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INTERPOLATION OF SUBSPACES AND APPLICATIONS TO EXPONENTIAL BASES

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Precise conditions are given under which the real interpolation space $[Y_0, X_1]_{\theta, p}$ coincides with a closed subspace of $[X_0, X_1]_{\theta, p}$ when Y_0 is a closed subspace of codimension one. This result is applied to the study of nonharmonic Fourier series in the Sobolev spaces $H^s(-\pi, \pi)$ with $0 < s < 1$. The main result looks like this: if $\{e^{i\lambda_n t}\}$ is an unconditional basis in $L^2(-\pi, \pi)$, then there exist two numbers s_0, s_1 such that for $s < s_0$ $\{e^{i\lambda_n t}\}$ forms an unconditional basis in $H^s(-\pi, \pi)$, and for $s_1 < s$ $\{e^{i\lambda_n t}\}$ forms an unconditional basis of a closed subspace in $H^s(-\pi, \pi)$ of codimension one. If $s_0 \leq s \leq s_1$, then the family $\{e^{i\lambda_n t}\}$ is not an unconditional basis in its span in $H^s(-\pi, \pi)$.

§1. Introduction

In this paper we shall apply a result on interpolation of subspaces to the study of exponential Riesz bases in Sobolev spaces.

In §2 we consider the comparison of the interpolation spaces $X_\theta := [X_0, X_1]_{\theta, p}$ and $Y_\theta := [Y_0, X_1]_{\theta, p}$ for $1 \leq p < \infty$, where Y_0 is a codimension one subspace of X_0 , say, $Y_0 = \ker \psi$ with $\psi \in X_0^*$. As far as we know, this problem was first formulated in [16, vol. 1, Chapter 1, Section 18] in 1968. As we show in Theorem 2.1, there are two indices $0 \leq \sigma_0 \leq \sigma_1 \leq 1$ that may be explicitly evaluated in terms of the K -functional of ψ and such that:

1. if $0 < \theta < \sigma_0$, then Y_θ is a closed subspace of codimension one in X_θ ;
2. if $\sigma_1 < \theta < 1$, then $Y_\theta = X_\theta$ with equivalence of norms;
3. if $\sigma_0 \leq \theta \leq \sigma_1$, then the norm on Y_θ is not equivalent to the norm on X_θ .

Let us discuss the history of this theorem. The special case of a Hilbert space of Sobolev type connected with elliptic boundary data was considered in [16], and in this case the critical indices σ_0 and σ_1 coincide. In the well-known case (see [16]) where $X_1 = L^2(0, \infty)$, $X_0 = W_2^1(0, \infty)$, and $Y_0 \subset W_2^1$ is the subspace of functions vanishing at the origin, this critical value is $\sigma_0 = \sigma_1 = 1/2$. Later, R. Wallsten [26]

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gave an example where the critical indices satisfy $\sigma_0 < \sigma_1$. The general problem was considered by J. Löfström in the paper [17], where some special cases of Theorem 2.1 were obtained. Later, in an unpublished (but web-posted) preprint from 1997, Löfström obtained most of the conclusion of Theorem 2.1; specifically, he obtained the same result except he did not treat the critical values $\theta = \sigma_0, \sigma_1$. The authors were not aware of Löfström's earlier work during the initial preparation of this article and our approach is rather different. A more general but closely related problem on interpolating subspaces of codimension one was considered recently in [12] and [10]. For general results on subcouple we refer to [9].

Next, we recall that a sequence $(e_n)_{n \in \mathbb{Z}}$ in a Hilbert space \mathcal{H} is called a *Riesz basic sequence* if there is a constant C such that for any finitely nonzero sequence $(a_n)_{n \in \mathbb{Z}}$ we have

$$\frac{1}{C} \left(\sum_{n \in \mathbb{Z}} |a_n|^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{n \in \mathbb{Z}} a_n e_n \right\| \leq C \left(\sum_{n \in \mathbb{Z}} |a_n|^2 \right)^{\frac{1}{2}}.$$

A *Riesz basis* for \mathcal{H} is a Riesz basic sequence whose closed linear span $[e_n]_{n \in \mathbb{Z}}$ is \mathcal{H} . A sequence (e_n) is an *unconditional basis* (respectively, an *unconditional basic sequence*) if $(e_n / \|e_n\|)_{n \in \mathbb{Z}}$ is a Riesz basis (respectively, a Riesz basic sequence).

In the second part of the paper we apply our interpolation result to study the basis properties of exponential families $\{e^{i\lambda_n t}\}$ in Sobolev spaces. These families appear in such fields of mathematics as the theory of dissipative operators (the Sz.-Nagy–Foias model), the Regge problem for resonance scattering, the theory of initial boundary value problems, control theory for distributed parameter systems, and signal processing; see, e.g., [22, 8, 13, 2, 25]. One of the most important problems arising in all of these applications is the question of the Riesz basis property of these families. In the space $L^2(-\pi, \pi)$ this problem was studied for the first time in the classical work of Paley and Wiener [23]. Now the problem has a complete solution [11, 20] on the basis of an approach suggested by B. S. Pavlov.

The principal result for Riesz bases can be formulated as follows [11].

Proposition 1.1. *The sequence $(e^{i\lambda_n t})_{n \in \mathbb{Z}}$ is a Riesz basis for $L^2(-\pi, \pi)$ if and only if $\sup |\Im \lambda_n| < \infty$,*

$$\inf_{k \neq j} |\lambda_k - \lambda_j| > 0, \quad (1.1)$$

and there is an entire function F of exponential type π (the generating function) with simple zeros at $(\lambda_n)_{n \in \mathbb{Z}}$ and such that for some y the weight $x \mapsto |F(x + iy)|^2$ satisfies the Muckenhoupt condition (A_2) (we shall write this as $|F|^2 \in (A_2)$):

$$\sup_{I \in \mathcal{J}} \left\{ \frac{1}{|I|} \int_I |F(x + iy)|^2 dx \frac{1}{|I|} \int_I |F(x + iy)|^{-2} dx \right\} < \infty,$$

where \mathcal{J} is the set of all intervals of the real axis.

In [20], a corresponding characterization was given for the exponential families that form an unconditional basis of $L^2(-\pi, \pi)$ when $\Im \lambda_n$ can be unbounded both from above and below.

Let us describe known results concerning exponential bases in Sobolev spaces. The first result in this direction was obtained by D. L. Russell in [24]. Russell studied the unconditional basis property for exponential families in the Sobolev spaces $H^m(-\pi, \pi)$ with $m \in \mathbb{Z}$.

Proposition 1.2 (see [24]). *Suppose $(e^{i\lambda_n t})_{n \in \mathbb{Z}}$ is a Riesz basis for $L^2(-\pi, \pi)$. Suppose $m \in \mathbb{N}$, and suppose $\mu_1, \dots, \mu_m \in \mathbb{C} \setminus \{\lambda_n : n \in \mathbb{Z}\}$ are distinct. Then $(e^{i\lambda_n t})_{n \in \mathbb{Z}} \cup (e^{i\mu_k t})_{k=1}^m$ is an unconditional basis of $H^m(-\pi, \pi)$. In particular, $(e^{i\lambda_n t})_{n \in \mathbb{Z}}$ is an unconditional basic sequence whose closed linear span has codimension m in $H^m(-\pi, \pi)$.*

In [21] the unconditional basis property for an exponential family was studied in $H^s(-\pi, \pi)$ for nonintegral s in the case where the λ_n are the eigenvalues of a Sturm–Liouville operator with a smooth potential.

Note that the generalization of the Levin–Golovin theorem for Sobolev spaces was obtained [3] by using “classical methods” of the entire function theory. Suppose $\{\lambda_n\}_{n \in \mathbb{Z}}$ are the zeros of an entire function F of exponential type π , (λ_n) is separated as in (1.1), and on some line $\{x + iy\}_{x \in \mathbb{R}}$ we have

$$C^{-1}(1 + |x|)^s \leq |F(x + iy)| \leq C(1 + |x|)^s.$$

Then the family $\{e^{i\lambda_k t}/(1 + |\lambda_k|)^s\}$ forms a Riesz basis in $H^s(-\pi, \pi)$. Notice that this result was applied to several controllability problems for the wave type equation (see [4]).

Recently Yu. Lyubarskii and K. Seip [19] established a necessary and sufficient criterion for the sampling/interpolation problem for weighted Paley–Wiener spaces, which gives a criterion for a sequence to be an unconditional basis in H^s . For the case where $\sup |\Im \lambda_n| < \infty$, the main result is the following.

Theorem 1.3. *$(e^{i\lambda_n t})_{n \in \mathbb{Z}}$ forms an unconditional basis in $H^s(-\pi, \pi)$ if and only if (λ_n) is separated (i.e., (1.1) is fulfilled) and for the generating function F we have $|F(x + iy)|^2/(1 + |x|^{2s}) \in (A_2)$ for some y .*

The main idea of the present paper is that if $(e^{i\lambda_n t})_{n \in \mathbb{Z}}$ forms a Riesz basis in $L^2(-\pi, \pi)$, then it also forms an unconditional basis of a subspace Y_0 in $H^1(-\pi, \pi)$ of codimension one. Then, by interpolation, we obtain the fact that $(e^{i\lambda_n t})_{n \in \mathbb{Z}}$ is an unconditional basis of the intermediate spaces $[Y_0, L^2]_{\theta, 2}$ for $0 < \theta < 1$. This approach was suggested in [7] by the first author. The main result of [7] is incorrect in the general case because of a mistake related to interpolation of subspaces. Here we correct this mistake.

We describe the results concerning unconditional bases in Sobolev spaces. One of our main results for Riesz bases is as follows.

Theorem 1.4. *Suppose $(e^{i\lambda_n t})_{n \in \mathbb{Z}}$ forms a Riesz basis of $L^2(-\pi, \pi)$. Suppose $(\lambda_n - n)_{n \in \mathbb{Z}}$ is bounded, and let $\delta_n = \Re \lambda_n - n$. Then there exist critical indices $0 < s_0 \leq s_1 < 1$ given by*

$$s_1 = \frac{1}{2} - \liminf_{\tau \rightarrow \infty} \inf_{t \geq 1} \frac{1}{\log \tau} \sum_{t < |n| \leq \tau t} \frac{\delta_n}{n}$$

and

$$s_0 = \frac{1}{2} - \limsup_{\tau \rightarrow \infty} \sup_{t \geq 1} \frac{1}{\log \tau} \sum_{t < |n| \leq \tau t} \frac{\delta_n}{n}$$

such that:

- (1) $(e_{\lambda_n})_{n \in \mathbb{Z}}$ is an unconditional basis of the Sobolev space H^s if and only if $0 \leq s < s_0$;
- (2) $(e_{\lambda_n})_{n \in \mathbb{Z}}$ is an unconditional basis of a closed subspace of H^s of codimension one if and only if $s_1 < s \leq 1$;
- (3) If $s_0 \leq s \leq s_1$, then (e_{λ_n}) is not an unconditional basic sequence.

This theorem is deduced from results of §§3 and 4. In §4 we in fact consider the more general situation for unconditional bases and give rather more technical results. The above Theorem 1.4 is the simplest case and follows by combining Theorem 4.2, Theorem 4.9 and Theorem 4.10. Our approach is based on estimates of the K -functional for the continuous linear functional on $H^1(-\pi, \pi)$ that annihilates each $e^{i\lambda_n x}$; the existence of such a functional is guaranteed by the result of Russell (Proposition 1.2). The estimates are in terms of the generating function F .

Once we have Theorem 1.4, it is easy to construct real sequences (λ_n) to show that s_0, s_1 can take any values in $(0, 1)$ such that $s_0 \leq s_1$. In the case of regular power behavior of F , i.e., if $|F(x + iy)| \sim (1 + |x|)^s$ for some $y \geq 0$, we have $s_1 = s_0 = s + \frac{1}{2}$.

The results for the entire scale $H^s(-\pi, \pi)$ can then be obtained by "shift", using the fact that the differentiation operator with appropriate conditions is an isomorphism between a one-codimensional subspace of H^m and H^{m-1} ; we shall not pursue this extension.

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§2. Interpolation of subspaces

Let (X_0, X_1) be a Banach couple with $X_0 \cap X_1$ dense in X_0, X_1 . If $0 < \theta < 1$ and $1 \leq p < \infty$, the real interpolation space $X_\theta = [X_0, X_1]_{\theta, p}$ is defined (see, e.g., [5]) to be the set of all $x \in X_0 + X_1$ such that

$$\|x\|_{X_\theta} = \left(\int_0^\infty t^{\theta p - 1} K(t, x)^p dt \right)^{\frac{1}{p}} < \infty,$$

where $K(t, x)$ is the K -functional. An equivalent definition [5, p. 314] (yielding an equivalent norm) can be given by using the J -method:

$$\|x\|_{X_\theta} = \inf \left\{ \left(\sum_{k \in \mathbb{Z}} \max\{\|x_k\|_0, 2^k \|x_k\|_1\}^p \right)^{\frac{1}{p}} : x = \sum_{k \in \mathbb{Z}} 2^{\theta k} x_k \right\},$$

where the series converges in $X_0 + X_1$.

Now suppose $0 \neq \psi \in X_0^*$, and let Y_0 be the kernel of ψ . We suppose also (only this case is interesting) that $Y_0 \cap X_1$ is dense in X_1 , i.e., ψ is not bounded in X_1 .

Let Y_θ be the corresponding spaces obtained by interpolating Y_0 and X_1 . Clearly $Y_\theta \subset X_\theta$ and the inclusion has norm one. It is easy to show that the closure of Y_θ in X_θ is either a subspace of codimension one when ψ is continuous on X_θ , or the whole of X_θ when ψ is not continuous.

Now, we introduce two important indices:

$$\sigma_1 = \lim_{\tau \rightarrow \infty} \sup_{0 < \tau t \leq 1} \frac{1}{\log \tau} \log \frac{K(\tau t, \psi)}{K(t, \psi)}$$

and

$$\sigma_0 = \lim_{\tau \rightarrow \infty} \inf_{0 < \tau t \leq 1} \frac{1}{\log \tau} \log \frac{K(\tau t, \psi)}{K(t, \psi)},$$

where $K(t, \psi) = K(t, \psi; X_0^*, X_1^*)$. From the multiplicative properties of the function $K(\tau t, \psi)/K(t, \psi)$ it is clear that these limits exist and $0 \leq \sigma_0 \leq \sigma_1 \leq 1$. Since $K(t, \psi)$ is bounded as $t \rightarrow \infty$, we can also write

$$\sigma_1 = \lim_{\tau \rightarrow \infty} \sup_{0 < t < \infty} \frac{1}{\log \tau} \log \frac{K(\tau t, \psi)}{K(t, \psi)}.$$

We observe that

$$\sup\{|\psi(x)| : \max\{\|x\|_0, t\|x\|_1\} \leq 1\} = K(t^{-1}, \psi).$$

We define a sequence $(w_n)_{n \in \mathbb{Z}}$ by

$$w_n = K(2^{-n}, \psi)^{-1}.$$

Notice that $\inf_{n \in \mathbb{Z}} w_n \geq \|\psi\|_{X_0^*}^{-1} > 0$ and that in general $w_n \leq w_{n+1} \leq 2w_n$. Now it is easy to see that

$$\begin{aligned} \sigma_1 &= \lim_{k \rightarrow \infty} \sup_n \frac{1}{k} \log_2 \frac{w_{n+k}}{w_n}, \\ \sigma_0 &= \lim_{k \rightarrow \infty} \inf_{n \geq 0} \frac{1}{k} \log_2 \frac{w_{n+k}}{w_n}. \end{aligned}$$

As mentioned in the Introduction, the following result is a slight improvement of a result of Löfström [18], who obtained the same statement by quite different arguments except for the critical indices $\theta = \sigma_0, \sigma_1$.

- Theorem 2.1.** 1. $Y_\theta = X_\theta$ (with equivalence of the norms) if and only if $\theta > \sigma_1$.
 2. Y_θ is a closed subspace of codimension one in X_θ if and only if $\theta < \sigma_0$.
 3. If $\sigma_0 \leq \theta \leq \sigma_1$, then Y_θ is not closed in X_θ .

We shall consider the weighted ℓ_p space $\ell_p(w)$ of all sequences $(\alpha_n)_{n \in \mathbb{Z}}$ such that

$$\|\alpha\| = \left(\sum_{k \in \mathbb{Z}} w_k^p |\alpha_k|^p \right)^{\frac{1}{p}}.$$

We shall use ζ_n for the standard basis vectors. On $\ell_p(w)$ we consider the shift operator $S((\alpha_n)) = (\alpha_{n-1})$. From the above remarks it is clear that S, S^{-1} are both bounded and $\|S\| \leq 2, \|S^{-1}\| = 1$. Furthermore, the spectral radius formula shows that 2^{σ_1} is the spectral radius $r(S)$ of S . Now let P_+ be the projection $P_+(\alpha) = (\delta_n \alpha_n)$, where $\delta_n = 1$ if $n \geq 0$ and $\delta_n = 0$ otherwise. It is easy to calculate

$$\|P_+ S^{-n}\| = \sup_{k \geq 0} \frac{w_k}{w_{n+k}},$$

and this implies that $r(P_+ S^{-1}) = 2^{-\sigma_0}$.

We need the following key lemma.

Lemma 2.2. Let $0 < \theta < 1$ and let $T_\theta = S - 2^\theta I$. Then:

1. T_θ is an isomorphism onto $\ell_p(w)$ if and only if $\sigma_1 < \theta$;
2. T_θ is an isomorphism onto a proper closed subspace if and only if $\theta < \sigma_0$. In this case the range of T_θ is the subspace of codimension one consisting of all α such that $\sum_{n \in \mathbb{Z}} 2^{n\theta} \alpha_n = 0$.

Proof. First, observe that if $\theta > \sigma_1$, then T_θ must be an isomorphism onto $\ell_p(w)$ since 2^θ exceeds the spectral radius of S . Furthermore, since the spectrum of S is rotation invariant, it is clear that T_{σ_1} cannot be an isomorphism onto $\ell_p(w)$. Also, we note that T_θ is always injective, and that if f_θ is a linear functional annihilating its range, then $f_\theta(\zeta_n) = c2^{n\theta}$ for some constant c , i.e., $f_\theta(\alpha) = \sum_{n \in \mathbb{Z}} 2^{n\theta} \alpha_n = 0$. This implies that the closure of the range is either the entire space or the subspace of codimension one when $\sum_{n \in \mathbb{Z}} 2^{n\theta q} w_n^{-q} < \infty$. Here $\frac{1}{p} + \frac{1}{q} = 1$ and the formula must be modified if $p = 1$.

Next, we show that if $\theta < \sigma_0$, then T_θ is an isomorphism onto a closed subspace of codimension one.

Let $E = [\{\zeta_n : n \leq -1\}]$ and $F = [\{\zeta_n : n \geq 1\}]$. We remark that $T_\theta(E)$ is easily seen to be closed, because T_θ is an isomorphism on the unweighted ℓ_p and w_n is bounded for $n \leq -1$. If we show $T_\theta(F)$ is closed, then we are done, because, clearly, this will imply that $T_\theta(E+F)$ is closed and this is a subspace of codimension one in the range. However, we have $2^{-\theta} > r(P_+ S^{-1})$, so that $2^{-\theta} - P_+ S^{-1}$ is an isomorphism. After restricting to F , this implies that $(2^{-\theta} - S^{-1})F$; consequently $T_\theta(F)$ is closed.

The proof is completed by showing that if $\theta \leq \sigma_1$ then, if T_θ has closed range, necessarily $\theta < \sigma_0$. Note first that it suffices to establish this for $\theta < \sigma_1$ since the set of operators with Fredholm index one is open. Suppose $\sigma_0 < \theta < \sigma_1$ and T_θ is closed. Then T_θ has a lower estimate $\|T_\theta \alpha\| \geq c\|\alpha\|$ for all α , where $c > 0$. Assume $w_{n+k} > 2^{n\theta} w_k$ for some $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Let $\alpha = (I + 2^{-\theta} S + \dots + 2^{-n\theta} S^n)^2 \zeta_k$; then $\|\alpha\| \geq n2^{-n\theta} w_{n+k}$. However,

$$\begin{aligned} \|T_\theta^2 \alpha\| &= 2^{2\theta} w_k + 2 \cdot 2^{(-n+1)\theta} w_{n+k+1} + 2^{-2n\theta} w_{2n+k+2} \\ &\leq 8 \max\{w_k, 2^{-n\theta} w_{n+k}, 2^{-2n\theta} w_{2n+k}\}. \end{aligned}$$

Let $v_n = 2^{-n\theta} w_n$. Then, if $nc^2 > 8$, we have

$$(nc^2 - 8)v_{k+n} \leq 8 \max\{v_k, v_{k+2n}\}.$$

In particular, if $nc^2 > 16$, then

$$v_{k+n} < \max\{v_k, v_{k+2n}\}.$$

Now, since $\theta < \sigma_1$, we can find $k \in \mathbb{Z}$ and $n > 16c^{-2}$ so that $w_{n+k} < 2^{n\theta} w_k$ or $v_{n+k} < v_k$. Iterating shows that $(v_{k+rn})_{r=0}^\infty$ is monotone increasing. Now for any large N and any $j \geq 0$ we have

$$\frac{w_{j+N}}{w_j} \geq \frac{w_{k+r_2 n}}{w_{k+r_1 n}} \geq 2^{n(r_2-r_1)\theta},$$

where r_1, r_2 are such that $k + (r_1 - 1)n \leq j \leq k + r_1 n$ and $k + r_2 n \leq j + N \leq k + (r_2 + 1)n$. This yields

$$\frac{w_{j+N}}{w_j} \geq 2^{(N-2n)\theta}.$$

Hence,

$$\inf_{j \geq 0} \frac{1}{N} \log_2 \frac{w_{j+N}}{w_j} \geq (1 - 2n/N)\theta.$$

Letting $N \rightarrow \infty$ gives $\sigma_0 \geq \theta$. To show that in fact $\theta < \sigma_0$, again it suffices to observe that the set of θ for which T_θ has Fredholm index one is open. •

Now we use Lemma 2.2 to establish our main result (see Theorem 2.1) on interpolating subspaces.

Proof. Suppose next that either (a) $\theta < \sigma_0$ or (b) $\theta > \sigma_1$. This implies the existence of a constant D such that $\|\alpha\| \leq D\|T_\theta \alpha\|$ for all $\alpha \in \ell_p(w)$; in case (a) T_θ maps onto the subspace of $\ell_p(w)$ defined by $f_\theta(\alpha) = \sum_{n \in \mathbb{Z}} 2^{n\theta} \alpha_n = 0$, while in case (b) T_θ is an isomorphism onto the entire space (see Lemma 2.2). We observe that in case (a) the linear functional ψ extends to a continuous linear functional on X_θ because

$$\sum_{n \in \mathbb{Z}} 2^{n\theta} K(2^n, \psi) < \infty.$$

Now suppose that $x \in X_\theta$ with $\|x\|_{X_\theta} = 1$ and with the additional assumption in case (a) that $\psi(x) = 0$. Then we may find $(x_n)_{n \in \mathbb{Z}}$ such that $\sum_{n \in \mathbb{Z}} 2^{\theta n} x_n = x$ and

$$\left(\sum_{k \in \mathbb{Z}} \max\{\|x_k\|_0, 2^k \|x_k\|_1\}^p \right)^{\frac{1}{p}} \leq 2.$$

Then

$$\left(\sum_{n \in \mathbb{Z}} |\psi(x_n)|^p w_n^p \right)^{\frac{1}{p}} \leq 2,$$

since

$$|\psi(x)| \leq w_n^{-1} \max\{\|x\|_0, 2^n \|x\|_1\}.$$

In case (a) we additionally have

$$\sum_{n \in \mathbb{Z}} 2^{n\theta} \psi(x_n) = 0.$$

Thus, we can find $\alpha \in \ell_p(w)$ with $T_\theta(\alpha) = (\psi(x_n))$ and $\|\alpha\| \leq 2D$. Then we can find $u_n \in X_0 \cap X_1$ such that $\max\{\|u_n\|_0, 2^n \|u_n\|_1\} \leq 2|\alpha_n|w_n$ and $\psi(u_n) = \alpha_n$. Let $v_n = u_{n-1} - 2^\theta u_n$. Then

$$\left(\sum_{k \in \mathbb{Z}} \max\{\|v_k\|, 2^k \|v_k\|_1\}^p \right)^{\frac{1}{p}} \leq 16D \|x\|_{X_\theta}.$$

Now, $\psi(v_n) = \alpha_{n-1} - 2^\theta \alpha_n = \psi(x_n)$ and $\sum_{n \in \mathbb{Z}} 2^{n\theta} v_n = 0$. Consequently,

$$x = \sum_{n \in \mathbb{Z}} 2^{\theta n} (x_n - v_n),$$

and so $x \in Y_\theta$ with $\|x\|_{Y_\theta} \leq (16D + 2)\|x\|_{X_\theta}$. This implies that in case (a) we have $Y_\theta = \{x : \psi(x) = 0, x \in X_\theta\}$ and in case (b) $Y_\theta = X_\theta$.

Next we consider the converse directions. Assume either (aa) ψ is continuous on X_θ and $Y_\theta = \{x : \psi(x) = 0, x \in X_\theta\}$ or (bb) $Y_\theta = X_\theta$. In either case there is a constant D such that if $x \in Y_\theta$ then $\|x\|_{Y_\theta} \leq D\|x\|_{X_\theta}$. Observe that in case (aa) the linear functional f_θ is continuous on $\ell_p(w)$, so that the range of T_θ is contained in the kernel of f_θ ; in case (bb) its range is dense.

Assume $\alpha = (\alpha_n)_{n \in \mathbb{Z}} \in \ell_p(w)$ with $\|\alpha\| = 1$; in case (aa) we also assume $f_\theta(\alpha) = 0$. First we find $x_n \in X_0 \cap X_1$ such that $\psi(x_n) = \alpha_n$ and $\max\{\|x_n\|_0, 2^n \|x_n\|_1\} \leq 2|\alpha_n|w_n$ for $n \in \mathbb{Z}$. Let $x = \sum_{n \in \mathbb{Z}} 2^{n\theta} x_n$; then $x \in X_\theta$ with $\|x\|_{X_\theta} \leq 2$. In case (aa) we additionally have the relation $\psi(x) = f_\theta(\alpha) = 0$. Now we can find $y_n \in Y_0 \cap X_1$ satisfying $\sum_{n \in \mathbb{Z}} 2^{n\theta} y_n = x$ and

$$\left(\sum_{k \in \mathbb{Z}} \max\{\|y_k\|, 2^k \|y_k\|_1\}^p \right)^{\frac{1}{p}} \leq 4D.$$

Putting $u_n = x_n - y_n$ and $v_n = \sum_{k=n+1}^{\infty} 2^{(k-n-1)\theta} u_k$, we obtain

$$\left(\sum_{k \in \mathbb{Z}} \max\{\|u_k\|, 2^k \|u_k\|_1\}^p \right)^{\frac{1}{p}} \leq 4D + 2.$$

We claim that

$$\left(\sum_{k \in \mathbb{Z}} \max\{\|v_k\|_0, 2^k \|v_k\|_1\}^p \right)^{\frac{1}{p}} \leq C_{\theta}(4D + 2), \tag{2.1}$$

where

$$C_{\theta} = \left(\sum_{k < 0} 2^{k\theta} + \sum_{k \geq 0} 2^{k(\theta-1)} \right).$$

To show (2.1) we note that

$$2^n \|v_n\|_1 \leq \sum_{k=n+1}^{\infty} 2^{(k-n-1)(\theta-1)} 2^k \|u_k\|_1$$

and (since $\sum 2^{n\theta} u_n = 0$)

$$\|v_n\|_0 \leq \sum_{k=-\infty}^n 2^{(k-n-1)\theta} \|u_k\|_0.$$

Let $\beta_n = \psi(v_n)$. Then $\beta \in \ell_p(w)$ and $\|\beta\| \leq C_{\theta}(4D + 2)$. But now $(T_{\theta}(\beta))_n = \psi(u_n) = \psi(x_n) = \alpha_n$, so that T_{θ} is an isomorphism onto the kernel of f_{θ} in case (aa) or onto $\ell_p(w)$ in case (bb). These two cases combined with the observation that Y_{θ} can only be a proper closed subspace of X_{θ} if ψ is continuous on X_{θ} complete the proof of the theorem. •

§3. Sobolev spaces

In this section we investigate a special case of the results of the preceding section for Sobolev spaces. These results are preparatory for §4 where we apply them to exponential bases. Let $L^2 = L^2(-\pi, \pi)$; the standard inner-product on $L^2(-\pi, \pi)$ will be denoted by

$$(f, g) = \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx,$$

and the standard norm on L^2 by $\|f\|$.

For $s > 0$ we define the Sobolev space $H^s(\mathbb{R})$ to be the space of all $f \in L_2(\mathbb{R})$ so that

$$\|f\|_{H^s}^2 := \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 (1 + |\xi|^{2s}) d\xi < \infty$$

(\hat{f} is the Fourier transform). We then define the Sobolev space $H^s = H^s(-\pi, \pi)$ to be the space of restrictions of $H^s(\mathbb{R})$ -functions to the interval $(-\pi, \pi)$ (with the obvious induced quotient norm). When $s = 1$, the space H^1 reduces to the space of all $f \in L^2(-\pi, \pi)$ such that $f' \in L^2$ under the (equivalent) norm:

$$\|f\|_1^2 = \int_{-\pi}^{\pi} |f(t)|^2 + |f'(t)|^2 dt < \infty.$$

Then, if $0 < s < 1$, we have $H^s = [H^1, L^2]_{1-s} = [H^1, L^2]_{1-s,2}$ (see [16]).

For $z \in \mathbb{C}$ we define $e_z(x) = e^{izx} \in L^2(-\pi, \pi)$. Now suppose $\psi \in (H^1)^*$; we define its Fourier transform $F = \hat{\psi}$ to be the entire function $F(z) := \psi(e_z)$ for $z \in \mathbb{C}$. Let us first identify $(H^1)^*$ via its Fourier transform.

Proposition 3.1. *Let F be an entire function. In order that there exist $\psi \in (H^1)^*$ with $F = \hat{\psi}$, it is necessary and sufficient that*

$$F \text{ be of exponential type } \leq \pi \tag{3.1}$$

and

$$\int_{-\infty}^{\infty} \frac{|F(x)|^2}{1+x^2} dx < \infty. \tag{3.2}$$

These conditions imply the estimate

$$\sup_{z \in \mathbb{C}} \frac{|F(z)|}{(1+|z|)e^{\pi|\Im z|}} < \infty. \tag{3.3}$$

Proof. These results follow immediately from the Paley-Wiener theorem once we observe that $\psi \in (H^1)^*$ if and only if ψ is of the form

$$\psi(f) = \alpha f(0) + \varphi(f')$$

where $\varphi \in (L^2)^*$. •

Consider H^1 with the inner product

$$\langle f, g \rangle_t = (f', g') + t^2(f, g),$$

where $t > 0$. We denote by $\|\psi\|_t$ the norm of ψ with respect to $\|\cdot\|_t$ where $\|f\|_t^2 = \langle f, f \rangle_t$, i.e., $\|\psi\|_t := \sup\{|\psi(f)| : \|f\|_t \leq 1\}$. Set

$$s_0 = 1 - \limsup_{\tau \rightarrow \infty} \sup_{t \geq 1} \frac{1}{\log \tau} \log \frac{\|\psi\|_t}{\|\psi\|_{\tau t}} \tag{3.4}$$

and

$$s_1 = 1 - \liminf_{\tau \rightarrow \infty} \inf_{t \geq 1} \frac{1}{\log \tau} \log \frac{\|\psi\|_t}{\|\psi\|_{\tau t}}. \tag{3.5}$$

We can specialize Theorem 2.1 to the particular case of interpolating between L^2 and H^1 .

Proposition 3.2. *Suppose $\psi \in (H^1)^*$ and $Y_0 = \{f \in H^1 : \psi(f) = 0\}$. Then:*

- (1) $(L^2, Y_0)_{s,2} = H^s$ if and only if $0 \leq s < s_0$;
- (2) $(L^2, Y_0)_{s,2}$ is a closed subspace of codimension one in H^s if and only if $s_1 < s \leq 1$.

Proof. We can apply Theorem 2.1 with $X_0 = H^1$ and $X_1 = L^2$. To estimate $K(t, \psi)$, we note that if $f \in H^1$ and $t \geq 1$, then

$$\max(\|f\|_1, t\|f\|) \leq \|f\|_t \leq \sqrt{2} \max(\|f\|_1, t\|f\|),$$

and so

$$\|\psi\|_t \leq K(t^{-1}, \psi) \leq \sqrt{2} \|\psi\|_t$$

for $t \geq 1$. Hence, we can describe the numbers σ_0, σ_1 of Theorem 2.1 by

$$\sigma_1 = \limsup_{\tau \rightarrow \infty} \sup_{t \geq 1} \frac{1}{\log \tau} \log \frac{\|\psi\|_t}{\|\psi\|_{\tau t}}$$

and

$$\sigma_0 = \liminf_{\tau \rightarrow \infty} \inf_{t \geq 1} \frac{1}{\log \tau} \log \frac{\|\psi\|_t}{\|\psi\|_{\tau t}}.$$

Since $\sigma_1 = 1 - s_0$ and $\sigma_0 = 1 - s_1$, this proves the proposition. •

Next, we turn to the problem of estimating $\|\psi\|_t$. The following lemma will be useful.

Lemma 3.3. *Suppose F satisfies (3.1) and (3.2). Then for any real t we have*

$$|F(it)| \leq |t|^{\frac{1}{2}} e^{\pi|t|} \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|F(x)|^2}{t^2 + x^2} dx \right)^{\frac{1}{2}}. \tag{3.6}$$

Proof. It suffices to consider $t > 0$. Then, by (3.3), $F(z)e^{i\pi z}(z + it)^{-1}$ is bounded and analytic in the upper half-plane, and so we have

$$F(it) = \frac{te^{\pi t}}{\pi} \int_{-\infty}^{\infty} \frac{F(x)}{x + it} \frac{e^{i\pi x}}{x - it} dx.$$

Applying the Cauchy–Bunyakovski inequality we prove the lemma. •

Now we can give an estimate for $\|\psi\|_t$ which essentially solves the problem of determining s_0 and s_1 .

Theorem 3.4. *There exist a constant C so that for $t \geq 2$ we have*

$$\frac{1}{C} \left(\int_{-\infty}^{\infty} \frac{|F(x)|^2}{x^2 + t^2} dx \right)^{\frac{1}{2}} \leq \|\psi\|_t \leq C \left(\int_{-\infty}^{\infty} \frac{|F(x)|^2}{x^2 + t^2} dx \right)^{\frac{1}{2}}. \quad (3.7)$$

Proof. We start with the remark that the functions $\{(2\pi)^{-\frac{1}{2}}(n^2 + t^2)^{-\frac{1}{2}}e_n : n \in \mathbb{Z}\}$ together with $(\frac{1}{2}(t \sinh 2\pi t)^{-\frac{1}{2}}(e_{it} + e_{-it}))$ form an orthonormal basis of H^1 for $\|\cdot\|_t$. Hence,

$$\|\psi\|_t^2 = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \frac{|F(n)|^2}{n^2 + t^2} + 4 \frac{|F(it) + F(-it)|^2}{t \sinh 2\pi t}. \quad (3.8)$$

By (3.6), the last term in (3.8) can be estimated by

$$\frac{|F(it) + F(-it)|^2}{t \sinh 2\pi t} \leq C^2 \int_{-\infty}^{\infty} \frac{|F(x)|^2}{t^2 + x^2} dx \quad (3.9)$$

for $t \geq 1$.

Now, if $-1 \leq \tau \leq 1$, then the map $T_\tau : H^1 \rightarrow H^1$ defined by $T_\tau f = e_\tau f$ satisfies $\|T_\tau\|_t \leq 2$ provided $t \leq 1$. Hence, if $\psi_\tau = T_\tau^* \psi$, we have $\frac{1}{2}\|\psi_\tau\|_t \leq \|\psi\|_t \leq 2\|\psi_\tau\|_t$. However, using (3.8) and (3.9) gives

$$\frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \frac{|F(n + \tau)|^2}{n^2 + t^2} \leq \|\psi_\tau\|_t^2 \leq \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \frac{|F(n + \tau)|^2}{n^2 + t^2} + C^2 \int_{-\infty}^{\infty} \frac{|F(x + \tau)|^2}{t^2 + x^2} dx.$$

Now by integrating for $0 \leq \tau \leq 1$ we obtain (3.7). •

§4. Application to nonharmonic Fourier series

At this point we turn our attention to exponential Riesz bases. Let $\Lambda = (\lambda_n)_{n \in \mathbb{Z}}$ be a sequence of complex numbers. For convenience we shall write $\sigma_n = \Re \lambda_n$ and $\tau_n = \Im \lambda_n$.

We suppose that $(e_{\lambda_n})_{n \in \mathbb{Z}}$ is an unconditional basis of L^2 , or equivalently, $((1 + |\tau_n|)^{\frac{1}{2}} e^{-\pi|\tau_n|} e_{\lambda_n})_{n \in \mathbb{Z}}$ is a Riesz basis of L^2 . Then this family is a *complete interpolating set* [25]. In particular, we have the *sampling condition*: there exists a constant D so that if $f \in L^2$, then

$$D^{-1} \|f\| \leq \left(\sum_{n \in \mathbb{Z}} (1 + |\tau_n|) e^{-2\pi|\tau_n|} |\hat{f}(\lambda_n)|^2 \right)^{\frac{1}{2}} \leq D \|f\| \quad (4.1)$$

(i.e., the latter family is a *frame*). We also note that it must satisfy a *separation condition*, i.e., for some $0 < \delta < 1$ we have

$$\frac{|\lambda_m - \lambda_n|}{1 + |\lambda_m - \bar{\lambda}_n|} \geq \delta, \quad m \neq n. \quad (4.2)$$

Then we can define an entire function F by

$$F(z) = \lim_{R \rightarrow \infty} \prod_{|\lambda_k| \leq R} (1 - z/\lambda_k). \tag{4.3}$$

The term $(1 - \lambda_k^{-1}z)$ is replaced by z if $\lambda_k = 0$. We call F the *generating function* for the unconditional basis (e_{λ_n}) .

Proposition 4.1 [23, 14, 7]. *The product (4.3) converges to an entire function of exponential type π and satisfies the integrability conditions (3.2), and*

$$\int_{-\infty}^{\infty} |F(x)|^2 dx = \infty. \tag{4.4}$$

We note that the inequality in (3.2) is necessary for the minimality of the family, and (4.4) for the completeness of (e_{λ_n}) . Also, since F satisfies (3.1) and (3.2), there exists $\psi \in (H^1)^*$ with $\hat{\psi} = F$. We remark that F is a Cartwright class function; then (see [14]) we have the Blaschke condition

$$\sum_{\lambda_n \neq 0} \frac{|\tau_n|}{|\lambda_n|^2} < \infty.$$

This implies that the families $(e_{\lambda_n})_{\Im \lambda_n > 0}$, $(e_{\lambda_n})_{\Im \lambda_n < 0}$ are minimal in $L^2(0, \infty)$, $L^2(-\infty, 0)$ correspondingly. Also we have $\sum_{\lambda_n \neq 0} \frac{1}{|\lambda_n|^2} < \infty$ (this follows from [15, p. 127]). Thus, we have a strong Blaschke condition

$$\sum_{\lambda_n \neq 0} \frac{1 + |\tau_n|}{|\lambda_n|^2} < \infty. \tag{4.5}$$

Now by the result of Russell, Proposition 1.2, the functions (e_{λ_n}) form an unconditional basis of a closed subspace Y_0 of H^1 of codimension one. Clearly, the kernel of ψ coincides with Y_0 . Hence, our above results (Proposition 3.2 and Theorem 3.4) apply to this case.

Theorem 4.2. *Suppose $(e_{\lambda_n})_{n \in \mathbb{Z}}$ is an unconditional basis of L^2 . Then:*

- (1) $(e_{\lambda_n})_{n \in \mathbb{Z}}$ is an unconditional basis of the Sobolev space H^s if and only if $0 \leq s < s_0$;
- (2) $(e_{\lambda_n})_{n \in \mathbb{Z}}$ is an unconditional basis of a closed subspace of H^s of codimension one if and only if $s_1 < s \leq 1$;
- (3) If $s_0 \leq s \leq s_1$, then (e_{λ_n}) is not an unconditional basic sequence.

Proof. By Russell's theorem and Proposition 1.2 above, $(e_{\lambda_n})_{n \in \mathbb{Z}}$ is an unconditional basis for the closed subspace Y_0 of codimension one that is the kernel of the linear functional ψ . Let v_n be the weight sequence $v_n = \frac{\sinh(2\pi\tau_n)}{\tau_n} = \|e_{\lambda_n}\|_{L^2}^2$, and let

$h_n = (1 + |\lambda_n|^2)v_n = \|e_{\lambda_n}\|_{H^1}^2$. From the basis property it follows that the map $V: \ell_2(h) \rightarrow Y_0$ defined by

$$V(\alpha) = \sum_{n \in \mathbb{Z}} \alpha_n e_{\lambda_n}$$

is an isomorphism (onto). Clearly, V is an isomorphism of $\ell_2(v)$ onto $L^2(-\pi, \pi) = Y_1$. Hence by interpolation V is an isomorphism of $\ell_2(v^{1-s}h^s)$ onto $Y_{1-s} = [Y_0, L^2]_{1-s,2}$. In other words, setting $q_n = v_n^{1-s}h_n^s = v_n(1 + |\lambda_n|^2)^s$, we obtain

$$C^{-1} \sum |\alpha_n|^2 q_n \leq \left\| \sum \alpha_n e_{\lambda_n} \right\|_{Y_{1-s}}^2 \leq C \sum |\alpha_n|^2 q_n,$$

and the almost normalized family $(e_{\lambda_n}/q_n^{1/2})_{n \in \mathbb{Z}}$ forms a Riesz basis in Y_{1-s} . Thus, if Y_{1-s} is a closed subspace in H^s , then (e_{λ_n}) forms an unconditional basic sequence in H^s also.

Next, we estimate $\|e_{\lambda_n}\|_{H^s}$ to have the inverse implication. In fact from interpolation between L^2 and H^1 we have

$$\|e_{\lambda_n}\|_{H^s} \leq C \|e_{\lambda_n}\|^{1-s} \|e_{\lambda_n}\|_1^s = C(v_n^{1-s}h_n^s)^{\frac{1}{2}},$$

where C depends only on s . Similarly, if we define $\phi_n(f) = (f, e_{\lambda_n})$, then the norm of ϕ_n in $(H^s)^*$ can be estimated by

$$\|\phi_n\|_{(H^s)^*} \leq C_1 \|\phi_n\|^{1-s} \|\phi_n\|_{(H^1)^*}^s = C_1(v_n^{1-s})^{1/2} (v_n^2/h_n)^{s/2} = C_1(v_n^{1+s}h_n^{-s})^{\frac{1}{2}}.$$

On the other hand, $\|e_{\lambda_n}\|_{H^s} \geq |\phi_n(e_{\lambda_n})|/\|\phi_n\|_{H^s}$, which gives $\|e_{\lambda_n}\|_{H^s} \geq C_1^{-1}(v_n^{1-s}h_n^s)^{1/2}$. Therefore, the norms $\|e_{\lambda_n}\|_{H^s}$ and $\|e_{\lambda_n}\|_{Y_{1-s}}$ are both equivalent to $\sqrt{q_n}$. Consequently, the assumption that (e_{λ_n}) is an unconditional basic sequence leads to the equivalence of the metrics in H^s and Y_{1-s} . •

Remark. It is easy to obtain a necessary and sufficient condition for an exponential family $(e_{\lambda_n})_{n \in \mathbb{Z}}$ that is complete and minimal in L^2 to be complete and/or minimal in H^s ; see [3]. To do this we relate the generating function F to the critical exponent s_Λ ,

$$s_\Lambda := \inf \left\{ s : \int_{-\infty}^{\infty} \frac{|F(x)|^2}{1 + |x|^{2s}} dx < \infty \right\} = \inf \{ s : \psi \in (H^s)^* \}.$$

Now $(e^{i\lambda_n t})$ is complete in $H^s(-\pi, \pi)$ for $s < s_\Lambda$ and is minimal for $s > s_\Lambda - 1$. The situation for $s = s_\Lambda$ or $s = s_\Lambda - 1$ depends on whether ψ is bounded in H^{s_Λ} . Note that $s_0 \leq s_\Lambda \leq s_1$ in general.

Thus, the family (e_{λ_n}) is minimal in H^s for $0 < s < 1$, and for any $(\alpha_n) \in l^2$, $\alpha \neq 0$, we have

$$0 < \left\| \sum \alpha_n e_{\lambda_n} / \sqrt{q_n} \right\|_{H^s}^2 \leq \left\| \sum \alpha_n e_{\lambda_n} / \sqrt{q_n} \right\|_{Y_{1-s}}^2 \leq C \sum |\alpha_n|^2.$$

We do not know whether $(e_{\lambda_n})_{n \in \mathbb{Z}}$ can be a conditional basis of H^s , for some appropriate ordering, when $s_0 \leq s \leq s_\Lambda$.

In order to get precise estimates of s_0 and s_1 we shall need an alternative formula for $\|\psi\|_t$ in this special case.

We introduce the function $\Phi(z)$ defined by $\Phi(z) = |F(z)|d(z, \Lambda)^{-1}$ for $z \notin \Lambda$ and $\Phi(\lambda_n) = |F'(\lambda_n)|$ for $n \in \mathbb{Z}$. The function Φ plays an important role in the known conditions for (e_{λ_n}) to be an unconditional basis [20, 19]. We call Φ the *carrier function* for (e_{λ_n}) .

The following lemma lists some useful properties.

Lemma 4.3. *Suppose $-\infty < t < \infty$. Then:*

- (i) *there is at most one $n \in \mathbb{Z}$ so that $|it - \lambda_n| < \frac{1}{2}\delta|t|$, where δ is the separation constant in (4.2). There is also at most one $n \in \mathbb{Z}$ so that $|it - \lambda_n| < \frac{1}{4}\delta|\lambda_n|$;*
- (ii) $|F(it)| \leq (|\lambda_0| + |t|)\Phi(it)$;
- (iii) *there is a constant C independent of t, n so that for every $n \in \mathbb{Z}$ we have*

$$|F(it)| \leq C(|\lambda_0| + |t|) \frac{|it - \lambda_n|}{(|\lambda_n|^2 + t^2)^{\frac{1}{2}}} \Phi(it);$$

(iv) *for $t \neq 0$, we have*

$$\Phi(it) \leq |t|^{-\frac{1}{2}} e^{\pi|t|} \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|F(x)|^2}{t^2 + x^2} dx \right)^{\frac{1}{2}}.$$

Proof. (i) Suppose m, n are distinct and $|it - \lambda_n|, |it - \lambda_m| < \frac{1}{2}\delta|t|$. Then $|\lambda_m - \lambda_n| < \delta t$, while

$$|\lambda_m - \bar{\lambda}_n| \geq |(\lambda_m - it) - \overline{(\lambda_n - it)} + 2it| \geq (2 - \delta)|t| > |t|.$$

Hence,

$$\frac{|\lambda_m - \lambda_n|}{1 + |\lambda_m - \bar{\lambda}_n|} < \delta,$$

which contradicts (4.2).

For the second part note that if $|it - \lambda_n| < \frac{1}{4}\delta|\lambda_n|$, then $|\lambda_n| < 2|t|$, so that $|it - \lambda_n| < \frac{1}{2}\delta|t|$.

(ii) is immediate from the fact that $d(it, \Lambda) \leq |\lambda_0| + t$.

(iii) If $|it - \lambda_n| < \frac{1}{2}\delta t$ then, by (i), $|it - \lambda_n|\Phi(it) = |F(it)|$ and $t^2 + |\lambda_n|^2 \leq 5t^2$. Let $|it - \lambda_n| \geq \frac{1}{2}\delta t$. Then

$$\frac{|it - \lambda_n|}{(|\lambda_n|^2 + t^2)^{1/2}} \geq \frac{|it - \lambda_n|}{|\lambda_n| + |t|} \geq \frac{|it - \lambda_n|}{|it - \lambda_n| + 2|t|} \geq c > 0.$$

Since $|\lambda_n| + |t| \geq d(it, \Lambda)$, we have (iii).

(iv) Let λ_n satisfy $|it - \lambda_n| = d(it, \Lambda)$. If t and τ_n have opposite signs or if $\tau_n = 0$, then $d(it, \Lambda) \geq t$, and so $\Phi(it) \leq t^{-1}|F(it)|$. If they have the same sign, define $G(z) = (z - \bar{\lambda}_n)(z - \lambda_n)^{-1}F(z)$ and note that

$$\Phi(it) = \frac{|G(it)|}{|it - \bar{\lambda}_n|} \leq t^{-1}|G(it)|.$$

Since $|G(x)| = |F(x)|$ for x real, we deduce (iv) from (3.6). •

Next, we show that the Blaschke condition (4.5) can be improved for Riesz bases.

Proposition 4.4. *If $(e_{\lambda_n})_{n \in \mathbb{Z}}$ is an unconditional basis of L^2 , then there is a constant C so that for any $0 < t < \infty$ we have*

$$\sum_{\lambda_n \neq 0} \frac{t(1 + |\tau_n|)}{|\lambda_n|^2 + t^2} \leq C. \tag{4.6}$$

Proof. We apply (4.1) to $e_{\pm it}$. Then

$$\sum_{n \in \mathbb{Z}} (1 + |\tau_n|) e^{-2\pi|\tau_n|} \left| \frac{\sin(\pi(\lambda_n \pm it))}{\lambda_n \pm it} \right|^2 \leq 4D^2 \|e_{\pm it}\|^2 = 4D^2 \frac{\sinh 2\pi t}{t}. \tag{4.7}$$

Now for each n there is a choice of sign so that

$$\left| \frac{\sin(\pi(\lambda_n \pm it))}{\lambda_n \pm it} \right| \geq \frac{|\sinh(\pi(|\tau_n| + t))|}{|\lambda_n| + t},$$

whence

$$\sum_{n \in \mathbb{Z}} (1 + |\tau_n|) e^{-2\pi|\tau_n|} \frac{\sinh(\pi(t + |\tau_n|))^2}{|\lambda_n|^2 + t^2} \leq 4D^2 \frac{\sinh 2\pi t}{t}.$$

This yields (4.6) for $t \geq 1$, and this extends to $t \geq 0$ in view of (4.5) and the fact that $\sum_{n \neq 0} |\lambda_n|^{-2} < \infty$. •

We shall also need a perturbation lemma.

Lemma 4.5. *Let (e_{λ_n}) and $(e_{\mu_n})_{n \in \mathbb{Z}}$ be two unconditional bases of L^2 . Suppose further that there is a constant C so that*

$$\sum_{n \in \mathbb{Z}} \frac{t|\mu_n - \lambda_n|}{|\mu_n||\lambda_n| + t^2} \leq C, \quad 1 < t < \infty.$$

Suppose Φ and Ψ are the carrier functions for (e_{λ_n}) and (e_{μ_n}) . Then there exist constants $B, T > 0$ so that if $t \geq T$, then

$$\frac{1}{B} \frac{\Psi(it)}{\Phi(it)} \leq \prod_{\substack{0 < |\lambda_n| \leq t \\ |\mu_n| \neq 0}} \frac{|\lambda_n|}{|\mu_n|} \leq B \frac{\Psi(it)}{\Phi(it)}.$$

Proof. Taking $t = \max(1, |\mu_n|^{\frac{1}{2}}|\lambda_n|^{\frac{1}{2}})$, we observe that

$$|\lambda_n - \mu_n| \leq C \max(1, 2|\mu_n|^{\frac{1}{2}}|\lambda_n|^{\frac{1}{2}})$$

for each n . Hence,

$$|\lambda_n| \leq |\mu_n| + 2C|\lambda_n|^{1/2}|\mu_n|^{1/2} + C \leq |\mu_n| + \frac{1}{2}|\lambda_n| + 2C^2|\mu_n| + C.$$

Writing a similar estimate for $|\mu_n|$ and setting $C_1 = 2 + 4C^2 > 1$, we get

$$|\lambda_n| \leq C_1(|\mu_n| + 1), \quad |\mu_n| \leq C_1(|\lambda_n| + 1). \tag{4.8}$$

Now let $c = \frac{1}{4} \min(\delta, \delta')$, where δ, δ' are the separation constants of $(\lambda_n)_{n \in \mathbb{Z}}$ and $(\mu_n)_{n \in \mathbb{Z}}$, respectively.

Next, we make the remark that there is a constant M so that if $|w|, |z| \leq 2C_1 + 1$ and $|1 - w|, |1 - z| \geq c$, then

$$|\log |1 - w| - \log |1 - z|| \leq M|w - z|. \tag{4.9}$$

We fix $T = |\mu_0| + |\lambda_0| + 2C_1$. Suppose that $t \geq T$, and let $p = p(t), q = q(t) \in \mathbb{Z}$ be chosen so that $|it - \lambda_p| = \min\{|it - \lambda_n| : n \in \mathbb{Z}\}$ and $|it - \mu_q| = \min\{|it - \mu_n| : n \in \mathbb{Z}\}$. It may happen that $p = q$. Note that, automatically,

$$|it - \lambda_p| \leq |t| + |\lambda_0| \leq 2t, \quad |it - \mu_q| \leq |t| + |\mu_0| \leq 2t. \tag{4.10}$$

Then if $n \neq p, q$ and $|\lambda_n| > t$, we have $|\mu_n| > \frac{1}{2}C_1^{-1}|\lambda_n|$, so that $|it - \mu_n| \leq |t| + |\mu_n| \leq (2C_1 + 1)|\mu_n|$. By Lemma 4.3 (i), we have $|it - \lambda_n| \geq c|\lambda_n|$ and $|it - \mu_n| \geq c|\mu_n|$. Hence,

$$\begin{aligned} |\log |it - \mu_n| - \log |it - \lambda_n| - \log |\mu_n| + \log |\lambda_n|| &\leq M \frac{t|\lambda_n - \mu_n|}{|\lambda_n||\mu_n|} \\ &\leq (2C_1 + 1)M \frac{t|\lambda_n - \mu_n|}{|\lambda_n||\mu_n| + t^2} \end{aligned}$$

by (4.9). Next, suppose $n \neq p, q$ and $|\lambda_n| \leq t$. Then $|\mu_n| \leq C_1(t + 1) \leq 2C_1t$. We also have $|it - \lambda_n|, |it - \mu_n| \geq c|t|$ and so

$$\begin{aligned} |\log |it - \mu_n| - \log |it - \lambda_n|| &\leq M \frac{|\lambda_n - \mu_n|}{t} \\ &\leq (2C_1 + 1)M \frac{t|\lambda_n - \mu_n|}{|\lambda_n||\mu_n| + t^2} \end{aligned}$$

by (4.9). Combining and summing over all $n \neq p, q$, we obtain

$$\log \frac{\Psi(it)}{\Phi(it)} = \delta(t) \log \left| \frac{it - \mu_p}{it - \lambda_q} \right| + \sum_{0 < |\lambda_n| \leq t} \log |\lambda_n| - \sum_{\substack{0 < |\lambda_n| \leq t \\ |\mu_n| \neq 0}} \log |\mu_n| + \gamma(t),$$

where $|\gamma(t)| \leq C(2C_1 + 1)M$ and $\delta(t)$ is 1 if $p \neq q$ and 0 if $p = q$.

To conclude we need only consider the case where $p \neq q$. In this case $|it - \mu_p|, |it - \lambda_q| \geq ct$. We also have $|\lambda_p|, |\mu_q| \leq 3t$ by (4.10); hence, by (4.8), $|\lambda_q|, |\mu_p| \leq C_1(3t + 1) \leq 4C_1t$. Thus, $|it - \mu_p|, |it - \lambda_q| \leq 5C_1t$. This concludes the proof. •

Lemma 4.6. *Suppose (e_{λ_n}) is an unconditional basis of L^2 . Then there exist constants B, T so that if $t \geq T$, then*

$$\frac{1}{B}\Phi(it) \leq \Phi(-it) \leq B\Phi(it).$$

Proof. This follows from Lemma 4.5 by taking $\mu_n = \bar{\lambda}_n$ in view of Lemma 4.4. •

The next theorem is the key step in the proof of our main result.

Theorem 4.7. *Suppose $(e_{\lambda_n})_{n \in \mathbb{Z}}$ is an unconditional basis of L^2 . Then there exist constants C and $T > 0$ so that if $t \geq T$, then*

$$C^{-1}t^{\frac{1}{2}}e^{-\pi t}\Phi(it) \leq \|\psi\|_t \leq Ct^{\frac{1}{2}}e^{-\pi t}\Phi(it). \quad (4.11)$$

Proof. The left-hand inequality in (4.11) is an immediate consequence of Lemma 4.3 (iv) and (3.7). We turn to the right-hand inequality.

First, we use Lemma 4.3(ii), (iii) and Lemma 4.6. There are constants $C, T > 1$ so that if $|t| \geq T$, then $\Phi(-it) \leq C\Phi(it)$, $|F(it)| \leq Ct\Phi(it)$, and

$$|F(it)| \leq Ct \frac{|\lambda_n - it|}{(|\lambda_n|^2 + t^2)^{\frac{1}{2}}} \Phi(it) \quad (4.12)$$

for every n .

Choose $g \in H^1$ so that $\psi(f) = \langle f, g \rangle_t$ for $f \in H^1$. Let h be the orthogonal projection with respect to $\langle \cdot, \cdot \rangle_t$ of g onto the subspace H_0^1 of all f so that $f(-\pi) = f(\pi) = 0$, and let $k = g - h$. Then $\|\psi\|_t^2 = \|k\|_t^2 + \|h\|_t^2$.

The orthogonal complement of H_0^1 (with respect to $\langle \cdot, \cdot \rangle_t$) is the 2-dimensional space with the orthonormal basis $\{e_{\pm it}/\|e_{\pm it}\|_t\}$. Hence,

$$k = \|e_{it}\|_t^{-2}(\overline{F(it)}e_{it} + \overline{F(-it)}e_{-it})$$

and

$$\|k\|_t^2 = \|e_{it}\|_t^{-2}(|F(it)|^2 + |F(-it)|^2).$$

Since $\|e_{it}\|_t^2 = 2t \sinh 2\pi t$, we deduce that

$$\|k\|_t \leq C_1 t^{\frac{1}{2}} \Phi(it) e^{-\pi t}, \quad t \geq T,$$

with a suitable constant C_1 . Therefore, it only remains to estimate $\|h\|_t$.

We first argue that

$$\begin{aligned} \langle e_z, k \rangle_t &= (2t \sinh 2\pi t)^{-1} (F(it) \langle e_z, e_{it} \rangle_t + F(-it) \langle e_z, e_{-it} \rangle_t) \\ &= \frac{i}{\sinh 2\pi t} (F(-it) \sin \pi(z - it) - F(it) \sin \pi(z + it)). \end{aligned}$$

Since $\psi(e_{\lambda_n}) = F(\lambda_n) = 0$ for $n \in \mathbb{Z}$, we obtain

$$\langle e_{\lambda_n}, h \rangle_t = \frac{i}{\sinh 2\pi t} (F(it) \sin \pi(\lambda_n + it) - F(-it) \sin \pi(\lambda_n - it)).$$

Now, if we use (4.12), we get an estimate valid for $t \geq T$:

$$|\langle e_{\lambda_n}, h \rangle_t| \leq C \Phi(it) \frac{t|\lambda_n + it||\lambda_n - it|}{(|\lambda_n|^2 + t^2)^{\frac{1}{2}} \sinh 2\pi t} \left(\left| \frac{\sin(\pi(\lambda_n - it))}{\lambda_n - it} \right| + \left| \frac{\sin(\pi(\lambda_n + it))}{\lambda_n + it} \right| \right).$$

Since $h \in H_0^1$, we then have

$$\langle e_{\lambda_n}, h \rangle_t = (\lambda_n^2 + t^2) \langle e_{\lambda_n}, h \rangle,$$

and we can rewrite the above estimate as

$$|\langle e_{\lambda_n}, h \rangle| \leq C \frac{\Phi(it)}{\sinh 2\pi t} \frac{1}{(|\lambda_n|^2 + t^2)^{1/2}} \left(\left| \frac{\sin(\pi(\lambda_n - it))}{\lambda_n - it} \right| + \left| \frac{\sin(\pi(\lambda_n + it))}{\lambda_n + it} \right| \right).$$

Now

$$\langle e_{\lambda_n}, th + h' \rangle = (t - i\lambda_n) \langle e_{\lambda_n}, h \rangle.$$

Next, we use the sampling inequality (4.1):

$$\|h\|_t^2 = \|th + h'\|_{L^2}^2 \leq D^2 \sum_{n \in \mathbb{Z}} (1 + |\tau_n|) e^{-2\pi|\tau_n|} |t - i\lambda_n| |\langle e_{\lambda_n}, h \rangle|^2.$$

However, we can combine with (4.7) to deduce that

$$\|th + h'\|_{L^2} \leq 4C^2 D^2 t^{\frac{1}{2}} \Phi(it) (\sinh 2\pi t)^{-\frac{1}{2}}$$

for $t \geq T$, which gives the conclusion. •

Now, we consider the case where (λ_n) is a small perturbation of the sequence $\mu_n = n$. For convenience we shall assume that $\lambda_n = 0$ can only occur when $n = 0$.

Theorem 4.8. *Suppose $(e_{\lambda_n})_{n \in \mathbb{Z}}$ is an unconditional basis of L^2 , and*

$$\sum_{n \neq 0} \frac{t|\lambda_n - n|}{n^2 + t^2} < C \tag{4.13}$$

for some constant C and all $t \geq 1$. Then

$$s_1 = \frac{1}{2} + \limsup_{\tau \rightarrow \infty} \sup_{t \geq 1} \frac{1}{\log \tau} \sum_{t < |n| \leq \tau t} \log \frac{|n|}{|\lambda_n|}$$

and

$$s_0 = \frac{1}{2} + \liminf_{\tau \rightarrow \infty} \inf_{t \geq 1} \frac{1}{\log \tau} \sum_{t < |n| \leq \tau t} \log \frac{|n|}{|\lambda_n|}.$$

Proof. In this case we compare the carrier function Φ for the basis (e_{λ_n}) with the carrier function Ψ for the basis (e_n) . Clearly, $\Psi(it) = |\sin \pi it|/\pi t$. We can next use Lemma 4.5 to estimate $\Phi(it)$, and then the theorem follows directly from Theorem 4.7 together with (3.5) and (3.4). •

Let us specialize to some important cases. Let $\delta_n = \Re \lambda_n - n = \sigma_n - n$.

Theorem 4.9. *Suppose $(e_{\lambda_n})_{n \in \mathbb{Z}}$ is an unconditional basis of L^2 such that $\sup |\delta_n| < \infty$ and $\sum_{n \neq 0} \tau_n^2 n^{-2} < \infty$. Then*

$$s_1 = \frac{1}{2} - \liminf_{\tau \rightarrow \infty} \inf_{t \geq 1} \frac{1}{\log \tau} \sum_{t < |n| \leq \tau t} \frac{\delta_n}{n} \tag{4.14}$$

and

$$s_0 = \frac{1}{2} - \limsup_{\tau \rightarrow \infty} \sup_{t \geq 1} \frac{1}{\log \tau} \sum_{t < |n| \leq \tau t} \frac{\delta_n}{n}. \tag{4.15}$$

Remark. In particular, relations (4.14) and (4.15) are true if $|\lambda_n - n|$ is bounded.

Proof. Combining Proposition 4.4 and the boundedness of (δ_n) gives us (4.13). Note that if $n \neq 0$, then

$$\log \frac{|n|}{|\lambda_n|} = -\log \left(1 + \frac{|\lambda_n - |n||}{|n|} \right).$$

Now, we have

$$\begin{aligned} \frac{|\lambda_n - |n||}{|n|} &= \left(1 + \frac{2\delta_n}{n} + \frac{\delta_n^2 + \tau_n^2}{n^2} \right)^{\frac{1}{2}} \\ &= \frac{\delta_n}{n} + \alpha_n, \end{aligned}$$

where

$$|\alpha_n| \leq C \frac{1 + \tau_n^2}{n^2}$$

for a suitable constant C . By (4.5) and the assumption of the theorem, this implies that $\sum_{n \neq 0} |\alpha_n| < \infty$ and yields the theorem. •

Before discussing examples, we observe one more property of s_0 and s_1 in this case, which uses recent results of [19] and the theory of A_2 -weights.

Theorem 4.10. *If $(e_{\lambda_n})_{n \in \mathbb{Z}}$ is an unconditional basis of L^2 , then $s_0 > 0$ and $s_1 < 1$.*

Proof. We shall use the relationship between the Riesz basis property and sampling/interpolation in the spaces of entire functions of exponential type. In the case of L^2 and the Paley-Wiener space, this relationship may be found in [25].

Consider the space $L^2_{\pi,s}$, of all entire functions having exponential type at most π and satisfying

$$\int_{-\infty}^{\infty} \frac{|f(\xi)|^2}{(1 + |\xi|)^{2s}} d\xi < \infty.$$

(Note that the Fourier transform of $L^2_{\pi,s}$ is the set of all distributions in $H^{-s}(\mathbb{R})$ supported on $[-\pi, \pi]$.) Now the formal adjoint of the map from $\ell_2(\mathbb{Z})$ to H^s

defined by $(\alpha_n) \mapsto \sum_{n \in \mathbb{Z}} \alpha_n (1 + |\tau_n|)^{-\frac{1}{2}} (1 + |\lambda_n|)^{-s} e_{\lambda_n}$ is the map from $L^2_{\pi, s}$ to $\ell_2(\mathbb{Z})$ given by $f \mapsto (f(\lambda_n) (1 + |\tau_n|)^{1/2} (1 + |\lambda_n|)^s e^{-\pi \tau_n})_{n \in \mathbb{Z}}$. Hence, $(e_{\lambda_n})_{n \in \mathbb{Z}}$ is an unconditional basic sequence (respectively, unconditional basis) if and only if $(\lambda_n)_{n \in \mathbb{Z}}$ is an interpolating sequence (respectively, complete interpolating sequence) in $L^2_{\pi, s}$.

Note that if (λ_n) is interpolating for $L^2_{\pi, s-1}$, then it is interpolating for $L^2_{\pi, s}$ by the simple device of considering functions of the form $f(z) = (z - \mu)g(z)$, where $\mu \notin \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ and $g \in L^2_{\pi, s-1}$.

It follows that our result can be proved by showing that $(\lambda_n)_{n \in \mathbb{Z}}$ is a complete interpolating sequence for $L^2_{\pi, s}$ for all $|s| < \epsilon$ for some $\epsilon > 0$. To do this, we note that, by the results of [19], this is equivalent to requiring that $(1 + |\xi|)^{2s} \Phi(\xi)^2$ be an A_2 -weight for $|s| < \epsilon$. Now Φ^2 is an A_2 -weight ([19] or [20]); hence, there exists $\eta > 0$ so that $\Phi^{2(1+\eta)}$ is an A_2 -weight (cf. [6, p. 262, Corollary 6.10]). Thus, the Hilbert transformation is bounded on both $L_2(\mathbb{R}, \Phi^{2(1+\eta)})$ and $L^2(\mathbb{R}, (1 + |\xi|)^{2\theta})$ for $0 < \theta < \frac{1}{2}$. It then follows by complex interpolation that $\Phi(\xi)^2 (1 + |\xi|)^{2s}$ is an A_2 -weight provided $|s| < \eta(1 + \eta)^{-1}$. •

Observe that these results imply Theorem 1.4.

Examples. We recall the classical theorem of Kadets (see, e.g., [11] or [15]) saying that if (λ_n) are real, then $\sup_n |\delta_n| < \frac{1}{4}$ is a sufficient condition for $(e_{\lambda_n})_{n \in \mathbb{Z}}$ to be a Riesz basis. First, we consider the case of regular behavior. For example, we can set $\delta_n = -\frac{1}{2}q \operatorname{sgn} n$ (see [1]). Then we obtain $s_1 = s_0 = \frac{1}{2} + q$. More generally, if for some $y > 0$ we have $C^{-1}(1 + |x|)^{2q} \leq |F(x + iy)| \leq C(1 + |x|)^{2q}$, we obtain $s_1 = s_0 = \frac{1}{2} + q$ (if we use the integral estimates of $\|\psi\|_t$, i.e., Theorem 3.4).

We can easily make sequences (δ_n) with $\sup |\delta_n| < \frac{1}{4}$ to exhibit any required behavior. In fact, if we put

$$b_n = \frac{1}{\log 2} \sum_{2^n < |k| \leq 2^{n+1}} \frac{\delta_k}{k},$$

then

$$s_0 = \frac{1}{2} - \lim_{N \rightarrow \infty} \frac{1}{N} \inf_{n \geq 1} \sum_{k=n+1}^{n+N} b_k$$

and

$$s_1 = \frac{1}{2} - \lim_{N \rightarrow \infty} \frac{1}{N} \sup_{n \geq 1} \sum_{k=n+1}^{n+N} b_k.$$

To be more specific, if $-\frac{1}{2} < p \leq q < \frac{1}{2}$, set

$$\delta_n = \frac{1}{2}q \operatorname{sgn} n \quad \text{for } 2^{(2^{2k})} < |n| \leq 2^{(2^{2k+1})},$$

$$\delta_n = \frac{1}{2}p \operatorname{sgn} n \quad \text{for } 2^{(2^{2k-1})} < |n| \leq 2^{(2^{2k})}.$$

Then

$$b_m = \frac{q}{\log 2} \sum_{2^{m+1}}^{2^{m+1}} \frac{1}{k} = q + o(1)$$

for $2^{2k} < m \leq 2^{2k+1}$, and

$$b_m = p + o(1)$$

for $2^{2k-1} < m \leq 2^{2k}$. Thus,

$$s_0 = \frac{1}{2} - q, \quad s_1 = \frac{1}{2} - p$$

(note that an example of irregular behavior was given in [1])

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