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Rational bases of $GL(2,\mathbb{R})$-comitants and of $GL(2,\mathbb{R})$-invariants for the planar system of differential equations with nonlinearities of the fourth degree

Stanislav Ciubotaru

Abstract. This paper is devoted to the construction of minimal rational bases of $GL(2,\mathbb{R})$-comitants and minimal rational bases of $GL(2,\mathbb{R})$-invariants for the bidimensional system of differential equations with nonlinearities of the fourth degree. For this system, three minimal rational bases of $GL(2,\mathbb{R})$-comitants and two minimal rational bases of $GL(2,\mathbb{R})$-invariants were constructed. It was established that any minimal rational basis of $GL(2,\mathbb{R})$-comitants contains 13 comitants and each minimal rational basis of $GL(2,\mathbb{R})$-invariants contains 11 invariants.

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1 Definitions and notations

Let us consider the system of differential equations with nonlinearities of the fourth degree

$$\frac{dx}{dt} = P_1(x,y) + P_4(x,y), \quad \frac{dy}{dt} = Q_1(x,y) + Q_4(x,y), \tag{1}$$

where $P_i(x,y),\ Q_i(x,y)$ are homogeneous polynomials of degree i in x and y with real coefficients.

The goal of this paper is to construct minimal rational bases of GL-comitants as well as GL-invariants for the above system. It is known (see for instance $[1,2]$) that invariant polynomials with respect to the group $GL(2,\mathbb{R})$ could be used to characterize some geometric proprieties of system (1). And clearly the knowledge of the elements of minimal rational bases essentially limits the number of invariant polynomials which could be used in the study of this system.

System (1) can be written in the following coefficient form:

$$\begin{aligned} \frac{dx}{dt} &= cx + dy + gx^4 + 4hx^3y + 6kx^2y^2 + 4lxy^3 + my^4, \\ \frac{dy}{dt} &= ex + fy + nx^4 + 4px^3y + 6qx^2y^2 + 4rxy^3 + sy^4. \end{aligned} \tag{2}$$

We denote by A the 14-dimensional coefficient space of system (1), by $\mathbf{a} \in A$ the vector of coefficients $\mathbf{a} = (c,d,e,f,g,h,k,l,m,n,p,q,r,s)$, by $\boldsymbol{q} \in \mathcal{Q} \subseteq Aff(2,\mathbb{R})$

a nondegenerate linear transformation of the phase plane of system (1), by $\mathbf{q}$ the transformation matrix and by $r_{\boldsymbol{q}}(\mathbf{a})$ a linear representation of coefficients of the transformed system in the space A.

Definition 1 (see [1, 2])**.** *A polynomial $\mathcal{K}(\mathbf{a},\mathbf{x})$ in coefficients of system* (1) *and coordinates of the vector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ is called a comitant of system* (1) *with respect to the group $\mathcal{Q}$ if there exists a function $\lambda : \mathcal{Q} \to \mathbb{R}$ such that*

$$\mathcal{K}(r_{\boldsymbol{q}}(\mathbf{a}), \mathbf{q}\mathbf{x}) \equiv \lambda(\boldsymbol{q})\mathcal{K}(\mathbf{a},\mathbf{x})$$

for every $\boldsymbol{q} \in \mathcal{Q}$, $\mathbf{a} \in A$ and $\mathbf{x} \in \mathbb{R}^2$.

If $\mathcal{Q}$ is the group $GL(2,\mathbb{R})$ of nondegenerate linear transformations

$$\mathbf{u} = \mathbf{q}\mathbf{x}, \quad \Delta_{\mathbf{q}} = \det \mathbf{q} \neq 0 \tag{3}$$

of the phase plane of system (1), where $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ is a vector of new phase variables and $\mathbf{q} = \begin{pmatrix} q_1^1 & q_2^1 \\ q_1^2 & q_2^2 \end{pmatrix}$ is the transformation matrix, then the comitant is called $GL(2,\mathbb{R})$-comitant or center-affine comitant. In what follows only $GL(2,\mathbb{R})$-comitants are considered. If a comitant does not depend on coordinates of the vector $\mathbf{x}$, then it is called invariant.

The function $\lambda(\boldsymbol{q})$ is called a multiplicator. It is known [1] that the function $\lambda(\boldsymbol{q})$ has the form $\lambda(\boldsymbol{q}) = \Delta_{\mathbf{q}}^{-\chi}$, where χ is an integer, which is called the weight of the comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$. If $\chi = 0$, then the comitant is called absolute, otherwise it is called relative.

According to [1] if a GL-comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$ is a non-homogeneous polynomial with respect to $\mathbf{x}$ and $\mathbf{a}$, then each its homogeneity is also a GL-comitant. So in what follows we shall consider only homogeneous invariant polynomials.

We say that a comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$ has the character $(\rho;\ \chi;\ \delta)$ if it has the weight χ, the degree δ with respect to coefficients of system (1) and the degree ρ with respect to coordinates of the vector $\mathbf{x}$.

Every comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$ of system (1) of the character $(\rho;\ \chi;\ \delta)$ can be represented in the form

$$\mathcal{K}(\mathbf{a},\mathbf{x}) = T_0(\mathbf{a})x^\rho + T_1(\mathbf{a})x^{\rho-1}y + \ldots + T_{\rho-1}(\mathbf{a})xy^{\rho-1} + T_\rho(\mathbf{a})y^\rho,$$

where $T_i(\mathbf{a})$ are polynomials in coefficients of the system. The polynomial $T_0(\mathbf{a})$ is called the semi-invariant of the comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$ and is denoted by $S\mathcal{K}(\mathbf{a})$. Thus,

$$S\mathcal{K}(\mathbf{a}) = \frac{1}{\rho!} \cdot \frac{\partial^\rho \mathcal{K}(\mathbf{a},\mathbf{x})}{\partial x^\rho}.$$

Definition 2. *A set $\mathcal{S}$ of comitants (invariants) is called a rational basis on $\mathcal{M} \subseteq A$ of comitants (invariants) for system (1) with respect to the group $\mathcal{Q}$ if any comitant (invariant) of system (1) with respect to the group $\mathcal{Q}$ can be expressed as a rational function of elements of the set $\mathcal{S}$.*

Definition 3. *A rational basis on $\mathcal{M} \subseteq A$ of comitants (invariants) for system (1) with respect to the group $\mathcal{Q}$ is called minimal if by the removal from it of any comitant (invariant) it ceases to be a rational basis.*

We say that $GL(2,\mathbb{R})$-comitants (invariants) of a set $\mathcal{S}$ are polynomial independent if there is no identity between them of the form $\mathcal{P}(\mathcal{K}_i) \equiv 0$, where $\mathcal{P}(\mathcal{K}_i)$ is a polynomial in elements of the set $\mathcal{S}$.

Definition 4 (see [3])**.** *Let φ and ψ be homogeneous polynomials in coordinates of the vector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ of the degrees ρ_1 and ρ_2, respectively. The polynomial*

$$(\varphi, \psi)^{(j)} = \frac{(\rho_1 - j)!(\rho_2 - j)!}{\rho_1!\rho_2!} \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{\partial^j \varphi}{\partial x^{j-i} \partial y^i} \frac{\partial^j \psi}{\partial x^i \partial y^{j-i}}$$

is called the transvectant of index j of polynomials φ and ψ.

Property 1 (see [4])**.** *If polynomials φ and ψ are $GL(2,\mathbb{R})$-comitants of system* (1) *with the characters $(\rho_\varphi;\ \chi_\varphi;\ \delta_\varphi)$ and $(\rho_\psi;\ \chi_\psi;\ \delta_\psi)$, respectively, then the transvectant of index $j \le \min\{\rho_\varphi, \rho_\psi\}$ is a $GL(2,\mathbb{R})$-comitant of system* (1) *with the character $(\rho_\varphi + \rho_\psi - 2j;\ \chi_\varphi + \chi_\psi + j;\ \delta_\varphi + d_\psi)$. If $j > \min\{\rho_\varphi, \rho_\psi\}$, then $(\varphi, \psi)^{(j)} = 0$.*

$GL(2,\mathbb{R})$-comitants of the first degree with respect to coefficients of system (1) have the form

$$R_i = P_i(x,y)y - Q_i(x,y)x,\ \ S_i = \frac{1}{i}\left(\frac{\partial P_i(x,y)}{\partial x} + \frac{\partial Q_i(x,y)}{\partial y}\right),\ \ i = 1,4.$$

By using the comitants R_i and S_i $(i = 1,4)$, and the notion of transvectant the following $GL(2,\mathbb{R})$-comitants and invariants of system (1) were constructed (in the list below, the bracket "$[\![$" is used in order to avoid placing the otherwise necessary parenthesis "(" (up to six)):

$$K_1 = R_4, \quad K_2 = S_4, \quad K_3 = (R_4, R_4)^{(4)}, \quad K_4 = (R_4, R_4)^{(2)},$$
$$K_5 = (R_4, S_4)^{(3)}, \quad K_6 = (R_4, S_4)^{(2)}, \quad K_7 = (R_4, S_4)^{(1)},$$
$$K_8 = (S_4, S_4)^{(2)}, \quad K_{10} = [\![R_4, R_4)^{(4)}, R_4)^{(1)},$$
$$K_{13} = [\![R_4, R_4)^{(2)}, R_4)^{(1)}, \quad K_{17} = [\![R_4, S_4)^{(3)}, S_4)^{(2)},$$
$$K_{18} = [\![R_4, S_4)^{(3)}, S_4)^{(1)}, \quad K_{21} = [\![S_4, S_4)^{(2)}, S_4)^{(1)}, \quad Q_1 = R_1,$$
$$Q_2 = S_1, \quad Q_3 = (R_4, R_1)^{(2)}, \quad Q_4 = (R_4, R_1)^{(1)}, \quad Q_5 = (S_4, R_1)^{(2)},$$
$$Q_6 = (S_4, R_1)^{(1)}, \quad Q_7 = (R_1, R_1)^{(2)}, \quad Q_{19} = [\![R_4, R_1)^{(2)}, R_1)^{(2)},$$
$$Q_{20} = [\![R_4, R_1)^{(2)}, R_1)^{(1)}, \quad Q_{21} = [\![S_4, R_1)^{(2)}, R_1)^{(1)},$$
$$Q_{43} = [\![R_4, R_1)^{(2)}, R_1)^{(2)}, R_1)^{(1)},$$
$$I_1 = S_1, \quad I_2 = (R_1, R_1)^{(2)}, \quad I_3 = [\![R_1, Q_5)^{(1)}, Q_5)^{(1)},$$

$$I_4 = [\![Q_{19}, R_1)^{(1)}, Q_{19})^{(1)},$$
$$J_1 = [\![R_4, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)},$$
$$J_2 = [\![S_4, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)},$$
$$J_3 = [\![R_4, R_1)^{(2)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)},$$
$$J_4 = [\![R_4, R_1)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)},$$
$$J_6 = [\![S_4, R_1)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)},$$
$$J_{19} = [\![R_4, R_1)^{(2)}, R_1)^{(2)}, Q_5)^{(1)},$$
$$J_{20} = [\![R_4, R_1)^{(2)}, R_1)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)},$$
$$J_{43} = [\![R_4, R_1)^{(2)}, R_1)^{(2)}, R_1)^{(1)}, Q_5)^{(1)},$$
$$\widetilde{J}_1 = [\![R_4, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)},$$
$$\widetilde{J}_2 = [\![S_4, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)},$$
$$\widetilde{J}_3 = [\![R_4, R_1)^{(2)}, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)},$$
$$\widetilde{J}_4 = [\![R_4, R_1)^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)},$$
$$\widetilde{J}_5 = [\![S_4, R_1)^{(2)}, Q_{19})^{(1)},$$
$$\widetilde{J}_6 = [\![S_4, R_1)^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)},$$
$$\widetilde{J}_{20} = [\![R_4, R_1)^{(2)}, R_1)^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)}, Q_{19})^{(1)},$$
$$\widetilde{J}_{21} = [\![S_4, R_1)^{(2)}, R_1)^{(1)}, Q_{19})^{(1)}.$$

2 Rational bases of $GL(2,\mathbb{R})$-comitants

2.1 The case $K_1 \not\equiv 0$ $(R_4 \not\equiv 0)$

Theorem 1. *The set of $GL(2,\mathbb{R})$-comitants*

$$\{\mathbf{K_1},\ K_2,\ K_3,\ K_4,\ K_5,\ K_6,\ K_7,\ K_{10},\ K_{13},\ Q_1,\ Q_2,\ Q_3,\ Q_4\} \tag{4}$$

is a minimal rational basis of $GL(2,\mathbb{R})$-comitants for system (1) *of differential equations with nonlinearities of the fourth degree on $\mathcal{M} = \{\, a \in A \mid K_1 \not\equiv 0\}$.*

Proof. Firstly we will show that the set of comitants $\{K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_{10}, K_{13}, Q_1, Q_2, Q_3, Q_4\}$ is a rational basis of $GL(2,\mathbb{R})$-comitants when $K_1 \not\equiv 0$. Let the $GL(2,\mathbb{R})$-comitant $K_1 \not\equiv 0$. By using the transformation:

$$\begin{aligned} u &= \frac{1}{5K_1(\mathbf{a},\mathbf{w})}\cdot\frac{\partial K_1(\mathbf{a},\mathbf{w})}{\partial w_1}\cdot x + \frac{1}{5K_1(\mathbf{a},\mathbf{w})}\cdot\frac{\partial K_1(\mathbf{a},\mathbf{w})}{\partial w_2}\cdot y,\\ v &= -w_2 x + w_1 y, \end{aligned} \tag{5}$$

where $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{R}^2$, system (1) can be brought to the system:

$$\begin{aligned}
\frac{du}{dt} &= \frac{K_1Q_2 + 2Q_4}{2K_1}u + \frac{-K_4Q_1 + 2K_1Q_3}{2K_1^2}v + \frac{4}{5}K_2u^4 + \\
&+ \frac{12K_7 + 10K_4}{5K_1}u^3v + \frac{-6K_2K_4 + 12K_1K_6 - 30K_{13}}{5K_1^2}u^2v^2 + \\
&+ \frac{10K_1^2K_3 - 15K_4^2 + 4K_1^2K_5 - 6K_4K_7 - 4K_2K_{13}}{5K_1^3}uv^3 + \\
&+ \frac{-K_1^2K_{10} + K_4K_{13}^2}{K_1^4}v^4, \qquad (6)\\
\frac{dv}{dt} &= -Q_1u + \frac{K_1Q_2 - 2Q_4}{2K_1}v - K_1u^4 + \frac{4}{5}K_2u^3v + \\
&+ \frac{12K_7 - 15K_4}{5K_1}u^2v^2 + \frac{-6K_2K_4 + 12K_1K_6 + 20K_{13}}{5K_1^2}uv^3 + \\
&+ \frac{-10K_1^2K_3 + 15K_4^2 + 16K_1^2K_5 - 24K_4K_7 - 16K_2K_{13}}{20K_1^3}v^4.
\end{aligned}$$

According to [5, Lemma 4] any $GL(2,\mathbb{R})$-comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$ of system (1) coincides with the semi-invariant $S\mathcal{K}$ of any comitant $\mathcal{K}$ calculated for system (6) in which coordinates of the vector $\mathbf{w}$ are replaced, respectively, with coordinates of the vector $\mathbf{x}$. In other words

$$\mathcal{K}(\mathbf{a},\mathbf{x}) = \frac{1}{\rho!} \cdot \frac{\partial^\rho \mathcal{K}\left(\mathbf{b}\left(w_1, w_2\right), \mathbf{u}\right)}{\partial u^\rho}\Bigg|_{\substack{w_1 = x \\ w_2 = y}}, \qquad (7)$$

where $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{q} \cdot \mathbf{x}$, $\mathbf{q}$ is the matrix of transformation (5) and $\mathbf{b}$ is the vector of coefficients of system (6). So any $GL(2,\mathbb{R})$-comitant $\mathcal{K}(\mathbf{a},\mathbf{x})$ can be represented as a rational function of comitants (4) where denominator is a nonnegative integer power of the comitant K_1. Thus, the set of comitatns (4) is a rational basis of $GL(2,\mathbb{R})$-comitants for system (1) on $\mathcal{M} = \{\mathbf{a} \in A \mid K_1 \not\equiv 0\}$.

Next we will show that this basis is minimal. Indeed, suppose the contrary that the rational basis (4) is not a minimal one. This means that among the comitants K_i and Q_j there exists a polynomial identity $\mathcal{P}(K_i, Q_j) \equiv 0$ in $\mathbb{R}[x,y]$. On the other hand, since each K_i or Q_j is well determined by its semi-invariant SK_i or SQ_j, respectively, we conclude that the identity $\mathcal{P}(SK_i, SQ_j) \equiv 0$ must also hold. To calculate the expressions for these semi-invariants, for simplicity we apply the following substitution: $c = \frac{D+F}{2}$, $d = E$, $e = -C$, $f = \frac{F-D}{2}$, $g = \frac{4P+H}{5}$, $h = \frac{K+2Q}{10}$, $k = \frac{3L+4R}{30}$, $l = \frac{M+S}{5}$, $m = N$, $n = -G$, $p = \frac{P-H}{5}$, $r = \frac{2R-L}{10}$, $s = \frac{4S-M}{5}$, $q = \frac{4Q-3K}{30}$.

By using these substitutions system (2) is written in the form:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{D+F}{2}x + Ey + \frac{4P+H}{5}x^4 + \frac{4Q+2K}{5}x^3y + \\
&\quad \frac{4R+3L}{5}x^2y^2 + \frac{4S+4M}{5}xy^3 + Ny^4, \\
\frac{dy}{dt} &= -Cx + \frac{F-D}{2}y - Gx^4 + \frac{4P-4H}{5}x^3y + \\
&\quad \frac{4Q-3K}{5}x^2y^2 + \frac{4R-2L}{5}xy^3 + \frac{4S-M}{5}y^4.
\end{aligned} \tag{8}$$

For system (8) the comitants R_1, S_1, R_4 and S_4 have the following form

$$\begin{aligned}
&R_1 = Cx^2 + Dxy + Ey^2, \quad S_1 = F \\
&R_4 = Gx^5 + Hx^4y + Kx^3y^2 + Lx^2y^3 + Mxy^4 + Ny^5, \\
&S_4 = Px^3 + Qx^2y + Rxy^2 + Sy^3,
\end{aligned} \tag{9}$$

For system (8) semi-invariants of comitants listed in the theorem have the form:

$$\begin{aligned}
SK_1 &= G, \\
SK_2 &= P, \\
SK_3 &= \frac{1}{50}(3K^2 - 8HL + 20GM), \\
SK_4 &= -\frac{1}{25}(2H^2 - 5GK), \\
SK_5 &= -\frac{1}{10}(LP - KQ + 2HR - 10GS), \\
SK_6 &= \frac{1}{30}(3KP - 4HQ + 10GR), \\
SK_7 &= -\frac{1}{15}(3HP - 5GQ), \\
SK_{10} &= -\frac{1}{250}(-3HK^2 + 8H^2L + 5GKL - 50GHM + 250G^2N), \\
SK_{13} &= -\frac{1}{250}(4H^3 - 15GHK + 25G^2L), \\
SQ_1 &= C, \\
SQ_2 &= F, \\
SQ_3 &= \frac{1}{10}(10EG - 2DH + CK), \\
SQ_4 &= \frac{1}{10}(5DG - 2CH).
\end{aligned} \tag{10}$$

Next in order to prove the impossibility of the polynomial identity $\mathcal{P}(SK_i, SQ_j) \equiv 0$ we use Table 1 in which the sign " $+$ " indicates that the respective parameter is contained in the expression of the semi-invariant SK_i or SQ_j, and the sign " $-$ " indicates that the respective parameter is missing from the expression of the semi-invariant SK_i or SQ_j.

Table 1

Semi-invariant	Parameters of system (8)													
	C	D	E	F	G	H	K	L	M	N	P	Q	R	S
SK_1	−	−	−	−	+	−	−	−	−	−	−	−	−	−
SK_2	−	−	−	−	−	−	−	−	−	−	+	−	−	−
SK_3	−	−	−	−	+	+	+	+	+	−	−	−	−	−
SK_4	−	−	−	−	+	+	+	−	−	−	−	−	−	−
SK_5	−	−	−	−	+	+	+	+	−	−	+	+	+	+
SK_6	−	−	−	−	+	+	+	−	−	−	+	+	+	−
SK_7	−	−	−	−	+	+	−	−	−	−	+	+	−	−
SK_{10}	−	−	−	−	+	+	+	+	+	+	−	−	−	−
SK_{13}	−	−	−	−	+	+	+	+	−	−	−	−	−	−
SQ_1	+	−	−	−	−	−	−	−	−	−	−	−	−	−
SQ_2	−	−	−	+	−	−	−	−	−	−	−	−	−	−
SQ_3	+	+	+	−	+	+	+	−	−	−	−	−	−	−
SQ_4	+	+	−	−	+	+	−	−	−	−	−	−	−	−

We observe that the parameter S is contained only in semi-invariant SK_5 and hence the identity $\mathcal{P}(SK_i, SQ_j) \equiv 0$ must be homogeneous in SK_5. This means that this semi-invariant could be removed from the list due to the parameter S and we denote this couple by $\langle SK_5, S\rangle$. Examining the remaining table after the removal of the line corresponding to the semi-invariant SK_5 and of the column corresponding to the parameter S, by the same reason we get the couple $\langle SK_6, R\rangle$ which allows us to remove the line corresponding to the semi-invariant SK_6 and the column corresponding to the parameter R. In the same way, we obtain the couples $\langle SK_7, Q\rangle$, $\langle SK_2, P\rangle$, $\langle SK_{10}, N\rangle$, $\langle SQ_3, E\rangle$, $\langle SQ_4, D\rangle$, $\langle SQ_2, F\rangle$, $\langle SQ_1, C\rangle$, $\langle SK_3, M\rangle$, $\langle SK_{13}, L\rangle$, $\langle SK_4, K\rangle$, $\langle SK_1, G\rangle$. It follows that the set of comitants listed in Theorem 1 are polynomial independent. So if by the removal from it of any comitant it ceases to be a rational basis. This proves that the set of comitants listed in Theorem 1 is a minimal rational basis of $GL(2,\mathbb{R})$-comitants for system (1). ☐

2.2 The case $\boldsymbol{K_2 \not\equiv 0}$ $\quad(\boldsymbol{S_4 \not\equiv 0})$

Theorem 2. *The set of* $GL(2,\mathbb{R})$*-comitants*

$$\{K_1,\ \mathbf{K}_2,\ K_5,\ K_6,\ K_7,\ K_8,\ K_{17},\ K_{18},\ K_{21},\ Q_1,\ Q_2,\ Q_5,\ Q_6\} \tag{11}$$

is a minimal rational basis of $GL(2,\mathbb{R})$*-comitants for system* (1) *of differential equations with nonlinearities of the fourth degree on* $\mathcal{M} = \{\mathbf{a} \in A \mid K_2 \not\equiv 0\}$.

Proof. Firstly we will show that the set of comitants $\{K_1, K_2, K_5, K_6, K_7, K_8, K_{17}, K_{18}, K_{21}, Q_1, Q_2, Q_5, Q_6\}$ is a rational basis of $GL(2,\mathbb{R})$-comitants when $K_2 \not\equiv 0$.

The proof of this theorem is completely the same as the proof of previous theorem. Let the $GL(2,\mathbb{R})$-comitant $K_2 \not\equiv 0$. By using the transformation:

$$u = \frac{1}{3K_2(\mathbf{a},\mathbf{w})}\cdot\frac{\partial K_2(\mathbf{a},\mathbf{w})}{\partial w_1}\cdot x + \frac{1}{3K_2(\mathbf{a},\mathbf{w})}\cdot\frac{\partial K_2(\mathbf{a},\mathbf{w})}{\partial w_2}\cdot y, \qquad v = -w_2 x + w_1 y, \tag{12}$$

system (1) can be brought to the system:

$$\begin{aligned}
\frac{du}{dt} &= \frac{K_2Q_2+2Q_6}{2K_2}u + \frac{-K_8Q_1+2K_2Q_5}{2K_2^2}v + \frac{4K_2^2-5K_7}{5K_2}u^4 + \\
&+\frac{4K_2K_6-2K_1K_8}{K_2^2}u^3v + \frac{-30K_2^2K_5+6K_2^2K_8+45K_7K_8-30K_1K_{21}}{5K_2^3}u^2v^2 + \\
&+\frac{-30K_2K_6K_8+15K_1K_8^2+20K_2^2K_{18}-4K_2^2K_{21}+20K_7K_{21}}{5K_2^4}uv^3 + \\
&+\frac{8K_2^2K_5K_8-9K_7K_8^2-4K_2^3K_{17}-4K_2K_6K_{21}+8K_1K_8K_{21}}{4K_2^5}v^4, \qquad (13)\\
\frac{dv}{dt} &= -Q_1u + \frac{K_2Q_2-2Q_6}{2K_2}v - K_1u^4 + \frac{4K_2^2+20K_7}{5K_2}u^3v + \\
&+\frac{-6K_2K_6+3K_1K_8}{K_2^2}u^2v^2 + \frac{20K_2^2K_5+6K_2^2K_8-30K_7K_8+20K_1K_{21}}{5K_2^3}uv^3 + \\
&+\frac{30K_2K_6K_8-15K_1K_8^2-20K_2^2K_{18}-16K_2^2K_{21}-20K_7K_{21}}{20K_2^4}v^4.
\end{aligned}$$

According to [5, Lemma 4] it follows that the set of comitants (11) forms a rational basis of $GL(2,\mathbb{R})$-comitants for system (1). The minimality results from the expressions of semi-invariants of comitants (11), calculated for system (8), which are the following:

$$\begin{aligned}
&SK_1 = G,\\
&SK_2 = P,\\
&SK_5 = -1/10(LP-KQ+2HR-10GS),\\
&SK_6 = 1/30(3KP-4HQ+10GR),\\
&SK_7 = -1/15(3HP-5GQ),\\
&SK_8 = -2/9(Q^2-3PR),\\
&SK_{17} = -1/30(30NP^2-10MPQ+2LQ^2+4LPR-3KQR+2HR^2-\\
&\qquad -3KPS+4HQS-10GRS), \qquad (14)\\
&SK_{18} = 1/30(6MP^2-4LPQ+KQ^2+3KPR-2HQR-\\
&\qquad -6HPS+10GQS),\\
&SK_{21} = -1/27(2Q^3-9PQR+27P^2S),\\
&SQ_1 = C,
\end{aligned}$$

$SQ_2 = F,$
$SQ_5 = 1/3(3EP - DQ + CR),$
$SQ_6 = 1/6(3DP - 2CQ).$

Next in order to prove the impossibility of the polynomial identity $\mathcal{P}(SK_i, SQ_j) \equiv 0$ we use Table 2 in which the sign " + " indicates that the respective parameter is contained in the expression of the semi-invariant SK_i or SQ_j, and the sign " − " indicates that the respective parameter is missing from the expression of the semi-invariant SK_i or SQ_j.

Table 2

	Parameters of system (8)													
Semi-invariant	C	D	E	F	G	H	K	L	M	N	P	Q	R	S
SK_1	−	−	−	−	+	−	−	−	−	−	−	−	−	−
SK_2	−	−	−	−	−	−	−	−	−	−	+	−	−	−
SK_5	−	−	−	−	+	+	+	+	−	−	+	+	+	+
SK_6	−	−	−	−	+	+	+	−	−	−	+	+	+	−
SK_7	−	−	−	−	+	+	−	−	−	−	+	+	−	−
SK_8	−	−	−	−	−	−	−	−	−	−	+	+	+	−
SK_{17}	−	−	−	−	+	+	+	+	+	+	+	+	+	+
SK_{18}	−	−	−	−	+	+	+	+	+	−	+	+	+	+
SK_{21}	−	−	−	−	−	−	−	−	−	−	+	+	+	+
SQ_1	+	−	−	−	−	−	−	−	−	−	−	−	−	−
SQ_2	−	−	−	+	−	−	−	−	−	−	−	−	−	−
SQ_5	+	+	+	−	−	−	−	−	−	−	+	+	+	−
SQ_6	+	+	−	−	−	−	−	−	−	−	+	+	−	−

In the same way as in the proof of Theorem 1 we obtain the couples $\langle SK_{17}, N\rangle$, $\langle SK_{18}, M\rangle$, $\langle SK_5, L\rangle$, $\langle SK_6, K\rangle$, $\langle SK_7, H\rangle$, $\langle SK_1, G\rangle$, $\langle SQ_2, F\rangle$, $\langle SQ_5, E\rangle$, $\langle SQ_6, D\rangle$, $\langle SQ_1, C\rangle$, $\langle SK_8, R\rangle$, $\langle SK_2, P\rangle$,$\langle SK_{21}, S\rangle$.

From Table 2 it follows that the comitants (11) are polynomial independent. $\square$

2.3 The case $\boldsymbol{Q_1 \not\equiv 0}$ $\quad(\boldsymbol{R_1 \not\equiv 0})$

Theorem 3. *The set of $GL(2,\mathbb{R})$-comitants*

$$\{K_1,\ K_2,\ \mathbf{Q}_1,\ Q_2,\ Q_3,\ Q_4,\ Q_5,\ Q_6,\ Q_7,\ Q_{19},\ Q_{20},\ Q_{21},\ Q_{43}\} \qquad (15)$$

is a minimal rational basis of $GL(2,\mathbb{R})$-comitants for system (1) *of differential equations with nonlinearities of the fourth degree on $\mathcal{M} = \{\mathbf{a} \in A \mid Q_1 \not\equiv 0\}$.*

Proof. Firstly we will show that the set of comitants $\{K_1, K_2, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_{19}, Q_{20}, Q_{21}, Q_{43}\}$ is a rational basis of $GL(2,\mathbb{R})$-comitants when $Q_1 \not\equiv 0$.

The proof of this theorem is completely the same as the proof of Theorem 1. Let the $GL(2,\mathbb{R})$-comitant $Q_1 \not\equiv 0$. By using the transformation:

$$u = \frac{1}{2Q_1(a,w)} \cdot \frac{\partial Q_1(a,w)}{\partial w_1} \cdot x + \frac{1}{2Q_1(a,w)} \cdot \frac{\partial Q_1(a,w)}{\partial w_2} \cdot y, \qquad v = -w_2 x + w_1 y, \tag{16}$$

system (1) can be brought to the system:

$$\begin{aligned}
\frac{du}{dt} &= \frac{Q_2}{2}u + \frac{Q_7}{2Q_1}v + \\
&+ \frac{4K_2Q_1 - 5Q_4}{5Q_1}u^4 + \frac{20Q_1Q_3 - 12Q_1Q_6 - 10K_1Q_7}{5Q_1^2}u^3v + \\
&+ \frac{12Q_1^2Q_5 - 6K_2Q_1Q_7 + 15Q_4Q_7 - 30Q_1Q_{20}}{5Q_1^3}u^2v^2 + \\
&+ \frac{-20Q_1Q_3Q_7 + 2Q_1Q_6Q_7 + 5K_1Q_7^2 + 20Q_1^2Q_{19} - 4Q_1^2Q_{21}}{5Q_1^4}uv^3 + \\
&+ \frac{4Q_1Q_7Q_{20} - Q_4Q_7^2 - 4Q_1^2Q_{43}}{4Q_1^5}v^4, \\
\frac{dv}{dt} &= -Q_1u + \frac{Q_2}{2}v - K_1u^4 + \\
&+ \frac{4K_2Q_1 + 20Q_4}{5Q_1}u^3v + \frac{15K_1Q_7 - 30Q_1Q_3 - 12Q_1Q_6}{5Q_1^2}u^2v^2 + \\
&+ \frac{12Q_1^2Q_5 - 6K_2Q_1Q_7 - 10Q_4Q_7 + 20Q_1Q_{20}}{5Q_1^3}uv^3 + \\
&+ \frac{20Q_1Q_3Q_7 + 8Q_1Q_6Q_7 - 5K_1Q_7^2 - 20Q_1^2Q_{19} - 16Q_1^2Q_{21}}{20Q_1^4}v^4.
\end{aligned} \tag{17}$$

According to [5, Lemma 4] it follows that the set of comitants (15) forms a rational basis of $GL(2,\mathbb{R})$-comitants for system (1).

The minimality results from expressions of semi-invariants of comitants (15), which are the following:

$$\begin{aligned}
&SK_1 = G, \\
&SK_2 = P, \\
&SQ_1 = C, \\
&SQ_2 = F, \\
&SQ_3 = 1/10(10EG - 2DH + CK), \\
&SQ_4 = 1/10(5DG - 2CH), \\
&SQ_5 = 1/3(3EP - DQ + CR), \\
&SQ_6 = 1/6(3DP - 2CQ),
\end{aligned} \tag{18}$$

$$SQ_7 = -1/2(D^2 - 4CE),$$
$$SQ_{19} = 1/10(10E^2G - 4DEH + D^2K + 2CEK - 2CDL + 2C^2M),$$
$$SQ_{20} = -1/20(-10DEG + 2D^2H + 4CEH - 3CDK + 2C^2L),$$
$$SQ_{21} = -1/6(-3DEP + D^2Q + 2CEQ - 3CDR + 6C^2S),$$
$$SQ_{43} = 1/20(10DE^2G - 4D^2EH - 4CE^2H + D^3K + 6CDEK - 4CD^2L - 4C^2EL + 10C^2DM - 20C^3N).$$

Next in order to prove the impossibility of the polynomial identity $\mathcal{P}(SK_i, SQ_j) \equiv 0$ we use Table 3 in which the sign " + " indicates that the respective parameter is contained in the expression of the semi-invariant SK_i or SQ_j, and the sign " − " indicates that the respective parameter is missing from the expression of the semi-invariant SK_i or SQ_j.

Table 3

Semi-invariant	Parameters of system (8)													
	C	D	E	F	G	H	K	L	M	N	P	Q	R	S
SK_1	−	−	−	−	+	−	−	−	−	−	−	−	−	−
SK_2	−	−	−	−	−	−	−	−	−	−	+	−	−	−
SQ_1	+	−	−	−	−	−	−	−	−	−	−	−	−	−
SQ_2	−	−	−	+	−	−	−	−	−	−	−	−	−	−
SQ_3	+	+	+	−	+	+	+	−	−	−	−	−	−	−
SQ_4	+	+	−	−	+	+	−	−	−	−	−	−	−	−
SQ_5	+	+	+	−	−	−	−	−	−	−	+	+	+	−
SQ_6	+	+	−	−	−	−	−	−	−	−	+	+	−	−
SQ_7	+	+	+	−	−	−	−	−	−	−	−	−	−	−
SQ_{19}	+	+	+	−	+	+	+	+	+	−	−	−	−	−
SQ_{20}	+	+	+	−	+	+	+	+	−	−	−	−	−	−
SQ_{21}	+	+	+	−	−	−	−	−	−	−	+	+	+	+
SQ_{43}	+	+	+	−	+	+	+	+	+	+	−	−	−	−

In the same way as in the proof of previous theorems we obtain the couples $\langle SQ_2, F\rangle$, $\langle SQ_{21}, S\rangle$, $\langle SQ_5, R\rangle$, $\langle SQ_6, Q\rangle$, $\langle SK_2, P\rangle$, $\langle SQ_{43}, N\rangle$, $\langle SQ_{19}, M\rangle$, $\langle SQ_{20}, L\rangle$, $\langle SQ_3, K\rangle$, $\langle SQ_4, H\rangle$, $\langle SK_1, G\rangle$, $\langle SQ_7, E\rangle$,$\langle SQ_1, C\rangle$.

From Table 3 it follows that the comitants (15) are polynomial independent. □

3 Rational bases of $GL(2,\mathbb{R})$-invariants

3.1 The case $I_3 \neq 0$

Theorem 4. *The set of* $GL(2,\mathbb{R})$*-invariants*

$$\{I_1,\ I_2,\ \mathbf{I}_3,\ J_1,\ J_2,\ J_3,\ J_4,\ J_6,\ J_{19},\ J_{20},\ J_{43}\} \tag{19}$$

is a minimal rational basis of $GL(2,\mathbb{R})$*-invariants for system* (1) *of differential equations with nonlinearities of the fourth degree on* $\mathcal{M} = \{\mathbf{a} \in A \mid I_3 \neq 0\}$.

Proof. Firstly we will show that the set of invariants $\{I_1, I_2, I_3, J_1, J_2, J_3, J_4, J_6, J_{19}, J_{20}, J_{43}\}$ is a rational basis of $GL(2,\mathbb{R})$-invariants when $I_3 \neq 0$.

By using the transformation:

$$u = \frac{1}{I_3(a,w)} \cdot \frac{\partial Q_{21}(a,w)}{\partial w_1} \cdot x + \frac{1}{I_3(a,w)} \cdot \frac{\partial Q_{21}(a,w)}{\partial w_2} \cdot y,$$
$$v = \frac{\partial Q_5(a,w)}{\partial w_1} \cdot x + \frac{\partial Q_5(a,w)}{\partial w_2} \cdot y, \tag{20}$$

system (1) can be brought to the system:

$$\begin{aligned}\frac{du}{dt} &= \frac{I_1}{2}u - \frac{I_2}{2I_3}v + \frac{4I_3J_2+5J_4}{5I_3}u^4 + \\ &\frac{-10I_2J_1 - 20I_3J_3 + 12I_3J_6}{5I_3^2}u^3v + \frac{-6I_2I_3J_2 - 15I_2J_4 - 30I_3J_{20}}{5I_3^3}u^2v^2 + \\ &\frac{-4I_3^3 + 5I_2^2J_1 + 20I_2I_3J_3 - 2I_2I_3J_6 + 20I_3^2J_{19}}{5I_3^4}uv^3 + \\ &+\frac{I_2^2J_4 + 4I_2I_3J_{20} + 4I_3^2J_{43}}{4I_3^5}v^4, \\ \frac{dv}{dt} &= I_3u + \frac{I_1}{2}v - J_1u^4 + \frac{4I_3J_2 - 20J_4}{5I_3}u^3v + \\ &+\frac{15I_2J_1 + 30I_3J_3 + 12I_3J_6}{5I_3^2}u^2v^2 + \frac{-6I_2I_3J_2 + 10I_2J_4 + 20I_3J_{20}}{5I_3^3}uv^3 + \\ &\frac{-16I_3^3 - 5I_2^2J_1 - 20I_2I_3J_3 - 8I_2I_3J_6 - 20I_3^2J_{19}}{20I_3^4}v^4.\end{aligned} \tag{21}$$

From system (21), it follows that any $GL(2,\mathbb{R})$-invariant of system (1) with $I_3 \neq 0$ can be represented as a rational function of invariants (19). So the set of $GL(2,\mathbb{R})$-invariants (19) forms a rational basis for system (1) with $I_3 \neq 0$.

To prove the minimality we write the expressions of invariants (19). By using the notation $U = \frac{1}{3}(EQ - DR + 3CS)$ and $V = \frac{1}{3}(3EP - DQ + CR)$ for the invariants (19) we have:

$$\begin{aligned}I_1 &= F, \\ I_2 &= \frac{1}{2}(-D^2 + 4CE), \\ I_3 &= -CU^2 + DUV - EV^2, \\ J_1 &= GU^5 - HU^4V + KU^3V^2 - LU^2V^3 + MUV^4 - NV^5, \\ J_2 &= PU^3 - QU^2V + RUV^2 - SV^3, \\ J_3 &= \frac{1}{10}[(10EG - 2DH + CK)U^3 + (-6EH + 3DK - 3CL)U^2V + \\ &+ (3EK - 3DL + 6CM)UV^2 + (-EL + 2DM - 10CN)V^3],\end{aligned}$$

$$
\begin{aligned}
J_4 = {} & \frac{1}{10}[(5DG-2CH)U^5+(-10EG-3DH+4CK)U^4V+\\
& +(8EH+DK-6CL)U^3V^2+(-6EK+DL+8CM)U^2V^3+\\
& +(4EL-3DM-10CN)UV^4+(-2EM+5DN)V^5],\\
J_6 = {} & \frac{1}{6}[(3DP-2CQ)U^3+(-6EP-DQ+4CR)U^2V+\\
& +(4EQ-DR-6CS)UV^2+(-2ER+3DS)V^3],\\
J_{19} = {} & \frac{1}{10}[(10E^2G-4DEH+D^2K+2CEK-2CDL+2C^2M)U+\\
& +(-2E^2H+2DEK-D^2L-2CEL+4CDM-10C^2N)V],\\
J_{20} = {} & \frac{1}{20}[(10DEG-2D^2H-4CEH+3CDK-2C^2L)U^3+\\
& +(-20E^2G+2DEH+D^2K+2CEK-5CDL+8C^2M)U^2V+\\
& +(8E^2H-5DEK+D^2L+2CEL+2CDM-20C^2N)UV^2+\\
& +(-2E^2K+3DEL-2D^2M-4CEM+10CDN)V^3],\\
J_{43} = {} & \frac{1}{20}[(10DE^2G-4D^2EH-4CE^2H+D^3K+6CDEK-4CD^2L-4C^2EL+\\
& +10C^2DM-20C^3N)U+(-20E^3G+10DE^2H-4D^2EK-4CE^2K+\\
& +D^3L+6CDEL-4CD^2M-4C^2EM+10C^2DN)V].
\end{aligned}
\tag{22}
$$

Next, we write the expressions of the highest power of S in the invariants (22), denoted by EI_i and EJ_i

$$
\begin{aligned}
& EI_1 = F,\\
& EI_2 = \frac{1}{2}(-D^2+4CE),\\
& EI_3 = -C^3,\\
& EJ_1 = C^5G,\\
& EJ_2 = C^3P,\\
& EJ_3 = \frac{1}{10}C^3(10EG-2DH+CK),\\
& EJ_4 = -\frac{1}{10}C^5(-5DG+2CH),\\
& EJ_6 = -\frac{1}{6}C^3(-3DP+2CQ),\\
& EJ_{19} = \frac{1}{10}C(10E^2G-4DEH+D^2K+2CEK-2CDL+2C^2M),\\
& EJ_{20} = -\frac{1}{20}C^3(-10DEG+2D^2H+4CEH-3CDK+2C^2L),\\
& EJ_{43} = -\frac{1}{20}C(-10DE^2G+4D^2EH+4CE^2H-D^3K-6CDEK+4CD^2L+\\
& \qquad +4C^2EL-10C^2DM+20C^3N).
\end{aligned}
\tag{23}
$$

In the following table the sign " + " indicates that the respective parameter is contained in the expression of the highest power of S in the invariants (22) and the sign " − " indicates that the respective parameter is missing from the expression of the highest power of S.

Table 4

Invariant	Parameters of system (8)												
	C	D	E	F	G	H	K	L	M	N	P	Q	R
I_1	−	−	−	+	−	−	−	−	−	−	−	−	−
I_2	+	+	+	−	−	−	−	−	−	−	−	−	−
I_3	+	−	−	−	−	−	−	−	−	−	−	−	−
J_1	+	−	−	−	+	−	−	−	−	−	−	−	−
J_2	+	−	−	−	−	−	−	−	−	−	+	−	−
J_3	+	+	+	−	+	+	+	−	−	−	−	−	−
J_4	+	+	−	−	+	+	−	−	−	−	−	−	−
J_6	+	+	−	−	−	−	−	−	−	−	+	+	−
J_{19}	+	+	+	−	+	+	+	+	+	−	−	−	−
J_{20}	+	+	+	−	+	+	+	+	−	−	−	−	−
J_{43}	+	+	+	−	+	+	+	+	+	+	−	−	−

According to Table 4 we obtain the couples $\langle J_6, Q\rangle$, $\langle J_2, P\rangle$, $\langle J_{43}, N\rangle$, $\langle J_{19}, M\rangle$, $\langle J_{20}, L\rangle$, $\langle J_3, K\rangle$, $\langle J_4, H\rangle$, $\langle J_1, G\rangle$, $\langle I_1, F\rangle$, $\langle I_2, E\rangle$, $\langle J_2, C\rangle$.

From Table 4 it follows that the invariants (19) are polynomial independent. □

3.2 The case $I_4 \neq 0$

Theorem 5. *The set of $GL(2,\mathbb{R})$-invariants*

$$\left\{ I_1,\ I_2,\ \mathbf{I}_4,\ \widetilde{J}_1,\ \widetilde{J}_2,\ \widetilde{J}_3,\ \widetilde{J}_4,\ \widetilde{J}_5,\ \widetilde{J}_6,\ \widetilde{J}_{20},\ \widetilde{J}_{21} \right\} \tag{24}$$

is a minimal rational basis of $GL(2,\mathbb{R})$-invariants for system (1) *of differential equations with nonlinearities of the fourth degree on $\mathcal{M} = \{\mathbf{a} \in A \mid I_4 \neq 0\}$.*

Proof. Firstly we will show that the set of invariants $\{I_1, I_2, I_4, \widetilde{J}_1, \widetilde{J}_2, \widetilde{J}_3, \widetilde{J}_4, \widetilde{J}_5, \widetilde{J}_6, \widetilde{J}_{20}, \widetilde{J}_{21}\}$ is a rational basis of $GL(2,\mathbb{R})$-invariants when $I_4 \neq 0$.

By using the transformation:

$$\begin{aligned} u &= \frac{1}{I_4(a,w)} \cdot \frac{\partial Q_{43}(a,w)}{\partial w_1} \cdot x + \frac{1}{I_4(a,w)} \cdot \frac{\partial Q_{43}(a,w)}{\partial w_2} \cdot y, \\ v &= \frac{\partial Q_{19}(a,w)}{\partial w_1} \cdot x + \frac{\partial Q_{19}(a,w)}{\partial w_2} \cdot y, \end{aligned} \tag{25}$$

system (1) can be brought to the system:

$$\frac{du}{dt} = \frac{I_1}{2}u - \frac{I_2}{2I_4}v + \frac{4I_4\widetilde{J}_2 + 5\widetilde{J}_4}{5I_4}u^4 + \frac{-10I_2\widetilde{J}_1 - 20I_4\widetilde{J}_3 + 12I_4\widetilde{J}_6}{5I_4^2}u^3v +$$

$$+\frac{6I_2I_4\widetilde{J}_2+15I_2\widetilde{J}_4+12I_4^2\widetilde{J}_5+30I_4\widetilde{J}_{20}}{5I_4^3}u^2v^2+$$
$$+\frac{5I_2^2\widetilde{J}_1+20I_2I_4\widetilde{J}_3-2I_2I_4\widetilde{J}_6-4I_4^2\widetilde{J}_{21}}{5I_4^4}uv^3+\frac{4I_4^3+I_2^2\widetilde{J}_4+4I_2I_4\widetilde{J}_{20}}{4I_4^5}v^4, \quad (26)$$
$$\frac{dv}{dt}=I_4u+\frac{I_1}{2}v-\widetilde{J}_1u^4+\frac{4I_4\widetilde{J}_2-20\widetilde{J}_4}{5I_4}u^3v+\frac{15I_2\widetilde{J}_1+30I_4\widetilde{J}_3+12I_4\widetilde{J}_6}{5I_4^2}u^2v^2+$$
$$+\frac{-6I_2I_4\widetilde{J}_2+10I_2\widetilde{J}_4-12I_4^2\widetilde{J}_5+20I_4\widetilde{J}_{20}}{5I_4^3}uv^3+$$
$$+\frac{-5I_2^2\widetilde{J}_1+20I_2I_4\widetilde{J}_3+8I_2I_4\widetilde{J}_6+16I_4^2\widetilde{J}_{21}}{20I_4^4}v^4.$$

From system (26), it follows that any $GL(2,\mathbb{R})$-invariant of system (1) with $I_4\neq 0$ can be represented as a rational function of invariants (24). So the set of $GL(2,\mathbb{R})$-invariants (24) forms a rational basis for system (1) with $I_4\neq 0$.

To prove the minimality we write the expressions of invariants (24). By using the notation $U=\frac{1}{10}(2E^2H-2DEK+D^2L+2CEL-4CDM+10C^2N)$ and $V=\frac{1}{10}(10E^2G-4DEH+D^2K+2CEK-2CDL+2C^2M)$ for the invariants (24) we have:

$$\begin{aligned}
I_1 &= F,\\
I_2 &= \frac{1}{2}(-D^2+4CE),\\
I_4 &= -CU^2+DUV-EV^2,\\
\widetilde{J}_1 &= GU^5-HU^4V+KU^3V^2-LU^2V^3+MUV^4-NV^5,\\
\widetilde{J}_2 &= PU^3-QU^2V+RUV^2-SV^3,\\
\widetilde{J}_3 &= \frac{1}{10}[(10EG-2DH+CK)U^3+(-6EH+3DK-3CL)U^2V+\\
&\quad+(3EK-3DL+6CM)UV^2+(-EL+2DM-10CN)V^3],\\
\widetilde{J}_4 &= \frac{1}{10}[(5DG-2CH)U^5+(-10EG-3DH+4CK)U^4V+\\
&\quad+(8EH+DK-6CL)U^3V^2+(-6EK+DL+8CM)U^2V^3+\\
&\quad+(4EL-3DM-10CN)UV^4+(-2EM+5DN)V^5], \qquad (27)\\
\widetilde{J}_5 &= \frac{1}{3}[(3EP-DQ+CR)U+(-EQ+DR-3CS)V],\\
\widetilde{J}_6 &= \frac{1}{6}[(3DP-2CQ)U^3+(-6EP-DQ+4CR)U^2V+\\
&\quad+(4EQ-DR-6CS)UV^2+(-2ER+3DS)V^3],\\
\widetilde{J}_{20} &= \frac{1}{20}[(10DEG-2D^2H-4CEH+3CDK-2C^2L)U^3+\\
&\quad+(-20E^2G+2DEH+D^2K+2CEK-5CDL+8C^2M)U^2V+
\end{aligned}$$

$$+(8E^2H-5DEK+D^2L+2CEL+2CDM-20C^2N)UV^2+$$
$$+(-2E^2K+3DEL-2D^2M-4CEM+10CDN)V^3],$$
$$\widetilde{J}_{21}=\frac{1}{6}[(3DEP-D^2Q-2CEQ+3CDR-6C^2S)U+$$
$$+(-6E^2P+3DEQ-D^2R-2CER+3CDS)V].$$

Table 5

Invariant	Parameters of system (8)												
	C	D	E	F	G	H	K	L	M	P	Q	R	S
I_1	−	−	−	+	−	−	−	−	−	−	−	−	−
I_2	+	+	+	−	−	−	−	−	−	−	−	−	−
I_4	+	−	−	−	−	−	−	−	−	−	−	−	−
$\widetilde{J}_1$	+	−	−	−	+	−	−	−	−	−	−	−	−
$\widetilde{J}_2$	+	−	−	−	−	−	−	−	−	+	−	−	−
$\widetilde{J}_3$	+	+	+	−	+	+	+	−	−	−	−	−	−
$\widetilde{J}_4$	+	+	−	−	+	+	−	−	−	−	−	−	−
$\widetilde{J}_5$	+	+	+	−	−	−	−	−	−	+	+	+	−
$\widetilde{J}_6$	+	+	−	−	−	−	−	−	−	+	+	−	−
$\widetilde{J}_{20}$	+	+	+	−	+	+	+	+	−	−	−	−	−
$\widetilde{J}_{21}$	+	+	+	−	−	−	−	−	−	+	+	+	+

Next, we write the expressions of the highest power of N in the invariants (24), denoted by EI_i and $E\widetilde{J}_i$

$$
\begin{aligned}
&EI_1=F,\\
&EI_2=\frac{1}{2}(-D^2+4CE),\\
&EI_4=C^5,\\
&E\widetilde{J}_1=C^{10}G,\\
&E\widetilde{J}_2=C^6P,\\
&E\widetilde{J}_3=\frac{1}{10}C^6(10EG-2DH+CK),\\
&E\widetilde{J}_4=-\frac{1}{10}C^{10}(-5DG+2CH),\\
&E\widetilde{J}_5=\frac{1}{3}C^2(3EP-DQ+CR),\\
&E\widetilde{J}_6=-\frac{1}{6}C^6(-3DP+2CQ),\\
&E\widetilde{J}_{20}=-\frac{1}{20}C^6(-10DEG+2D^2H+4CEH-3CDK+2C^2L),\\
&E\widetilde{J}_{21}=-\frac{1}{6}C^2(-3DEP+D^2Q+2CEQ-3CDR+6C^2S).
\end{aligned}
\tag{28}
$$

In Table 5 the sign " + " indicates that the respective parameter is contained in the expression of the highest power of N in the invariants (27) and the sign " − " indicates that the respective parameter is missing from the expression of the highest power of N.

According to Table 5 we obtain the couples $\left\langle \widetilde{J}_{21}, S\right\rangle$, $\left\langle \widetilde{J}_{5}, R\right\rangle$, $\left\langle \widetilde{J}_{6}, Q\right\rangle$, $\left\langle \widetilde{J}_{2}, P\right\rangle$, $\left\langle \widetilde{J}_{20}, L\right\rangle$, $\left\langle \widetilde{J}_{3}, K\right\rangle$, $\left\langle \widetilde{J}_{4}, H\right\rangle$, $\left\langle \widetilde{J}_{1}, G\right\rangle$, $\langle I_1, F\rangle$, $\langle I_2, E\rangle$, $\left\langle \widetilde{J}_{2}, C\right\rangle$.

From Table 5 it follows that the invariants (24) are polynomial independent. □

The aim of constructing systems of the form (6), (13), (17), (21), (26) the minimal rational basis of $GL(2,\mathbb{R})$-comitants (4), (11), (15) and the minimal rational basis of $GL(2,\mathbb{R})$-invariants (19), (24), is to use them in the qualitative study of systems (1), for example in establishing the invariant center conditions (center-focus problem) [6]. For the center-focus problem, when system (1) satisfies the conditions $I_1 = 0$, $I_2 > 0$, by a linear transformation and time scaling system (1) can be brought to the form

$$\begin{aligned} \frac{dx}{dt} &= y + gx^4 + 4hx^3y + 6kx^2y^2 + 4lxy^3 + my^4, \\ \frac{dy}{dt} &= -x + nx^4 + 4px^3y + 6qx^2y^2 + 4rxy^3 + sy^4. \end{aligned} \tag{29}$$

For this system the first two Lyapunov quantities have the form:

$$\begin{aligned} G_8 = \frac{1}{16}(&-7gh - 18hk - 3gl - 18kl - 3hm - 7lm + 7gn + 3kn + 8hp + 7np+ \\ &+ 3gq - 3mq + 18pq - 8lr + 3nr + 18qr - 3ks - 7ms + 3ps + 7rs), \end{aligned} \tag{30}$$

$$\begin{aligned} G_{14} = \frac{1}{23040}(&121121g^3h + 92952gh^3 + 579516g^2hk + 234576h^3k + 866556ghk^2+ \\ &+ 436752hk^3 + 55419g^3l + 199032gh^2l + 414072g^2kl + 635472h^2kl+ \\ &+ 720900gk^2l + 393984k^3l + 132776ghl^2 + 572976hkl^2 + 18888gl^3+ \\ &+ 165168kl^3 + 109545g^2hm + 39096h^3m + 340704ghkm + 272052hk^2m+ \\ &+ 134135g^2lm + 158040h^2lm + 390264gklm + 299772k^2lm + 182280hl^2m+ \\ &+ 64232l^3m + 33831ghm^2 + 58644hkm^2 + 38277glm^2 + 60048klm^2+ \\ &+ 3807hm^3 + 1161lm^3 - 121121g^3n - 7470gh^2n - 351561g^2kn+ \\ &+ 139428h^2kn - 321066gk^2n - 72792k^3n - 48588ghln + 254376hkln- \\ &- 32318gl^2n + 125508kl^2n - 65376g^2mn + 29970h^2mn - 123066gkmn- \\ &- 36774k^2mn + 94068hlmn + 60354l^2mn - 14175gm^2n - 5589km^2n- \\ &- 35181ghn^2 + 648hkn^2 - 23859gln^2 + 15660kln^2 + 5589hmn^2+ \\ &+ 14175lmn^2 - 1161gn^3 - 3807kn^3 + 23034g^2hp - 97344h^3p+ \\ &+ 313896ghkp + 347400hk^2p + 40242g^2lp - 162048h^2lp + 428040gklp+ \\ &+ 487080k^2lp - 80320hl^2p - 11520l^3p + 46452ghmp + 113256hkmp+ \end{aligned}$$

$$+149092glmp+279240klmp+2970hm^2p+27666lm^2p-324699g^2np-$$
$$-118440h^2np-662136gknp-330300k^2np-146544hlnp-54728l^2np-$$
$$-125730gmnp-123552kmnp-14175m^2np-31050hn^2p-27666ln^2p-$$
$$-1161n^3p-144360ghp^2-107184hkp^2+5928glp^2+149616klp^2-$$
$$-40008hmp^2+54728lmp^2-267810gnp^2-292836knp^2-60354mnp^2-$$
$$-73408hp^3+11520lp^3-64232np^3-30849g^3q+252324gh^2q-$$
$$-49158g^2kq+703080h^2kq+73008gk^2q+128304k^3q+310056ghlq+$$
$$+1324944hklq+102468gl^2q+684072kl^2q+25665g^2mq+155844h^2mq+$$
$$+100188gkmq+121392k^2mq+438120hlmq+292836l^2mq+15201gm^2q+$$
$$+35802km^2q+3807m^3q-200160ghnq+75384hknq-185496glnq+$$
$$+143784klnq+66744hmnq+123552lmnq-64179gn^2q-35802kn^2q+$$
$$+5589mn^2q-431796g^2pq-454320h^2pq-923256gkpq-530064k^2pq-$$
$$-550560hlpq-149616l^2pq-122928gmpq-143784kmpq-15660m^2pq-$$
$$-351144hnpq-279240lnpq-60048n^2pq-538980gp^2q-684072kp^2q-$$
$$-125508mp^2q-165168p^3q+7740ghq^2+401760hkq^2-15084glq^2+$$
$$+530064klq^2+189540hmq^2+330300lmq^2-259758gnq^2-121392knq^2+$$
$$+36774mnq^2-713160hpq^2-487080lpq^2-299772npq^2-189432gq^3-$$
$$-128304kq^3+72792mq^3-393984pq^3+168018g^2hr+666504ghkr+$$
$$+614952hk^2r+191146g^2lr+104256h^2lr+749928gklr+713160k^2lr+$$
$$+173568hl^2r+73408l^3r+126756ghmr+251208hkmr+184020glmr+$$
$$+351144klmr+22194hm^2r+31050lm^2r-219297g^2nr-36504h^2nr-$$
$$-443952gknr-189540k^2nr+20016hlnr+40008l^2nr-93582gmnr-$$
$$-66744kmnr-5589m^2nr-22194hn^2r-2970ln^2r-3807n^3r-$$
$$-49776ghpr+99936hkpr+193328glpr+550560klpr-20016hmpr+$$
$$+146544lmpr-419316gnpr-438120knpr-94068mnpr-173568hp^2r+$$
$$+80320lp^2r-182280np^2r-364176g^2qr-239760h^2qr-755784gkqr-$$
$$-401760k^2qr-99936hlqr+107184l^2qr-91080gmqr-75384kmqr-$$
$$-648m^2qr-251208hnqr-113256lnqr-58644n^2qr-1012968gpqr-$$
$$-1324944kpqr-254376mpqr-572976p^2qr-614952hq^2r-347400lq^2r-$$
$$-272052nq^2r-436752q^3r+93816ghr^2+239760hkr^2+197256glr^2+$$
$$+454320klr^2+36504hmr^2+118440lmr^2-173106gnr^2-155844knr^2-$$
$$-29970mnr^2-104256hpr^2+162048lpr^2-158040npr^2-518724gqr^2-$$
$$-703080kqr^2-139428mqr^2-635472pqr^2+97344lr^3-39096nr^3-$$
$$-234576qr^3+11250g^3s+200274gh^2s+96999g^2ks+518724h^2ks+$$

$$
\begin{aligned}
&+236754gk^2s+189432k^3s+282868ghls+1012968hkls+104322gl^2s+\\
&+538980kl^2s+114479g^2ms+173106h^2ms+308670gkms+259758k^2ms+\\
&+419316hlms+267810l^2ms+34470gm^2s+64179km^2s+1161m^3s-\\
&-98214ghns+91080hkns-107026glns+122928klns+93582hmns+\\
&+125730lmns-34470gn^2s-15201kn^2s+14175mn^2s-83769g^2ps-\\
&-197256h^2ps-98592gkps+15084k^2ps-193328hlps-5928l^2ps+\\
&+107026gmps+185496kmps+23859m^2ps-184020hnps-149092lnps-\\
&-38277n^2ps-104322gp^2s-102468kp^2s+32318mp^2s-18888p^3s+\\
&+159288ghqs+755784hkqs+98592glqs+923256klqs+443952hmqs+\\
&+662136lmqs-308670gnqs-100188knqs+123066mnqs-749928hpqs-\\
&-428040lpqs-390264npqs-236754gq^2s-73008kq^2s+321066mq^2s-\\
&-720900pq^2s-114479g^2rs-93816h^2rs-159288gkrs-7740k^2rs+\\
&+49776hlrs+144360l^2rs+98214gmrs+200160kmrs+35181m^2rs-\\
&-126756hnrs-46452lnrs-33831n^2rs-282868gprs-310056kprs+\\
&+48588mprs-132776p^2rs-666504hqrs-313896lqrs-340704nqrs-\\
&-866556q^2rs-200274gr^2s-252324kr^2s+7470mr^2s-199032pr^2s-\\
&-92952r^3s+114479ghs^2+364176hks^2+83769gls^2+431796kls^2+\\
&+219297hms^2+324699lms^2-114479gns^2-25665kns^2+65376mns^2-\\
&-191146hps^2-40242lps^2-134135nps^2-96999gqs^2+49158kqs^2+\\
&+351561mqs^2-414072pqs^2-168018hrs^2-23034lrs^2-109545nrs^2-\\
&-579516qrs^2-11250gs^3+30849ks^3+121121ms^3-55419ps^3-\\
&-121121rs^3)\Big). \qquad (31)
\end{aligned}
$$

By using system (21) with $I_3 \neq 0$, $I_2 \neq 0$ and $I_1 = 0$, the first two Lyapunov quantities have the form:

$$G_8 = \frac{3I_2^2I_3J_2J_{19} + 2I_2I_3^2J_{20} + 2I_2^3J_2J_3 + 7I_3^3J_{43} + 4I_2^2J_6J_{20} + 2I_2I_3J_6J_{43}}{I_2^4I_3^3}, \qquad (32)$$

$$
\begin{aligned}
G_{14} = \frac{1}{900I_2^9I_3^6}\big(&12096I_2^5I_3^3J_1J_2 + 456288I_2^3I_3^5J_2J_{19} + 1872I_2^5I_3^2J_1J_2J_{19}+\\
&+225360I_2^3I_3^4J_2J_{19}^2 - 12960I_2^5I_3J_1J_2J_{19}^2 + 43875I_2^3I_3^3J_2J_{19}^3+\\
&+11424I_2^6I_3J_2^3J_{19} + 379648I_2^2I_3^6J_{20} - 39312I_2^4I_3^3J_1J_{20}+\\
&+335880I_2^2I_3^5J_{19}J_{20} - 84240I_2^4I_3^2J_1J_{19}J_{20} + 113400I_2^2I_3^4J_{19}^2J_{20}+\\
&+12800I_2^5I_3^2J_2^2J_{20} - 46224I_2^5I_3J_2^2J_{19}J_{20} - 68256I_2^4I_3^2J_2J_{20}^2+\\
&+9720I_2^4I_3J_2J_{19}J_{20}^2 - 25920I_2^3I_3^2J_{20}^3 + 319168I_2^4I_3^4J_2J_3-
\end{aligned}
$$

$$
\begin{aligned}
&-11232I_2^6I_3J_1J_2J_3+155880I_2^4I_3^3J_2J_3J_{19}-12960I_2^6J_1J_2J_3J_{19}+\\
&+60750I_2^4I_3^2J_2J_3J_{19}^2+7616I_2^7J_2^3J_3-22032I_2^3I_3^4J_3J_{20}-\\
&-38880I_2^5I_3J_1J_3J_{20}-84240I_2^3I_3^3J_3J_{19}J_{20}-34560I_2^6J_2^2J_3J_{20}+\\
&+32400I_2^5J_2J_3J_{20}^2-10800I_2^5I_3^2J_2J_3^2+40500I_2^5I_3J_2J_3^2J_{19}-\\
&-51840I_2^4I_3^2J_3^2J_{20}+16200I_2^6J_2J_3^3+13824I2^3I_3^5J_4+\\
&+79560I_2^3I_3^4J_4J_{19}+89100I_2^3I_3^3J_4J_{19}^2+3456I_2^6I_3J_2^2J_4+\\
&+3744I_2^6J_2^2J_4J_{19}-22464I_2^5I_3J_2J_4J_{20}-25920I_2^5J_{19}J_2J_4J_{20}-\\
&-38880I_2^4I_3J_4J_{20}^2+39312I_2^4I_3^3J_3J_4+84240I_2^4I_3^2J_3J_4J_{19}+\\
&+19440I_2^5I_3J_3^2J_4+1126592I_2I_3^7J_{43}-43992I_2^3I_3^4J_1J_{43}+\\
&+459540I_2I_3^6J_{19}J_{43}-132840I_2^3I_3^3J_1J_{19}J_{43}-182250I_2I_3^5J_{19}^2J_{43}+\\
&+25792I_2^4I_3^3J_2^2J_{43}+3744I_2^6J_1J_2^2J_{43}-20304I_2^4I_3^2J_2^2J_{19}J_{43}-\\
&-54000I_2^3I_3^3J_2J_{20}J_{43}-25920I_2^5J_1J_2J_{20}J_{43}+87480I_2^3I_3^2J_2J_{19}J_{20}J_{43}-\\
&-174960I_2^2I_3^3J_{20}^2J_{43}-180072I_2^2I_3^5J_3J_{43}-71280I_2^4I_3^2J_1J_3J_{43}-\\
&-473040I_2^2I_3^4J_3J_{19}J_{43}-7920I_2^5I_3J_2^2J_3J_{43}+6480I_2^4I_3J_2J_3J_{20}J_{43}-\\
&-165240I_2^3I_3^3J_3^2J_{43}+41184I_2^4I_3^2J_2J_4J_{43}+51840I_2^4I_3J_2J_4J_{19}J_{43}-\\
&-142560I_2^3I_3^2J_{20}J_4J_{43}+25920I_2^5J_2J_3J_4J_{43}-64224I_2^2I_3^4J_2J_{43}^2-\\
&-25920I_2^4I_3J_1J_2J_{43}^2-39690I_2^2I_3^3J_2J_{19}J_{43}^2-220320I_2I_3^4J_{20}J_{43}^2-\\
&-50220I_2^3I_3^2J_2J_3J_{43}^2-74520I_2^2I_3^3J_4J_{43}^2+23220I_3^5J_{43}^3-6912I_2^6I_3J_1J_2J_6+\\
&+14208I_2^4I_3^3J_2J_6J_{19}-7488I_2^6J_1J_2J_6J_{19}+68400I_2^4I_3^2J_2J_6J_{19}^2+\\
&+629376I_2^3I_3^4J_6J_{20}+22464I_2^5I_3J_1J_6J_{20}+364176I_2^3I_3^3J_6J_{19}J_{20}+\\
&+25920I_2^5J_1J_6J_{19}J_{20}+66420I_2^3I_3^2J_6J_{19}^2J_{20}+15232I_2^6J_2^2J_6J_{20}-\\
&-69120I_2^5J_2J_6J_{20}^2+64800I_2^4J_6J_{20}^3-2048I_2^5I_3^2J_2J_3J_6+\\
&+84960I_2^5I_3J_{19}J_2J_3J_6+91584I_2^4I_3^2J_3J_6J_{20}+123120I_2^4I_3J_3J_6J_{19}J_{20}+\\
&+34560I_2^6J_2J_3^2J_6+32400I_2^5J_3^2J_6J_{20}+24192I_2^4I_3^3J_4J_6+\\
&+3744I_2^4I_3^2J_4J_6J_{19}-25920I_2^4I_3J_4J_6J_{19}^2-22464I_2^5I_3J_3J_4J_6-\\
&-25920I_2^5J_{19}J_3J_4J_6+360384I_2^2I_3^5J_6J_{43}-41184I_2^4I_3^2J_1J_6J_{43}+\\
&+284256I_2^2I_3^4J_6J_{19}J_{43}-51840I_2^4I_3J_1J_6J_{19}J_{43}-2430I_2^2I_3^3J_6J_{19}^2J_{43}+\\
&+7616I_2^5I_3J_2^2J_6J_{43}-46656I_2^4I_3J_2J_6J_{20}J_{43}+19440I_2^3I_3J_6J_{20}^2J_{43}+\\
&+1584I_2^3I_3^3J_3J_6J_{43}-25920I_2^5J_1J_3J_6J_{43}-119880I_2^3I_3^2J_{19}J_3J_6J_{43}-\\
&-48600I_2^4I_3J_3^2J_6J_{43}+14976I_2^5J_2J_4J_6J_{43}-51840I_2^4J_{20}J_4J_6J_{43}-\\
&-13536I_2^3I_3^2J_2J_6J_{43}^2-42120I_2^2I_3^2J_6J_{20}J_{43}^2-51840I_2^3I_3J_4J_6J_{43}^2-\\
&-26460I_2I_3^3J_6J_{43}^3+22848I_2^5I_3J_2J_6^2J_{19}+35328I_2^4I_3^2J_6^2J_{20}+\\
&+92448I_2^4I_3J_6^2J_{19}J_{20}+15232I_2^6J_2J_3J_6^2+69120I_2^5J_3J_6^2J_{20}-\\
&-6912I_2^5I_3J_4J_6^2-7488I_2^5J_4J_6^2J_{19}+63360I_2^3I_3^3J_6^2J_{43}-
\end{aligned}
$$

$$- 7488I_2^5J_1J_6^2J_{43} + 40608I_2^3I_3^2J_6^2J_{19}J_{43} + 15840I_2^4I_3J_3J_6^2J_{43}+$$
$$+ 30464I_2^5J_6^3J_{20} + 15232I_2^4I_3J_6^3J_{43}). \quad (33)$$

We conclude that the number of terms in expressions (32) (6 terms) and (33) (126 terms) is less than the number of terms in expressions (30) (20 terms) and (31) (346 terms), respectively. Moreover, the expressions (32) and (33) given via invariants, can be used for any system of the form (1) with $I_3 \neq 0$, $I_2 \neq 0$ and $I_1 = 0$, while the expressions of the form (30) and (31) can be used only for system of the form (29).

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Stanislav Ciubotaru
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
E-mail: *stanislav.ciubotaru@yahoo.com*

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