed number which we shall specify in what follows.

$$\begin{aligned}E_1(t) = \int_{B_R} u\,dz &\leqslant \biggl(\int_{B_R} u^{\nu q}|D_L\rho|^q\,dz\biggr)^{\frac{1}{\nu q}}\biggl(\int_{B_R}|D_L\rho|^{-q\frac{1}{\nu q-1}}\,dz\biggr)^{\frac{\nu q-1}{\nu q}}\\ &= I_1^{\frac{1}{\nu q}}I_2^{\frac{\nu q-1}{\nu q}}.\end{aligned}$$

We use Proposition 3:

$$I_1 \leqslant \frac{R^q}{a(R)} \int_{B_R} \frac{u^{\nu q} |D_L \rho|^q a(\rho)}{\rho^q}\, dz \leqslant C \frac{R^q}{a(R)} \int_{\mathbb{R}^{N+M}} a(\rho(z)) |D_L u^\nu|^q\, dz.$$

We consider the second factor using the change of variables (2.5) in Proposition 4. We have

$$\begin{aligned} I_2 &= \int_{B_R} |D_L \rho|^{-q \frac{1}{\nu q - 1}}\, dz = \int_{B_R} \left(\frac{\rho^\alpha}{|x|^\alpha} \right)^{q \frac{1}{\nu q - 1}} J(\rho, \varphi)\, d\rho\, d\varphi_1 \cdots d\varphi_{M+N-1} \\ &= \int \frac{J(\rho, \varphi)\, d\rho\, d\varphi}{\big((\cos \varphi_{M+N-1} \cdots \cos \varphi_1)^2 + \cdots + (\cos \varphi_{M+N-1} \cdots \sin \varphi_{M-1})^2 \big)^{\frac{\alpha q}{2(\alpha+1)(\nu q - 1)}}} \\ &= \int_0^R \rho^{Q-1}\, d\rho \iint_\Psi \Phi(\varphi)\, d\varphi_1 \cdots d\varphi_{M+N-1} \leqslant C R^Q, \end{aligned}$$

where

$$\begin{aligned} \Phi(\varphi) = {} & \frac{\cos^{\frac{1}{\alpha+1}} \varphi_2 \cos^{\frac{2}{\alpha+1}} \varphi_3 \cdots \cos^{\frac{M-2}{\alpha+1}} \varphi_{M-1} \cos^{\frac{M-1}{\alpha+1}} \varphi_M \cos^{\frac{M}{\alpha+1}} \varphi_{M+1}}{(\cos \varphi_{M+N-1} \cdots \cos \varphi_{M+1})^{\frac{\alpha q}{(\alpha+1)(\nu q-1)}} (\sin \varphi_1 \cdots \sin \varphi_{M-1})^{\frac{\alpha}{\alpha+1}}} \\ & \times \frac{\cos^{\frac{M}{\alpha+1}+1} \varphi_{M+2} \cdots \cos^{\frac{M}{\alpha+1}+N-2} \varphi_{M+N-1}}{\cos \varphi_M^{\frac{\alpha q}{(\alpha+1)(\nu q-1)} + \frac{\alpha}{\alpha+1}} (\cos \varphi_1 \cdots \cos \varphi_{M-1})^{\frac{\alpha}{\alpha+1}}}. \end{aligned}$$

Now the integral with respect to $d\varphi$ can have singularities only for $\varphi_j = \varphi_{01j}$ because $\sin \varphi_{01j} = 0$, $j = M-1, \dots, M+N-1$, and $\varphi_j = \varphi_{02j}$, since $\cos \varphi_{02j} = 0$, $j = M, \dots, M+N-1$. We will look at these cases. In a neighbourhood of φ_{01j} the integrand $\Phi(\varphi)$ is equivalent to

$$\frac{1}{\sin(\varphi_j - \varphi_{01j})^{\frac{\alpha}{\alpha+1}}} \sim \frac{1}{(\varphi_j - \varphi_{01j})^{\frac{\alpha}{\alpha+1}}},$$

that is, it has a removable singularity. In a neighbourhood of φ_{02M} the integrand $\Phi(\varphi)$ is equivalent to $(\varphi_M - \varphi_{02M})^{\frac{M-1}{\alpha+1} - \frac{\alpha q}{(\alpha+1)(\nu q-1)} - \frac{\alpha}{\alpha+1}}$, and the singularity is removable if $M - \alpha q/(\nu q - 1) > 0$. In a neighbourhood of φ_{02j}, $j = M+1, \dots, M+N-1$ the integrand satisfies

$$\Phi(\varphi) \sim (\varphi_j - \varphi_{02j})^{\frac{M}{\alpha+1} + (j-M-1) - \frac{\alpha q}{(\alpha+1)(\nu q-1)}},$$

and the integral is removable if the condition $M - \alpha q/(\nu q - 1) > 0$ holds. In a neighbourhood of φ_{02j}, $j=1, \dots, M-1$, the integrand $\Phi(\varphi) \sim (\varphi_j - \varphi_{02j})^{\frac{j-1}{\alpha+1} - \frac{\alpha}{\alpha+1}}$, and the singularity is removable. Hence the integral with respect to $d\varphi$ converges. Thus

$$E_1(t) \leqslant C \left(\frac{R^q}{a(R) f(t)} \int_{\mathbb{R}^{N+M}} a(\rho(z)) f(t) |D_L u^\nu|^q\, dz \right)^{\frac{1}{\nu q}} R^{Q \frac{\nu q - 1}{\nu q}}. \tag{5.1}$$

Integrating the equation over $\mathbb{R}^{N+M}$ and carrying over the derivatives in the second term we obtain

$$\frac{d}{d\tau} \int_{\mathbb{R}^{N+M}} u\, dz = - \int_{\mathbb{R}^{N+M}} a(\rho(z)) f(\tau) |D_L u^\nu|^q\, dz. \tag{5.2}$$

Then we have

$$\int_{\mathbb{R}^{N+M}} u(\tau)\,dz \leqslant C\frac{R^{\frac{q}{\nu q}+Q\frac{\nu q-1}{\nu q}}}{f(\tau)^{\frac{1}{\nu q}}a(R)^{\frac{1}{\nu q}}}\left(-\frac{d}{d\tau}\int_{\mathbb{R}^{N+M}} u(\tau)\,dz\right)^{\frac{1}{\nu q}} + E_2(\tau). \tag{5.3}$$

We shall find an estimate for $E_2(\tau)$ in the case $\nu = 1$. The equation takes the following form:

$$u_t = \operatorname{div}_L\big(|D_L u|^{\lambda-1}D_L u\big) - a(\rho(z))f(t)|D_L u|^q,$$
$$(z,t)\in S_T = \mathbb{R}^{N+M}\times(0,T_r), \qquad \lambda < q < \lambda+1.$$

We take the cutoff function

$$\xi(z) = \begin{cases} 0 & \text{for } \rho(z) < \dfrac{R}{2}, \\ 1 & \text{for } \rho(z) > R, \end{cases} \qquad 0\leqslant \xi\leqslant 1, \quad |D_L\xi|\leqslant \frac{C}{R}.$$

Multiplying the equation by ξ^s with $s = q/(q-\lambda) > \lambda+1$ and integrating by parts over $\mathbb{R}^{N+M}$ we obtain

$$\frac{d}{d\tau}\int_{\mathbb{R}^{N+M}} u\xi^s\,dz + \int_{\mathbb{R}^{N+M}} a(\rho(z))f(\tau)|D_L u|^q\xi^s\,dz \leqslant \frac{C}{R}\int_{\frac{R}{2}\leqslant\rho\leqslant R}|D_L u|^\lambda\xi^{s-1}\,dz.$$

To estimate the right-hand side we use Hölder's inequality with $p = q/\lambda = s/(s-1)$ and Young's inequality:

$$\begin{aligned}
&\frac{C}{R}\int_{\frac{R}{2}\leqslant\rho\leqslant R}|D_L u|^\lambda\xi^{s-1}\,dz \\
&\quad\leqslant \frac{C}{R}\left(\int_{\frac{R}{2}\leqslant\rho\leqslant R} a(\rho)f(\tau)|D_L u|^q\xi^s\,dz\right)^{\frac{q}{\lambda}}\left(\int_{\frac{R}{2}\leqslant\rho\leqslant R}[a(\rho)f(\tau)]^{-\frac{\lambda}{q-\lambda}}\,dz\right)^{\frac{q-\lambda}{q}} \\
&\quad\leqslant \frac{1}{2}\int_{\mathbb{R}^{N+M}} a(\rho(z))f(\tau)|D_L u|^q\xi^s\,dz + \frac{C}{R^{\frac{q}{q-\lambda}}}\int_{\frac{R}{2}\leqslant\rho\leqslant R}[a(\rho)f(\tau)]^{-\frac{\lambda}{q-\lambda}}\,dz.
\end{aligned}$$

Then we obtain the differential inequality

$$\frac{d}{d\tau}\int_{\mathbb{R}^{N+M}} u\xi^s\,dz \leqslant C\frac{R^{Q-\frac{q}{q-\lambda}}}{a(R)^{\frac{\lambda}{q-\lambda}}f(\tau)^{\frac{\lambda}{q-\lambda}}}.$$

We integrate it over $(0,\tau)$:

$$\begin{aligned}
\int_{\mathbb{R}^{N+M}} u(z,\tau)\xi^s\,dz &\leqslant \int_{\mathbb{R}^{N+M}} u_0\xi^s\,dz + C\frac{R^{Q-\frac{q}{q-\lambda}}}{a(R)^{\frac{\lambda}{q-\lambda}}}\int_0^\tau f(\eta)^{-\frac{\lambda}{q-\lambda}}\,d\eta \\
&\leqslant \int_{\mathbb{R}^{N+M}} u_0\xi^s\,dz + C\frac{\tau R^{Q-\frac{q}{q-\lambda}}}{a(R)^{\frac{\lambda}{q-\lambda}}f(\tau)^{\frac{\lambda}{q-\lambda}}} \equiv \Upsilon(R,\tau)\leqslant\Upsilon(R,t), \quad 0<t_1<\tau<t.
\end{aligned}$$

Hence

$$E_2(R,\tau)\leqslant\Upsilon(R,t). \tag{5.4}$$

Combining the estimates for E_1 and E_2 we obtain

$$E(\tau) \leqslant \Upsilon(R,t) + C\frac{R^{\frac{q}{\nu q}+Q\frac{\nu q-1}{\nu q}}}{f(\tau)^{\frac{1}{\nu q}}a(R)^{\frac{1}{\nu q}}}\left(-\frac{d}{d\tau}E(\tau)\right)^{\frac{1}{\nu q}}. \tag{5.5}$$

Now we find an estimate for $E_2(\tau)$ in the case $\lambda = 1$. Then the equation takes the following form:

$$u_t = \operatorname{div}_L(D_L u) - a(\rho(z))f(t)|D_L u^\nu|^q,$$
$$(z,t) \in S_T = \mathbb{R}^{N+M} \times (0,T_r), \qquad 1 < q < 2, \quad \nu q > 1.$$

We take the cutoff function

$$\xi(z) = \begin{cases} 0 & \text{for } \rho(z) < R, \\ 1 & \text{for } \rho(z) > 2R, \end{cases} \qquad 0 \leqslant \xi \leqslant 1, \quad |D_L\xi| \leqslant \frac{C}{R}, \quad |\operatorname{div}_L(D_L\xi)| \leqslant \frac{C}{R^2}.$$

Multiplying by ξ^2 we integrate by parts over the set $\mathbb{R}^{N+M} \times (0,\tau)$. This yields

$$\begin{aligned}\int_{\mathbb{R}^{N+M}} u(\tau)\xi^2\,dz &+ \int_0^\tau \int_{\mathbb{R}^{N+M}} a(\rho(z))f(\eta)|D_L u^\nu|^q \xi^2\,dzd\eta \\ &= \int_{\mathbb{R}^{N+M}} u_0\xi^2\,dz + \int_0^\tau \int_{\mathbb{R}^{N+M}} u\cdot \operatorname{div}_L(D_L\xi^2)\,dz\,d\eta \\ &\leqslant \int_{\rho>R} u_0\,dz + \frac{C}{R}\int_0^\tau \int_{2R>\rho>R} u\,dz\,d\eta.\end{aligned}$$

From (5.1) we obtain the estimate

$$\begin{aligned}&\frac{C}{R^2}\int_0^\tau \int_{2R>\rho} u\,dz\,d\eta \\ &\quad\leqslant \frac{C}{R^2}\int_0^\tau \left(\frac{R^q}{a(R)f(\eta)}\int_{\mathbb{R}^{N+M}} a(\rho(z))f(\eta)|D_L u^\nu|^q\,dz\right)^{\frac{1}{\nu q}} R^{Q\frac{\nu q-1}{\nu q}}\,d\eta \\ &\quad\leqslant C\frac{R^{\frac{Q(\nu q-1)+q-2\nu q}{\nu q}}\tau^{\frac{\nu q-1}{\nu q}}}{a(R)^{\frac{1}{\nu q}}f(\tau)^{\frac{1}{\nu q}}}\left(\int_0^\tau \int_{\mathbb{R}^{N+M}} a(\rho(z))f(\eta)|D_L u^\nu|^q\,dz\,d\eta\right)^{\frac{1}{\nu q}}.\end{aligned}$$

Young's inequality yields

$$\begin{aligned}\int_{\rho>2R} u(\tau)\,dz &\leqslant \int_{\rho>R} u_0\,dz \\ &\qquad + C\frac{R^{\frac{Q(\nu q-1)+q-2\nu q}{\nu q}}\tau^{\frac{\nu q-1}{\nu q}}}{a(R)^{\frac{1}{\nu q}}f(\tau)^{\frac{1}{\nu q}}}\left(\int_0^\tau \int_{\mathbb{R}^{N+M}} a(\rho(z))f(\eta)|D_L u^\nu|^q\,dz\,d\eta\right)^{\frac{1}{\nu q}} \\ &\leqslant \int_{\rho>R} u_0\,dz + \frac{1}{2}\int_0^\tau \int_{\mathbb{R}^{N+M}} a(\rho(z))f(\eta)|D_L u^\nu|^q\,dz\,d\eta + C\frac{R^{\frac{Q(\nu q-1)+q-2\nu q}{\nu q-1}}\tau}{a(R)^{\frac{1}{\nu q-1}}f(\tau)^{\frac{1}{\nu q-1}}} \\ &\leqslant \int_{\rho>R} u_0\,dz + \frac{1}{2}\left(\int_{\mathbb{R}^{N+M}} u(\tau)\,dz - \int_{\mathbb{R}^{N+M}} u_0\,dz\right) + C\frac{R^{\frac{Q(\nu q-1)+q-2\nu q}{\nu q-1}}\tau}{a(R)^{\frac{1}{\nu q-1}}f(\tau)^{\frac{1}{\nu q-1}}}.\end{aligned}$$

In our notation

$$E_2(\tau) \leqslant \frac{1}{2}\int_{\rho>R} u_0\,dz + \frac{1}{2}E(\tau) + C\frac{R^{\frac{Q(\nu q-1)+q-2\nu q}{\nu q-1}}\tau}{a(R)^{\frac{1}{\nu q-1}}f(\tau)^{\frac{1}{\nu q-1}}}.$$

Similarly to the previous case we set

$$\Upsilon(R,\tau) \equiv \frac{1}{2}\int_{\rho>R} u_0\,dz + C\frac{R^{\frac{Q(\nu q-1)+q-2\nu q}{\nu q-1}}\tau}{a(R)^{\frac{1}{\nu q-1}}f(\tau)^{\frac{1}{\nu q-1}}} \leqslant \Upsilon(R,t), \qquad 0<t_1<\tau<t.$$

Combining the estimates for E_1 and E_2 we obtain

$$E(\tau) \leqslant \frac{1}{2}E(\tau) + \Upsilon(R,t) + C\frac{R^{\frac{q}{\nu q}+Q\frac{\nu q-1}{\nu q}}}{f(\tau)^{\frac{1}{\nu q}}a(R)^{\frac{1}{\nu q}}}\biggl(-\frac{d}{d\tau}E(\tau)\biggr)^{\frac{1}{\nu q}}.$$

In both cases ($\lambda=1$ and $\nu=1$) estimate (5.5) holds. We set $y(\tau)=E(\tau)-\Upsilon(R,t)$ in it, integrate the result from 0 to t and solve the inequality obtained:

$$y(\tau) \leqslant C\frac{R^{\frac{q}{\nu q}+Q\frac{\nu q-1}{\nu q}}}{f(\tau)^{\frac{1}{\nu q}}a(R)^{\frac{1}{\nu q}}}(-y_\tau(\tau))^{\frac{1}{\nu q}}, \qquad y(t) \leqslant C\frac{R^{Q+\frac{q}{\nu q-1}}}{a(R)^{\frac{1}{\nu q-1}}t^{\frac{1}{\nu q-1}}f(t)^{\frac{1}{\nu q-1}}}.$$

Hence for $\nu=1$ we have

$$\int_{\mathbb{R}^{N+M}} u(t)\,dz \leqslant \int_{\rho>R} u_0\,dz + C\frac{tR^{Q-\frac{q}{q-\lambda}}}{a(R)^{\frac{\lambda}{q-\lambda}}f(t)^{\frac{\lambda}{q-\lambda}}} + C\frac{R^{Q+\frac{q}{q-1}}}{a(R)^{\frac{1}{q-1}}t^{\frac{1}{q-1}}f(t)^{\frac{1}{q-1}}}.$$

The last two terms coincide up to a constant coefficient if

$$R=R(t)\colon\ R^H a(R)^{\lambda-1} = \frac{t^{q-\lambda}}{f(t)^{\lambda-1}}.$$

This completes the proof of inequality (1.13) for $\nu=1$. In the case $\lambda=1$ we obtain

$$\int_{\mathbb{R}^{N+M}} u(t)\,dz \leqslant \int_{\rho>R} u_0\,dz + C\frac{R^{\frac{Q(\nu q-1)+q-2\nu q}{\nu q-1}}t}{a(R)^{\frac{1}{\nu q-1}}f(t)^{\frac{1}{\nu q-1}}} + C\frac{R^{Q+\frac{q}{\nu q-1}}}{a(R)^{\frac{1}{\nu q-1}}t^{\frac{1}{\nu q-1}}f(t)^{\frac{1}{\nu q-1}}}.$$

The last two terms coincide up to a constant coefficient if $R=\sqrt{t}$. The proof of Theorem 3 is complete.

§ 6. The proof of Theorem 4

We integrate (1.1) over the ball $B_{Z(t)}$. Then we have

$$\frac{d}{dt}\int_{B_{Z(t)}} u\,dz = -\int_{B_{Z(t)}} a(\rho(z))f(t)|D_L u^\nu|^q\,dz.$$

We use inequality (5.1), which holds for $M-\alpha q/(\nu q-1)>0$. We have

$$\int_{B_{Z(t)}} u(z,t)\,dz \leqslant \biggl(\int_{B_{Z(t)}} u^{\nu q}|D_L\rho|^q\,dz\biggr)^{\frac{1}{\nu q}}\biggl(\int_{B_{Z(t)}} |D_L\rho|^{-q\frac{1}{\nu q-1}}\,dz\biggr)^{\frac{\nu q-1}{\nu q}}$$

$$\leqslant CZ(t)^{Q\frac{\nu q-1}{\nu q}}\biggl(\frac{Z(t)^q}{a(Z(t))f(t)}\int_{B_{Z(t)}} a(\rho(z))f(t)|D_L u^\nu|^q\,dz\biggr)^{\frac{1}{\nu q}}.$$

Combining our estimates yields

$$\frac{d}{dt}\int_{B_{Z(t)}} u\,dz \leqslant -C\biggl(\int_{B_{Z(t)}} u\,dz\biggr)^{\nu q} a(Z(t))Z(t)^{-Q(\nu q-1)-q}f(t).$$

We set $y(t)=\displaystyle\int_{B_{Z(t)}} u(z,t)\,dz$; then

$$\frac{dy(t)}{dt}y^{-\nu q}(t) \leqslant -Ca(Z(t))Z(t)^{-Q(\nu q-1)-q}f(t).$$

Integrating this over (t_0,t) we obtain

$$y(t) \leqslant C\biggl(\int_{t_0}^{t}\frac{a(Z(\tau))f(\tau)}{Z(\tau)^{Q(\nu q-1)+q}}\,d\tau\biggr)^{-\frac{1}{\nu q-1}},$$

$$\int_{B_{Z(t)}} u\,dz \leqslant C\biggl(\int_{t_0}^{t}\frac{a(Z(\tau))f(\tau)}{Z(\tau)^{Q(\nu q-1)+q}}\,d\tau\biggr)^{-\frac{1}{\nu q-1}}.$$

Under the integral sign we have the product of $f(\tau)$ and the function $a(Z)/Z^{Q(\nu q-1)+q}$ which decreases in Z.

Assume that (1.8) holds; then $Z(t)\leqslant Ct^{\frac{1}{K}}$ for sufficiently large t, so that

$$\frac{a(Z(\tau))f(\tau)}{Z(\tau)^{Q(\nu q-1)+q}} \geqslant \frac{a(\tau^{\frac{1}{K}})f(\tau)}{\tau^{\frac{Q(\nu q-1)+q}{K}}}.$$

Using the last inequality we obtain

$$\int_{t_0}^{t}\frac{a(Z(\tau))f(\tau)}{Z(\tau)^{Q(\nu q-1)+q}}\,d\tau \geqslant C\int_{t_0}^{t}\frac{a(\tau^{\frac{1}{K}})f(\tau)}{\tau^{\frac{Q(\nu q-1)+q}{K}}}\,d\tau,$$

$$\int_{\mathbb{R}^{N+M}} u\,dz \leqslant C\biggl(\int_{t_0}^{t}\frac{a(\tau^{\frac{1}{K}})f(\tau)}{\tau^{\frac{Q(\nu q-1)+q}{K}}}\,d\tau\biggr)^{-\frac{1}{\nu q-1}}.$$

In the case when there exist $C>0$ and $\varrho\in(0,1)$ such that $C\varrho\leqslant\omega(t)\leqslant C$, for sufficiently large t, $t\geqslant t_0=\|u_0\|_{1,\mathbb{R}^{N+M}}^{\lambda-1}/R_0^K$, we obtain

$$\frac{a(Z(\tau))f(\tau)}{Z(\tau)^{Q(\nu q-1)+q}} \geqslant C\frac{a(\tau^{\frac{1}{K}})f(\tau)}{\tau^{\frac{Q(\nu q-1)+q}{K}}} = C\frac{1}{\tau}, \qquad \int_{B_{Z(t)}} u\,dz \leqslant C\biggl(\int_{t_0}^{t}\tau^{-1}\,d\tau\biggr)^{-\frac{1}{\nu q-1}}.$$

The proof is complete.

§ 7. The proof of Theorem 5

Lemma 7.1. *Let u be a solution of equation (1.1). Then $u(z,t_0)\neq 0$ for all $z\in\mathbb{R}^{N+M}$ and each finite $t_0>0$.*

Proof. Assume that there exists t_0 such that $\|u(t_0)\|_1=0$ and $\|u(t)\|_1>0$ for $t<t_0$. Let ϑ, $0<\vartheta<1$, be a parameter. We set

$$E_\vartheta(t)=\int_{\mathbb{R}^{N+M}} u(t)^{1+\vartheta}\,dz.$$

Then integrating the equation by parts over the whole of $\mathbb{R}^{N+M}$ we obtain (5.2); we will find an estimate for the integral on the right-hand side using Hölder's inequality:

$$\begin{aligned}\int_{\mathbb{R}^{N+M}} a(\rho(z))f(\tau)|D_L u^\nu|^q\,dz &= \int_{\mathbb{R}^{N+M}} a(\rho(z))f(t)|\nu u^{\nu-1}D_L u|^q u^{\frac{(\vartheta-1)q}{\lambda+1}} u^{-\frac{(\vartheta-1)q}{\lambda+1}}\,dz\\ &\leqslant C\biggl(\int_{\mathbb{R}^{N+M}} u^{\vartheta-1}|D_L u|^{\lambda+1}\,dz\biggr)^{\frac{q}{\lambda+1}}\\ &\quad\times\biggl(\int_{\mathbb{R}^{N+M}} a(\rho(z))^{\frac{\lambda+1}{\lambda+1-q}} f(t)^{\frac{\lambda+1}{\lambda+1-q}}[u^{(\nu-1)q-\frac{(\vartheta-1)q}{\lambda+1}}]^{\frac{\lambda+1}{\lambda+1-q}}\,dz\biggr)^{\frac{\lambda+1-q}{\lambda+1}}\\ &= CH_1^{\frac{q}{\lambda+1}}H_2^{\frac{\lambda+1-q}{\lambda+1}}.\end{aligned}$$

We shall estimate each factor. Multiplying (1.1) by u^ϑ and integrating over $\mathbb{R}^{N+M}$ we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^{N+M}} u^{1+\vartheta}\,dz+\int_{\mathbb{R}^{N+M}} u^{\vartheta-1}|D_L u|^{\lambda+1}\,dz+\int_{\mathbb{R}^{N+M}} a(\rho(z))f(t)|D_L u^\nu|^q u^\vartheta\,dz = 0. \tag{7.1}$$

Hence

$$H_1 \leqslant -C\frac{d}{d\tau}\int_{\mathbb{R}^{N+M}} u^{1+\vartheta}\,dz.$$

To estimate the second factor we use Proposition 7:

$$\begin{aligned}H_2 &= \int_{\mathbb{R}^{N+M}} a(\rho(z))^{\frac{\lambda+1}{\lambda+1-q}} f(t)^{\frac{\lambda+1}{\lambda+1-q}} u^{\frac{(\nu-1)q(\lambda+1)}{\lambda+1-q}-\frac{(\vartheta-1)q}{\lambda+1-q}-1+1}\,dz\\ &\leqslant \int_{\mathbb{R}^{N+M}} u(z,t)a(C\|u_0\|_1^{\frac{\lambda+1}{K}}t^{\frac{1}{K}})f(t)^{\frac{\lambda+1}{\lambda+1-q}}\|u\|_{\infty,\mathbb{R}^{N+M}}^{\frac{H-\vartheta q}{\lambda+1-q}}\,dz\\ &= \|u(t)\|_{1,\mathbb{R}^{N+M}}a(C\|u_0\|_1^{\frac{\lambda+1}{K}}t^{\frac{1}{K}})f(t)^{\frac{\lambda+1}{\lambda+1-q}}(C\|u_0\|_1^{\frac{\lambda+1}{K}}t^{-\frac{Q}{K}})^{\frac{H-\vartheta q}{\lambda+1-q}}\\ &\leqslant C(t_0,\|u_0\|_1)\|u(t)\|_{1,\mathbb{R}^{N+M}}.\end{aligned}$$

We have

$$\begin{aligned}H_1^{\frac{q}{\lambda+1}}H_2^{\frac{\lambda+1-q}{\lambda+1}} &\leqslant C\biggl(-\frac{d}{dt}E_\vartheta(t)\biggr)^{\frac{q}{\lambda+1}}(C\|u(t)\|_{1,\mathbb{R}^{N+M}})^{\frac{\lambda+1-q}{\lambda+1}}\\ &\leqslant C_\sigma\|u(t)\|_{1,\mathbb{R}^{N+M}}-\sigma\frac{d}{dt}E_\vartheta(t);\end{aligned}$$

we shall specify σ in what follows. Now,

$$\frac{d}{d\tau}\|u(\tau)\|_{1,\mathbb{R}^{N+M}} \geqslant \sigma\frac{d}{d\tau}E_\vartheta(\tau)-C_\sigma\|u(\tau)\|_{1,\mathbb{R}^{N+M}}.$$

Integrating this inequality over the interval (t,t_0) with t close to t_0, $t<t_0$ yields

$$\|u(t_0)\|_{1,\mathbb{R}^{N+M}}-\|u(t)\|_{1,\mathbb{R}^{N+M}} \geqslant \sigma E_\vartheta(t_0)-\sigma E_\vartheta(t)-C_\sigma\int_t^{t_0}\|u(\tau)\|_{1,\mathbb{R}^{N+M}}\,d\tau.$$

It is obvious that $\|u(\tau)\|_{1,\mathbb{R}^{N+M}}$ is decreasing since its derivative is negative (see (5.2)), so we have

$$\begin{aligned}\|u(t)\|_{1,\mathbb{R}^{N+M}} &\leqslant \sigma E_\vartheta(t) + C_\sigma(t_0-t)\|u(t)\|_{1,\mathbb{R}^{N+M}} \\ &\leqslant \sigma C^\vartheta \|u(t)\|_{1,\mathbb{R}^{N+M}} + C_\sigma(t_0-t)\|u(t)\|_{1,\mathbb{R}^{N+M}}.\end{aligned}$$

We pick σ such that $\sigma C(t_0, \|u_0\|, 1)^\vartheta < 1$ and divide the inequality by $\|u(t)\|_{1,\mathbb{R}^{N+M}} > 0$. This yields $1 \leqslant \sigma C(t_0, \|u_0\|, 1)^\vartheta + C_\sigma(t_0-t)$, and letting $t \to t_0$ we arrive at a contradiction. The proof is complete.

Integrating equation (1.1) over the space $\mathbb{R}^{N+M} \times (t_1, t)$, similarly to the proof of Lemma 7.1 we obtain

$$\begin{aligned}\|u(t_1)\|_{1,\mathbb{R}^{N+M}} - \|u(t)\|_{1,\mathbb{R}^{N+M}} &= \int_{t_1}^t \int_{\mathbb{R}^{N+M}} a(\rho(z)) f(\tau) |D_L u^\nu|^q \, dz \, d\tau \\ &= \int_{t_1}^t \int_{\mathbb{R}^{N+M}} a(\rho(z)) f(\tau) |\nu u^{\nu-1} D_L u|^q u^{\frac{(\vartheta-1)q}{\lambda+1}} u^{-\frac{(\vartheta-1)q}{\lambda+1}} \, dz \, d\tau \\ &\leqslant C\Big(\int_{t_1}^t \int_{\mathbb{R}^{N+M}} u^{\vartheta-1} |D_L u|^{\lambda+1} \, dz \, d\tau\Big)^{\frac{q}{\lambda+1}} \\ &\quad \times \Big(\int_{t_1}^t \int_{\mathbb{R}^{N+M}} a(\rho(z))^{\frac{\lambda+1}{\lambda+1-q}} f(\tau)^{\frac{\lambda+1}{\lambda+1-q}} \big[u^{(\nu-1)q - \frac{(\vartheta-1)q}{\lambda+1}}\big]^{\frac{\lambda+1}{\lambda+1-q}} \, dz \, d\tau\Big)^{\frac{\lambda+1-q}{\lambda+1}} \\ &= C P_1^{\frac{q}{\lambda+1}} P_2^{\frac{\lambda+1-q}{\lambda+1}}.\end{aligned}$$

Using (7.1) we see that

$$\begin{aligned}P_1 = \int_{t_1}^t \int_{\mathbb{R}^{N+M}} u^{\vartheta-1} |D_L u|^{\lambda+1} \, dz \, d\tau &\leqslant C\|u^{1+\vartheta}(t_1)\|_{1,\mathbb{R}^{N+M}} \\ &\leqslant C\|u(t_1)\|_{1,\mathbb{R}^{N+M}} \|u_0\|_1^{\frac{(\lambda+1)\vartheta}{K}} t_1^{-\frac{Q\vartheta}{K}}.\end{aligned}$$

Assume that (1.8) holds; then for sufficiently large t we have the estimate $Z(t) \leqslant Ct^{\frac{1}{K}} \leqslant C\omega(t)t^{\frac{1}{K}}$. Using Proposition 8 and the law of mass conservation for the problem without damping we obtain

$$\begin{aligned}P_2 &= \int_{t_1}^t \int_{\mathbb{R}^{N+M}} a(\rho(z))^{\frac{\lambda+1}{\lambda+1-q}} f(\tau)^{\frac{\lambda+1}{\lambda+1-q}} u^{\frac{(\nu-1)q(\lambda+1)}{\lambda+1-q} - \frac{(\vartheta-1)q}{\lambda+1-q}} \, dz \, d\tau \\ &\leqslant \|u(t_1)\|_{1,\mathbb{R}^{N+M}} \int_{t_1}^t C a(\omega(\tau)\tau^{\frac{1}{K}})^{\frac{\lambda+1}{\lambda+1-q}} f(\tau)^{\frac{\lambda+1}{\lambda+1-q}} \|u(\tau)\|_{\infty,\mathbb{R}^{N+M}}^{\frac{H-\vartheta q}{\lambda+1-q}} \, d\tau \\ &\leqslant \|u(t_1)\|_{1,\mathbb{R}^{N+M}} \int_{t_1}^t C a(\omega(\tau)\tau^{\frac{1}{K}})^{\frac{\lambda+1}{\lambda+1-q}} f(\tau)^{\frac{\lambda+1}{\lambda+1-q}} \tau^{-\frac{Q(H-\vartheta q)}{K(\lambda+1-q)}} \, d\tau.\end{aligned}$$

From (4.5) we see that

$$a(\omega(\tau)\tau^{\frac{1}{K}})^{\frac{\lambda+1}{\lambda+1-q}} f(\tau)^{\frac{\lambda+1}{\lambda+1-q}} \tau^{-\frac{Q(H-\vartheta q)}{K(\lambda+1-q)}} = \frac{\tau^{\frac{\vartheta q Q}{K(\lambda+1-q)}}}{\tau \omega^{\frac{H(\lambda+1)}{(\lambda-1)(\lambda+1-q)}}(\tau)}.$$

If ϑ is taken sufficiently small, then

$$
\begin{gathered}
a(\tau^{\frac{1}{K}})^{\frac{\lambda+1}{\lambda+1-q}} f(\tau)^{\frac{\lambda+1}{\lambda+1-q}} \tau^{-\frac{Q(H-\vartheta q)}{K(\lambda+1-q)}} \\
= \tau^{-1-\epsilon\frac{H(\lambda+1)}{(\lambda-1)(\lambda+1-q)}} \tau^{\frac{\vartheta q Q}{K(\lambda+1-q)}} = \tau^{-1-\left(\epsilon\frac{H(\lambda+1)}{(\lambda-1)(\lambda+1-q)}-\frac{\vartheta q Q}{K(\lambda+1-q)}\right)}, \\
-\left(\epsilon\frac{H(\lambda+1)}{(\lambda-1)(\lambda+1-q)}-\frac{\vartheta q Q}{K(\lambda+1-q)}\right) < 0, \\
P_2 \leqslant \|u(t_1)\|_{1,\mathbb{R}^{N+M}} C(\|u_0\|_1) t_1^{-\left(\epsilon\frac{H(\lambda+1)}{(\lambda-1)(\lambda+1-q)}-\frac{\vartheta q Q}{K(\lambda+1-q)}\right)}, \\
\|u(t_1)\|_{1,\mathbb{R}^{N+M}} \leqslant \|u(t)\|_{1,\mathbb{R}^{N+M}} + C\|u(t_1)\|_{1,\mathbb{R}^{N+M}} t_1^{-\epsilon\frac{H}{\lambda-1}}.
\end{gathered}
$$

Hence for sufficiently large t_1 we obtain

$$
\|u(t_1)\|_{1,\mathbb{R}^{N+M}} \leqslant \|u(t)\|_{1,\mathbb{R}^{N+M}} + \frac{1}{2}\|u(t_1)\|_{1,\mathbb{R}^{N+M}}.
$$

In other words, we have proved that $\|u(t)\|_{1,\mathbb{R}^{N+M}} \geqslant \frac{1}{2}\|u(t_1)\|_{1,\mathbb{R}^{N+M}}$, $t > t_1$. By Lemma 7.1,

$$
\frac{1}{2}\|u(t_1)\|_{1,\mathbb{R}^{N+M}} \equiv C > 0
$$

because $t_1 > 0$. The proof is complete.

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