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# Socially Acceptable Values for Cooperative TU Games

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**Abstract.** In the framework of the solution theory for cooperative transferable utility games, a value is called socially acceptable with reference to a certain basis of games if, for each relevant game, the payoff to any productive player covers the payoff to any non-productive player. Firstly, it is shown that two properties called desirability and monotonicity are sufficient to guarantee social acceptability of type $I$. Secondly, the main goal is to investigate and characterize the subclass of efficient, linear, and symmetric values that are socially acceptable for any of three types (with clear affinities to simple unanimity games).

**Keywords:** cooperative game, unanimity game, socially acceptable value, Shapley value, solidarity value, egalitarian value

**Mathematics Subject Classification 2000:** 91A12

## 1. Introduction and notions

Formally, a *transferable utility game* (or cooperative game or coalitional game with side payments) is a pair $\langle N, v\rangle$, where $N$ is a finite set of at least two *players* and $v : 2^N \to \mathbb{R}$ is a *characteristic function* satisfying $v(\emptyset) = 0$. An element of $N$ (notation: $i \in N$) and a nonempty subset $S$ of $N$ (notation: $S \subseteq N$ or $S \in 2^N$ with $S \neq \emptyset$) is called a *player* and *coalition* respectively, and the real number $v(S)$ is called the *worth* of coalition $S$. A TU game $\langle N, v\rangle$ is called *monotonic* if $v(S) \leq v(T)$ for all $S, T \subseteq N$ with $S \subseteq T$. The size (cardinality) of coalition $S$ is denoted by $|S|$ or, if no ambiguity is possible, by $s$. Particularly, $n$ denotes the size of the player set $N$. Let $\mathcal{G}_N$ denote the linear space consisting of all games with fixed player set $N$. Given two games $\langle N, v\rangle$, $\langle N, w\rangle$, and two scalars $\beta, \delta \in \mathbb{R}$, their *linear combination* $\langle N, \beta \cdot v + \delta \cdot w\rangle$ is defined by $(\beta \cdot v + \delta \cdot w)(S) = \beta \cdot v(S) + \delta \cdot w(S)$ for all $S \subseteq N$.

The solution part of cooperative game theory deals with the allocation problem of how to divide, for any game $\langle N, v\rangle$, the worth $v(N)$ of the grand coalition $N$ among the players. The traditional one-point solution concepts associate, with every game, a single allocation called the value of the game. Formally, a *value* on $\mathcal{G}_N$ is a function $\psi$ that assigns a single payoff vector $\psi(N, v) = (\psi_i(N, v))_{i \in N} \in \mathbb{R}^N$ to every TU game $\langle N, v\rangle$. The so-called *value* $\psi_i(N, v)$ of player $i$ in the TU game $\langle N, v\rangle$ represents an assessment by $i$ of his gains for participating in the game. For instance, the *egalitarian value* $\psi^{EG}$ allocates the same payoff to every player in that $\psi_i^{EG}(N, v) = \frac{v(N)}{n}$ for all games $\langle N, v\rangle$ and all $i \in N$. Throughout the paper we restrict ourselves to the class of efficient, linear, and symmetric values.

**Definition 1.** A value $\psi$ on $\mathcal{G}_N$ is said to possess

(i) *efficiency*, if $\sum_{i\in N} \psi_i(N,v) = v(N)$ for all games $\langle N,v\rangle$;
(ii) *linearity*, if $\psi(N,\beta\cdot v+\delta\cdot w) = \beta\cdot\psi(N,v)+\delta\cdot\psi(N,w)$ for all games $\langle N,v\rangle$, $\langle N,w\rangle$, and all scalars $\beta,\delta\in\mathbb{R}$;
(iii) *symmetry*, if $\psi_{\pi(i)}(N,\pi v) = \psi_i(N,v)$ for all games $\langle N,v\rangle$, all $i\in N$, and every permutation $\pi$ on $N$. Here the game $\langle N,\pi v\rangle$ is defined by $(\pi v)(\pi S) := v(S)$ for all $S\subseteq N$.

Our main goal is to develop the notion of social acceptability on the class of efficient, linear, and symmetric values. Undoubtedly, the Shapley value (Shapley, 1953) is the most appealing value of this class, whereas the solidarity value introduced in (Nowak and Radzik, 1994) has clear affinities to the Shapley value. In fact, these clear affinities have been stressed in Calvo's approach (Calvo, 2008) to non-transferable utility (NTU) games (inclusive of TU games) by introducing the so-called "random marginal NTU value" and "random removal NTU value" as the NTU counterparts of the Shapley TU value and the solidarity TU value, respectively, in the sense that pairwise coincidence of values happens to occur on the class of TU games. Surprisingly, it turns out that the solidarity value and the various social acceptability notions are well-matched. In order to review similar axiomatizations of both the Shapley value and the solidarity value, we recall three essential properties of values for TU games.

**Definition 2.** A value $\psi$ on $\mathcal{G}_N$ possesses

(i) *substitution property*, if $\psi_i(N,v) = \psi_j(N,v)$ for all games $\langle N,v\rangle$, all pairs $i,j\in N$, such that players $i$ and $j$ are *substitutes* in the game $\langle N,v\rangle$, i.e., $v(S\cup\{i\}) = v(S\cup\{j\})$ for all $S\subseteq N\backslash\{i,j\}$;
(ii) *null player property*, if $\psi_i(N,v)=0$ for all games $\langle N,v\rangle$, all $i\in N$, such that player $i$ is a *null player* in the game $\langle N,v\rangle$, i.e., $v(S\cup\{i\}) = v(S)$ for all $S\subseteq N\backslash\{i\}$;
(iii) *A-null player property*, if $\psi_i(N,v)=0$ for all games $\langle N,v\rangle$, all $i\in N$, such that player $i$ is a *A-null player* in the game $\langle N,v\rangle$, i.e., $\sum_{k\in S}\Big[v(S)-v(S\backslash\{k\})\Big] = 0$ for all $S\subseteq N$ with $i\in S$.

It is well-known that the symmetry property implies the substitution property. (Shapley, 1953) and (Nowak and Radzik, 1994) proved that there exists a unique value on $\mathcal{G}_N$ satisfying the following four properties: efficiency, linearity, symmetry, and either null player property or A-null player property. In fact, the explicit formulas for the *Shapley value* $\psi^{Sh}(N,v) = (\psi_i^{Sh}(N,v))_{i\in N}$ and the *solidarity value* $\psi^{Sol}(N,v) = (\psi_i^{Sol}(N,v))_{i\in N}$ are as follows (Shapley, 1953; Roth, 1988; Driessen, 1988; Nowak and Radzik, 1994): for all $i\in N$

$$\psi_i^{Sh}(N,v) = \sum_{S\subseteq N\backslash\{i\}} \frac{1}{n\cdot\binom{n-1}{s}}\cdot\Big[v(S\cup\{i\})-v(S)\Big], \qquad (1)$$

or equivalently,

$$\psi_i^{Sh}(N,v) = \sum_{\substack{T\subseteq N,\\ T\ni i}} \frac{1}{n\cdot\binom{n-1}{t-1}} \cdot \Big[v(T) - v(T\backslash\{i\})\Big];$$

$$\psi_i^{Sol}(N,v) = \sum_{\substack{T\subseteq N,\\ T\ni i}} \frac{1}{n\cdot\binom{n-1}{t-1}} \cdot \frac{1}{t} \cdot \sum_{k\in T}\Big[v(T) - v(T\backslash\{k\})\Big]. \tag{2}$$

According to the so-called "Equivalence Theorem" concerning the class of efficient, linear, and symmetric values, the following equivalent interpretations will be exploited throughout the remainder of this paper (cf. (Driessen and Radzik, 2002), (Driessen and Radzik, 2003), (Ruiz et al., 1998)).

**Theorem 1.** *The next four statements for a value $\psi$ on $\mathcal{G}_N$ are equivalent.*

*(i) $\psi$ verifies efficiency, linearity, and symmetry;*

*(ii) There exists a unique collection of constants $\{\rho_k\}_{k=1}^n$ with $\rho_n = 1$ such that, for every $n$-person game $\langle N,v\rangle$ with at least two players, the value payoff vector $(\psi_i(N,v))_{i\in N}$ is of the following form (cf. (Ruiz et al., 1998), Lemma 9, page 117): for all $i\in N$*

$$\psi_i(N,v) = \sum_{\substack{S\subseteq N,\\ S\ni i}} \frac{\rho_s}{s}\cdot v(S) - \sum_{\substack{S\subseteq N,\\ S\not\ni i}} \frac{\rho_s}{n-s}\cdot v(S); \tag{3}$$

*(iii) There exists a unique collection of constants $\mathcal{B} = \{b_k\}_{k=1}^n$ with $b_n = 1$ such that, for every $n$-person game $\langle N,v\rangle$ with at least two players, the value payoff vector $(\psi_i(N,v))_{i\in N}$ is of the following form (cf. (Driessen and Radzik, 2002), (Driessen and Radzik, 2003)): for all $i\in N$*

$$\psi_i(N,v) = \sum_{S\subseteq N\backslash\{i\}} \frac{1}{n\cdot\binom{n-1}{s}} \cdot \Big[b_{s+1}\cdot v(S\cup\{i\}) - b_s\cdot v(S)\Big]; \tag{4}$$

*(iv) There exists a unique collection of constants $\mathcal{B} = \{b_k\}_{k=1}^n$ with $b_n = 1$ such that $\psi(N,v) = \psi^{Sh}(N,\mathcal{B}v)$ for every $n$-person game $\langle N,v\rangle$ with at least two players. Here the $n$-person game $\langle N,\mathcal{B}v\rangle$, called $\mathcal{B}$-*scaled game*, is defined by $(\mathcal{B}v)(S) = b_s\cdot v(S)$ for all $S\subseteq N$, $S\neq\emptyset$.*

By straightforward computations, the reader may verify that the expression on the right hand of (3) agrees with the one on the right hand of (4) by choosing $b_k = \binom{n}{k}\cdot\rho_k$ for all $k = 1,2,\ldots,n$. Clearly, the expression on the right hand of (4) reduces to the Shapley value payoff (1) of player $i$ in the $n$-person game $\langle N,v\rangle$ itself (denoted by $\psi = \psi^{Sh}$) whenever $b_k = 1$ for all $k = 1,2,\ldots,n$, that is $\rho_k = \binom{n}{k}^{-1} = \frac{k!\cdot(n-k)!}{n!}$.

**Remark 1.** Let $\psi$ be an efficient, linear, and symmetric value on $\mathcal{G}_N$ of the form (4) with reference to the collection of constants $\mathcal{B} = \{b_k\}_{k=1}^n$ with $b_n = 1$. Fix two players $i\in N$, $j\in N$, $i\neq j$. Without going into details, by distinguishing between coalitions containing none, one or both players, straightforward calculations yield the next relationship about the difference between the value payoffs of both players:

$$\psi_j(N,v) - \psi_i(N,v) = \sum_{S \subseteq N \setminus \{i,j\}} \frac{\gamma(n-1,s)}{n} \cdot b_{s+1} \cdot \Big[ v(S \cup \{j\}) - v(S \cup \{i\}) \Big] \quad (5)$$

where $\gamma(n-1,s) = \frac{s! \cdot (n-2-s)!}{(n-1)!}$ for all $s = 0,1,2,\ldots,n-2$.

Generally speaking, in view of (1), the right hand of (4) equals the Shapley value payoff $Sh_i(N, \mathcal{B}v)$ of player $i$ in the $\mathcal{B}$-scaled game $\langle N, \mathcal{B}v \rangle$. In summary, the Equivalence Theorem 1 states that a value $\psi$ is efficient, linear, and symmetric if and only if the $\psi$-value of a game coincides with the Shapley value of the $\mathcal{B}$-scaled game (denoted by $\psi(N,v) = \psi^{Sh}(N, \mathcal{B}v)$). We call $\psi$ the per-capita Shapley value whenever $b_s = \frac{1}{s}$ for all $s = 1,2,\ldots,n-1$. It appears that the solidarity value $\psi^{Sol}(N,v)$ of the form (2) arises whenever $b_s = \frac{1}{s+1}$ for all $s = 1,2,\ldots,n-1$, that is $\rho_s = \frac{s! \cdot (n-s)!}{n! \cdot (s+1)}$. As a last, but appealing example, $\psi$ is called a *discount Shapley value* if there exists a discount factor $0 < \delta \leq 1$ such that the value payoff $\psi(N,v)$ is of the form (4) with reference to the collection of constants $b_s = \delta^{n-s}$ for all $s = 1,2,\ldots,n$, that is the larger the coalition size, the larger the discount factor $\delta^{n-s}$ of the worth $v(S)$ of any coalition $S$.

**Remark 2.** For future purposes, we list a number of combinatorial (in)equalities.

$$\sum_{k=t}^{n-1} \binom{k}{t} \cdot \frac{1}{k} = t^{-1} \cdot \binom{n-1}{t} \qquad \text{for all } t = 1,2,\ldots,n-1. \quad (6)$$

$$\sum_{k=t}^{n-1} \binom{k}{t} \cdot \frac{n}{k^2 \cdot (k+1)} \leq t^{-1} \cdot \binom{n-1}{t} \qquad \text{for all } t = 1,2,\ldots,n-1. \quad (7)$$

$$\sum_{k=t}^{n-1} \binom{k}{t} \cdot \frac{n}{k \cdot (n-k) \cdot (n-k+1)} \geq t^{-1} \cdot \binom{n-1}{t} \qquad \text{for all } t = 1,2,\ldots,n-1. \quad (8)$$

The proofs of both (6) and (7) proceed by induction on $n$ and are left to the reader. In fact, (8) applied to $t = 1$ reduces to an equality since it concerns a telescoping sum. For all $t \geq 2$, the expression on the left hand of (8) applied to $k = n-1$ already covers the single term on the right hand.

## 2. Socially Acceptable Values of Three Types

Any linear value $\psi$ on $\mathcal{G}_N$ is fully determined by the value payoffs of games that form a basis of $\mathcal{G}_N$. It is well-known that the collection of simple *unanimity games* $\mathcal{U} = \{\langle N, u_T \rangle \mid T \subseteq N, T \neq \emptyset\}$ forms a $(2^n - 1)$-dimensional basis of $\mathcal{G}_N$. Here the $\{0,1\}$-*unanimity game* $\langle N, u_T \rangle$ is defined by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise. In order to simplify forthcoming mathematical expressions, we prefer to deal with the adapted collection of non-simple unanimity games $\langle N, u_T^t \rangle$, $T \subseteq N$, $T \neq \emptyset$, given by $u_T^t(S) = t$ if $T \subseteq S$, and $u_T^t(S) = 0$ otherwise. Throughout this paper we aim to investigate the value payoffs for *productive players* of $T$ in comparison with *non-productive players* of $N \backslash T$.

**Definition 3.** A value $\psi$ on $\mathcal{G}_N$ is called *socially acceptable of type I* if the collection of value payoffs $(\psi_k(N,u_T^t))_{k\in N}$ of any adapted unanimity game $\langle N,u_T^t\rangle$ are such that, for all $T\subseteq N$, $T\neq\emptyset$, every productive player of $T$ receives at least as much as every non-productive player of $N\backslash T$, that is

$$\psi_i(N,u_T^t)\geq\psi_j(N,u_T^t)\geq 0 \qquad \text{for all } i\in T\text{, and all } j\in N\backslash T. \tag{9}$$

**Remark 3.** Since non-productive players $j\in N\backslash T$ are null players in the adapted unanimity game $\langle N,u_T^t\rangle$, their Shapley value payoff $\psi_j^{Sh}(N,u_T^t)=0$, whereas productive players $i\in T$ are treated as substitutes who allocate the worth $u_T(N)=t$ equally in that $\psi_i^{Sh}(N,u_T^t)=1$. Without going into details, it is possible to derive from (2) that the solidarity value payoffs for these adapted unanimity games are bounded such that for all $T\subseteq N$, $T\neq\emptyset$,

$$0<\psi_j^{Sol}(N,u_T^t)<\tfrac{t}{n} \qquad \text{if } j\in N\backslash T \qquad \textit{and} \qquad \tfrac{t}{n}<\psi_i^{Sol}(N,u_T^t)<1 \qquad \text{if } i\in T.$$

In words, the egalitarian value, the Shapley value and the solidarity value are socially acceptable of type $I$ in that these three linear values favour, in a weak or strict sense, the productive players to the non-productive players of any (adapted) unanimity game. We remark that non-linear values like the nucleolus (Schmeidler, 1969) and the $\tau$-value (Tijs, 1981) are also socially acceptable in that, for simple unanimity games, both of them coincide with the Shapley value. As already mentioned, we restrict ourselves to the class of efficient, linear, and symmetric values.

**Definition 4.** Let the collection $\mathcal{W}=\{\langle N,w_T\rangle \mid T\subseteq N, T\neq\emptyset\}$ of *coalition-size dependent unanimity games* be defined by $w_T(S)=\frac{s}{t}\cdot\binom{s}{t}^{-1}$ if $T\subseteq S$, and $w_T(S)=0$ otherwise.
A value $\psi$ on $\mathcal{G}_N$ is called *socially acceptable of type II* if the collection of value payoffs $(\psi_k(N,w_T))_{k\in N}$ of any coalition-size dependent unanimity game $\langle N,w_T\rangle$ are such that, for all $T\subseteq N$, $T\neq\emptyset$, every productive player of $T$ receives at least as much as every non-productive player of $N\backslash T$, that is

$$\psi_i(N,w_T)\geq\psi_j(N,w_T)\geq 0 \qquad \text{for all } i\in T\text{, and all } j\in N\backslash T. \tag{10}$$

**Remark 4.** The $(2^n-1)$-dimensional collection $\mathcal{W}$ of coalition-size dependent unanimity games forms a basis of $\mathcal{G}_N$ since, for any TU game $\langle N,v\rangle$, its game representation is given by $v=\sum_{T\subseteq N}\alpha_T^v\cdot w_T$, where $\alpha_T^v=v(T)$ if $|T|=1$ and $\alpha_T^v=v(T)-\sum_{k\in T}\frac{v(T\backslash\{k\})}{|T|-1}$ if $|T|\geq 2$. Notice that $w_T(T)=1$ and further, $w_T(N)<1$ if and only if $1<t<n$.
In this setting, a player $i$ is called a *scale dummy* in the game $\langle N,v\rangle$ if, for all $S\subseteq N$ with $|S|\geq 2$ containing $i$, it holds $\sum_{k\in S}v(S\backslash\{k\})=(|S|-1)\cdot v(S)$. Particularly, any player $j\in N\backslash T$ is a scale dummy in the coalition-size dependent unanimity game $\langle N,w_T\rangle$.

**Definition 5.** Let the collection $\mathcal{Z}=\{\langle N,z_T^t\rangle \mid T\subsetneqq N\}$ of *complementary unanimity games* be defined by $z_T^t(S)=t$ if $S\cap T=\emptyset$, $S\neq\emptyset$, and $z_T(S)=0$ otherwise. Note that $z_T^t(N)=0$ whenever $T\neq\emptyset$. In case $T=\emptyset$, then $z_\emptyset^0(S)=1$ for all $S\subseteq N$, $S\neq\emptyset$, and all players are substitutes in the unitary game $\langle N,z_\emptyset^0\rangle$.
A value $\psi$ on $\mathcal{G}_N$ is called *socially acceptable of type III* if the collection of value payoffs $(\psi_k(N,z_T^t))_{k\in N}$ of any complementary unanimity game $\langle N,z_T^t\rangle$ are such

that, for all $T \subsetneqq N$, $T \neq \emptyset$, every player of $T$ (considered as an enemy) receives at most as much as every player of $N\backslash T$ (considered as a friend), of which the payoff is bounded above by the ratio of the number of enemies to the number of players, that is

$$\psi_i(N, z_T^t) \leq \psi_j(N, z_T^t) \leq \frac{t}{n} \qquad \text{for all } i \in T, \text{ and all } j \in N\backslash T. \tag{11}$$

This paper is organized as follows. In Sections 3. and 4. we investigate and characterize the class of efficient, linear, and symmetric values that verify the social acceptability. In Section 3. it is shown that two additional properties called desirability and monotonicity are sufficient to guarantee social acceptability of type $I$ because of unitary conditions $0 \leq b_k \leq 1$ for all $k = 1, 2, \dots, n-1$. In Section 4. the main goal is, given an efficient, linear, and symmetric value $\psi$, to determine the exact conditions for social acceptability of each of three types, in terms of column sums of suitably chosen $n \times n$ lower triangular matrices $A^\psi$, $B^\psi$, and $C^\psi$ respectively. Section 5. contains some concluding remarks. Throughout this paper we deal with efficient, linear, and symmetric values in such a way that the value representation (4) with reference to the collection of constants $\mathcal{B} = \{b_k\}_{k=1}^n$ is the most appropriate tool.

## 3. A Sufficient Property for Social Acceptability of Values

To start with, we list the following two properties of values that turn out to be sufficient for social acceptability of type $I$.

**Definition 6.** Let $\psi$ be a value on $\mathcal{G}_N$.

(i) $\psi$ satisfies *desirability* if $\psi_j(N, v) \leq \psi_i(N, v)$ whenever player $j$ is *less desirable* than player $i$ in the game $\langle N, v\rangle$, that is $v(S \cup \{j\}) \leq v(S \cup \{i\})$ for all $S \subseteq N\backslash\{i, j\}$.

(ii) $\psi$ satisfies *monotonicity* if $\psi_i(N, v) \geq 0$ for all $i \in N$ and every *monotonic game* $\langle N, v\rangle$.

**Theorem 2.** *If a value $\psi$ on $\mathcal{G}_N$ verifies both desirability and monotonicity, then $\psi$ is socially acceptable of type $I$.*

*Proof.* Suppose a value $\psi$ on $\mathcal{G}_N$ verifies both desirability and monotonicity. Let $T \subseteq N$, $T \neq \emptyset$, $i \in T$, $j \in N\backslash T$. Since $u_T^t(S \cup \{j\}) = 0 \leq u_T^t(S \cup \{i\})$ for all $S \subseteq N\backslash\{i, j\}$, we obtain that player $j$ is less desirable than player $i$ in the adapted unanimity game $\langle N, u_T^t\rangle$. ¿From the desirability property of $\psi$, we derive $\psi_j(N, u_T^t) \leq \psi_i(N, u_T^t)$. Because the adapted unanimity game $\langle N, u_T^t\rangle$ is monotonic, it follows from the monotonicity property of $\psi$ that $\psi_k(N, u_T^t) \geq 0$ for all $k \in N$. So, $\psi$ is socially acceptable of type $I$. □

Neither the coalition-size dependent unanimity games $\langle N, w_T\rangle$ nor the complementary unanimity games $\langle N, z_T^t\rangle$ are monotonic games, so the latter proof does not apply in their context. Next we show that the two properties of desirability and monotonicity are equivalent to $[0, 1]$ boundedness for the underlying collection of constants associated with any efficient, linear, and symmetric value.

**Theorem 3.** *Let $\psi$ be an efficient, linear, and symmetric value on $\mathcal{G}_N$ of the form (4) with reference to a collection of constants $\mathcal{B} = \{b_k\}_{k=1}^n$ with $b_n = 1$.*

*(i) $\psi$ verifies desirability if and only if $b_k \geq 0$ for all $k = 1, 2, \ldots, n-1$.*
*(ii) $\psi$ verifies desirability and monotonicity if and only if $0 \leq b_k \leq 1$ for all $k = 1, 2, \ldots, n-1$.*

*Proof.* (i). If $b_k \geq 0$ for all $k = 1, 2, \ldots, n-1$, then the desirability property of $\psi$ follows immediately from (5). In order to prove the converse statement, suppose $\psi$ verifies desirability. Fix any two players $i \in N$, $j \in N$, and any coalition $T \subseteq N \backslash \{i, j\}$. Define the $n$-person game $\langle N, w\rangle$ by $w(T \cup \{i\}) = 1$ and $w(S) = 0$ for all $S \subseteq N$, $S \neq T \cup \{i\}$. On the one hand, from (5) we derive $\psi_j(N, w) - \psi_i(N, w) = -\frac{\gamma(n-1,t)}{n} \cdot b_{t+1}$. On the other, player $j$ is less desirable than player $i$ in the game $\langle N, w\rangle$, and so, the desirability property of $\psi$ implies $\psi_j(N, w) \leq \psi_i(N, w)$. We conclude that $b_{t+1} \geq 0$ for all $t = 0, 1, \ldots, n-2$. This proves the statement in part (i).

(ii) Suppose $\psi$ verifies monotonicity. Let $k = 1, 2, \ldots, n-1$ and fix player $i \in N$. Define the $n$-person game $\langle N, u\rangle$ by $u(S) = 1$ if either $i \in S$ and $s \geq k+1$ or $i \notin S$ and $s \geq k$, and $u(S) = 0$ otherwise. On the one hand, the game $\langle N, u\rangle$ is monotonic and so, the monotonicity property of $\psi$ implies $\psi_i(N, u) \geq 0$. On the other, from (4) we derive

$$\psi_i(N, u) = \sum_{S \subseteq N \backslash \{i\}} \frac{1}{n \cdot \binom{n-1}{s}} \cdot \Big[ b_{s+1} \cdot u(S \cup \{i\}) - b_s \cdot u(S) \Big]$$

$$= \sum_{\substack{S \subseteq N \backslash \{i\},\\ s \geq k}} \frac{b_{s+1} - b_s}{n \cdot \binom{n-1}{s}} = \sum_{s=k}^{n-1} \binom{n-1}{s} \cdot \frac{b_{s+1} - b_s}{n \cdot \binom{n-1}{s}} = \frac{b_n - b_k}{n}.$$

Recall $b_n = 1$. We obtain that $\psi_i(N, u) = \frac{1-b_k}{n} \geq 0$ and hence, $b_k \leq 1$ for all $k = 1, 2, \ldots, n-1$. The technical proof of the converse statement is postponed till the end of Section 5.. □

Unfortunately, in the setting of efficient, linear, and symmetric values, it turns out that both the desirability and monotonicity conditions are not necessary for the value to be socially acceptable. That is, the class of socially acceptable values strictly contains the class of values verifying the desirability and monotonicity properties. In the next section we provide a full characterization of socially acceptable values of each of three types.

## 4. Characterizations of Socially Acceptable Values

In the setting of values satisfying the substitution property, it suffices to distinguish two types of players, called productive players (members of a certain coalition $T$) and non-productive players (nonmembers of $T$), respectively. For any efficient value $\psi$ on $\mathcal{G}_N$ satisfying the substitution property, the efficiency condition applied to the adapted unanimity game $\langle N, u_T^t\rangle$ reduces to the equality $t \cdot \psi_i(N, u_T^t) + (n-t) \cdot \psi_j(N, u_T^t) = t$ for all $t = 1, 2, \ldots, n$, for all $i \in T$, $j \in N \backslash T$. Consequently, by (9), an efficient and symmetric value $\psi$ on $\mathcal{G}_N$ is socially acceptable of type $I$ if and only if

$$\frac{t}{n} \leq \psi_i(N, u_T^t) \leq 1 \qquad \text{for all } T \subseteq N,\ T \neq \emptyset, \text{ all } i \in T. \tag{12}$$

**Theorem 4.** *Let $\psi$ be an efficient, linear, and symmetric value on $\mathcal{G}_N$ of the form (4) with reference to the collection of constants $\mathcal{B} = \{b_k\}_{k=1}^n$ with $b_n = 1$. With the* *value $\psi$, there is associated the $n \times n$ lower triangular matrix $A^\psi$ of which the rows are indexed by the coalition size $k$, and the columns by the number $t$ of productive players in the adapted unanimity games, such that each matrix entry $[A^\psi]_{k,t}$ is given by $[A^\psi]_{k,t} = \binom{k}{t} \cdot \frac{b_k}{k}$ if $t \leq k \leq n$, and $[A^\psi]_{k,t} = 0$ otherwise. Then the value $\psi$ is socially acceptable of type I if and only if the sum of the entries in each column (except for the entry in the last row $n$) of $A^\psi$ is not less than zero, and not more than $t^{-1} \cdot \binom{n-1}{t}$ with reference to its $t$-th column. That is,*

$$0 \leq \sum_{k=t}^{n-1} [A^\psi]_{k,t} \leq t^{-1} \cdot \binom{n-1}{t} \qquad \text{for all } t = 1, 2, \ldots, n-1. \tag{13}$$

*Proof.* Fix $T \subsetneq N$, $T \neq \emptyset$, and $i \in T$. Then $u_T^t(S) = 0$ for all $S \subseteq N \backslash \{i\}$. From (4) and some combinatorial calculations, we derive

$$\begin{aligned}
\psi_i(N, u_T^t) &= \sum_{S \subseteq N\backslash\{i\}} \frac{1}{n \cdot \binom{n-1}{s}} \cdot \Big[ b_{s+1} \cdot u_T^t(S \cup \{i\}) - b_s \cdot u_T^t(S) \Big] \\
&= \sum_{S \subseteq N\backslash\{i\}} \frac{b_{s+1} \cdot u_T^t(S \cup \{i\})}{n \cdot \binom{n-1}{s}} = \sum_{T\backslash\{i\} \subseteq S \subseteq N\backslash\{i\}} \frac{t \cdot b_{s+1}}{n \cdot \binom{n-1}{s}} \\
&= \sum_{s=t-1}^{n-1} \binom{n-t}{s-t+1} \cdot \frac{t \cdot b_{s+1}}{n \cdot \binom{n-1}{s}} = \sum_{k=t}^{n} \binom{n-t}{k-t} \cdot \frac{t \cdot b_k}{n \cdot \binom{n-1}{k-1}} \\
&= t \cdot \sum_{k=t}^{n} \frac{\binom{k}{t}}{\binom{n}{t}} \cdot \frac{b_k}{k} = t \cdot \binom{n}{t}^{-1} \cdot \sum_{k=t}^{n} [A^\psi]_{k,t}
\end{aligned}$$

From this we conclude that (12) holds if and only if $\frac{t}{n} \leq t \cdot \binom{n}{t}^{-1} \cdot \sum_{k=t}^{n} [A^\psi]_{k,t} \leq 1$ if and only if $0 \leq t \cdot \binom{n}{t}^{-1} \cdot \sum_{k=t}^{n-1} [A^\psi]_{k,t} \leq 1 - \frac{t}{n}$ or equivalently, (13) holds. □

Each non-zero entry of the $k$-th row of matrix $A^\psi$ is proportional to the average expression $\frac{b_k}{k}$. Clearly, the egalitarian value is socially acceptable of type $I$ since it arises as one extreme case whenever the whole matrix $A^\psi$, except for its bottom row, equals zero (or equivalently, $b_k = 0$ for all $k = 1, 2, \ldots, n-1$). The Shapley value $\psi^{Sh}$, associated with the unitary collection $b_k = 1$ for all $k = 1, 2, \ldots, n$, arises as the second extreme case in that the inequalities on the right hand of (13) are met as combinatorial equalities (to be verified by induction on the number $n$ of players). Any linear combination $\psi^\beta = (1-\beta) \cdot \psi^{EG} + \beta \cdot \psi^{Sh}$ is of the form (4) with reference to a constant collection $b_k = \beta$ for all $k = 1, 2, \ldots, n-1$, and such value $\psi^\beta$ is socially acceptable of type $I$ if and only if $0 \leq \beta \leq 1$. Moreover, the solidarity value $\psi^{Sol}$ is socially acceptable of type $I$ since its associated collection $b_k = \frac{1}{k+1} \leq 1$ for all $k = 1, 2, \ldots, n-1$.
Generally speaking, (13) applied to $t = n-1$ and $t = 1$ respectively require $0 \leq b_{n-1} \leq 1$ and $0 \leq \sum_{k=1}^{n-1} b_k \leq n-1$. In case $n = 3$, the social acceptability condition (13) reduces to both $0 \leq b_2 \leq 1$ and $0 \leq b_1 + b_2 \leq 2$, whereas, in case $n = 4$, (13) reduces to $0 \leq b_2 \leq 1$, $0 \leq b_1 + b_2 + b_3 \leq 3$, together with $b_2 + 2 \cdot b_3 \leq 3$.

Further, observe that a $n \times n$ lower triangular matrix $A = [A]_{k,t}$ induces an efficient, linear, and symmetric value on $\mathcal{G}_N$ of the form (4) with reference to a collection of constants $\{b_k\}_{k=1}^n$ with $b_n = 1$ provided that, for all $1 \le k \le n-1$, $\binom{k}{t}^{-1} \cdot [A]_{k,t}$ is the same for all $1 \le t \le k$.

In the context of values satisfying the substitution property, the efficiency condition applied to the coalition-size dependent unanimity game $\langle N, w_T \rangle$ reduces to the equality $t \cdot \psi_i(N, w_T) + (n-t) \cdot \psi_j(N, w_T) = \frac{n}{t} \cdot \binom{n}{t}^{-1}$ for all $t = 1, 2, \ldots, n$, for all $i \in T$, $j \in N \backslash T$. Thus, by (10), an efficient and symmetric value $\psi$ on $\mathcal{G}_N$ is socially acceptable of type $II$ if and only if

$$\frac{1}{t} \cdot \binom{n}{t}^{-1} \le \psi_i(N, w_T) \le \frac{n}{t^2} \cdot \binom{n}{t}^{-1} \qquad \text{for all } T \subseteq N,\ T \ne \emptyset, \text{ all } i \in T. \tag{14}$$

**Theorem 5.** *Let $\psi$ be an efficient, linear, and symmetric value on $\mathcal{G}_N$ of the form (4) with reference to the collection of constants $\mathcal{B} = \{b_k\}_{k=1}^n$ with $b_n = 1$. With the value $\psi$, there is associated the $n \times n$ lower triangular matrix $B^\psi$ of which the rows are indexed by the coalition size $k$, and the columns by the number $t$ of productive players in the coalition-size dependent unanimity games, such that each matrix entry $[B^\psi]_{k,t}$ is given by $[B^\psi]_{k,t} = b_k$ if $t \le k \le n$, and $[B^\psi]_{k,t} = 0$ otherwise. Then the value $\psi$ is socially acceptable of type $II$ if and only if the sum of the entries in each column (except for the entry in the last row $n$) of $B^\psi$ is not less than zero, and not more than $\frac{n-t}{t}$ with reference to its $t$-th column. That is,*

$$0 \le \sum_{k=t}^{n-1} [B^\psi]_{k,t} \le \frac{n-t}{t} \qquad \textit{for all } t = 1, 2, \ldots, n-1. \tag{15}$$

*Proof.* The same proof technique applies as before by modifying the choice of the basis of $\mathcal{G}_N$. Fix $T \subsetneq N$, $T \ne \emptyset$, and $i \in T$. Then $w_T(S) = 0$ for all $S \subseteq N \backslash \{i\}$. In the current framework, from (4) and some combinatorial calculations, we derive

$$\begin{aligned}
\psi_i(N, w_T) &= \sum_{S \subseteq N \backslash \{i\}} \frac{b_{s+1} \cdot w_T(S \cup \{i\})}{n \cdot \binom{n-1}{s}} = \sum_{T \backslash \{i\} \subseteq S \subseteq N \backslash \{i\}} \frac{s+1}{t} \cdot \binom{s+1}{t}^{-1} \cdot \frac{b_{s+1}}{n \cdot \binom{n-1}{s}} \\
&= \sum_{s=t-1}^{n-1} \binom{n-t}{s-t+1} \cdot \frac{s+1}{t} \cdot \binom{s+1}{t}^{-1} \cdot \frac{b_{s+1}}{n \cdot \binom{n-1}{s}} = \sum_{k=t}^{n} \binom{n-t}{k-t} \cdot \frac{k}{t} \cdot \binom{k}{t}^{-1} \cdot \frac{b_k}{n \cdot \binom{n-1}{k-1}} \\
&= t^{-1} \cdot \binom{n}{t}^{-1} \cdot \sum_{k=t}^{n} b_k = t^{-1} \cdot \binom{n}{t}^{-1} \cdot \sum_{k=t}^{n} [B^\psi]_{k,t}
\end{aligned}$$

From this we conclude that (14) holds if and only if $t^{-1} \cdot \binom{n}{t}^{-1} \le t^{-1} \cdot \binom{n}{t}^{-1} \cdot \sum_{k=t}^{n} [B^\psi]_{k,t} \le \frac{n}{t^2} \cdot \binom{n}{t}^{-1}$ if and only if $1 \le \sum_{k=t}^{n} [B^\psi]_{k,t} \le \frac{n}{t}$ or equivalently, (15) holds. □

Clearly, the egalitarian value is socially acceptable of type $II$, whereas the Shapley value, associated with the unitary collection, fails to be of type $II$. The extreme case in that the inequalities of (15) are met as combinatorial equalities happens for the collection of constants $b_k = \frac{n}{k \cdot (k+1)}$ for all $k = 1, 2, \ldots, n-1$, because of its telescoping sum. Consequently, the solidarity value $\psi^{Sol}$ is socially acceptable of type $II$

since its associated collection $b_k = \frac{1}{k+1} \leq \frac{n}{k\cdot(k+1)}$ for all $k = 1, 2, \ldots, n-1$. Due to the development of the theory about social acceptability of type $II$, we end up with the introduction of an appealing value on $\mathcal{G}_N$ transforming an $n$-person game $\langle N, v\rangle$ into its per-capita game $\langle N, v_{pc}\rangle$, applying the solidarity value and finally, repairing efficiency in a multiplicative fashion. In formula, $\psi(N, v) = n\cdot\psi^{Sol}(N, v_{pc})$ where the characteristic function of the per-capita game $\langle N, v_{pc}\rangle$ is defined by $v_{pc}(S) = \frac{v(S)}{|S|}$ for all $S \subseteq N$, $S \neq \emptyset$.

In the framework of values satisfying the substitution property, the efficiency condition applied to the complementary unanimity game $\langle N, z^t_T\rangle$ reduces to the equality $t \cdot \psi_i(N, z^t_T) + (n-t) \cdot \psi_j(N, z^t_T) = 0$ for all $t = 1, 2, \ldots, n-1$, for all $i \in T$, $j \in N\backslash T$. Thus, by (11), an efficient and symmetric value $\psi$ on $\mathcal{G}_N$ is socially acceptable of type $III$ if and only if

$$\frac{t}{n} - 1 \leq \psi_i(N, z^t_T) \leq 0 \qquad \text{for all } T \subsetneq N,\ T \neq \emptyset, \text{ all } i \in T. \tag{16}$$

**Theorem 6.** *Let $\psi$ be an efficient, linear, and symmetric value on $\mathcal{G}_N$ of the form (4) with reference to the collection of constants $\mathcal{B} = \{b_k\}_{k=1}^n$ with $b_n = 1$. With the value $\psi$, there is associated the $n \times n$ lower triangular matrix $C^\psi$ of which the rows are indexed by the coalition size $k$, and the columns by the number $t$ of productive players in the complementary unanimity games, such that each matrix entry $[C^\psi]_{k,t}$ is given by $[C^\psi]_{k,t} = \binom{k}{t} \cdot \frac{b_{n-k}}{k}$ if $t \leq k \leq n-1$, and $[C^\psi]_{k,t} = 0$ otherwise. Then the value $\psi$ is socially acceptable of type $III$ if and only if the sum of the entries in each column (except for the entry in the last row $n$) of $C^\psi$ is not less than zero, and not more than $t^{-1} \cdot \binom{n-1}{t}$ with reference to its $t$-th column. That is,*

$$0 \leq \sum_{k=t}^{n-1} [C^\psi]_{k,t} \leq t^{-1} \cdot \binom{n-1}{t} \qquad \text{for all } t = 1, 2, \ldots, n-1. \tag{17}$$

*Proof.* The same proof technique applies as before by modifying the choice of the basis of $\mathcal{G}_N$. Fix $T \subsetneq N$, $T \neq \emptyset$, and $i \in T$. Then $z^t_T(S \cup \{i\}) = 0$ for all $S \subseteq N\backslash\{i\}$. In the current framework, from (4) and some combinatorial calculations, we derive

$$\begin{aligned}\psi_i(N, z^t_T) &= \sum_{\substack{S \subseteq N\backslash\{i\},\\ S \neq \emptyset}} \frac{-b_s \cdot z^t_T(S)}{n\cdot\binom{n-1}{s}} = -\sum_{\substack{\emptyset \neq S \subseteq N\backslash\{i\},\\ S \cap T = \emptyset}} \frac{t\cdot b_s}{n\cdot\binom{n-1}{s}} = -\sum_{\substack{S \subseteq N\backslash T,\\ S \neq \emptyset}} \frac{t\cdot b_s}{n\cdot\binom{n-1}{s}} \\ &= -\sum_{s=1}^{n-t} \binom{n-t}{s} \cdot \frac{t\cdot b_s}{n\cdot\binom{n-1}{s}} = -\sum_{s=1}^{n-t} \frac{\binom{n-s}{t}}{\binom{n}{t}} \cdot \frac{t\cdot b_s}{n-s} \\ &= -t \cdot \sum_{k=t}^{n-1} \frac{\binom{k}{t}}{\binom{n}{t}} \frac{b_{n-k}}{k} = -t \cdot \binom{n}{t}^{-1} \cdot \sum_{k=t}^{n-1} [C^\psi]_{k,t}\end{aligned}$$

From this we conclude that (16) holds if and only if $\frac{t}{n} - 1 \leq -t\cdot\binom{n}{t}^{-1} \cdot \sum_{k=t}^{n-1} [C^\psi]_{k,t} \leq 0$ if and only if $0 \leq \sum_{k=t}^{n-1} [C^\psi]_{k,t} \leq t^{-1} \cdot \binom{n-1}{t}$ or equivalently, (17) holds. □

**Remark 5.** The well-known notion of the *dual game* $\langle N, v^*\rangle$ of a TU game $\langle N, v\rangle$ is defined by $v^*(S) = v(N) - v(N\backslash S)$ for all $S \subseteq N$. Particularly, $v^*(N) = v(N)$ as well as $(v^*)^*(S) = v(S)$ for all $S \subseteq N$. The interrelationship between any adapted unanimity game $\langle N, u_T^t\rangle$ and any complementary unanimity game $\langle N, z_T^t\rangle$ is given by $z_T^t(S) = t - (u_T^t)^*(S)$ for all $S \subseteq N$, $S \neq \emptyset$. Due to its efficiency, linearity, symmetry, and self-duality (expressing $\psi(N, v^*) = \psi(N, v)$ for all games $\langle N, v\rangle$), the Shapley values of both types of games are related by $\psi_i^{Sh}(N, z_T^t) = \frac{t}{n} - \psi_i^{Sh}(N, u_T^t)$ for all $i \in N$. Thus, $\psi_i^{Sh}(N, z_T^t) = \frac{t}{n} - 1$ if $i \in T$, whereas $\psi_j^{Sh}(N, z_T^t) = \frac{t}{n}$ if $j \in N\backslash T$.
Both the egalitarian value and the Shapley value are socially acceptable of type $III$ as the two extreme cases in that the inequalities in (17) are met as equalities. Notice the similarity of both conditions (13) and (17), while the underlying matrix entries $[A^\psi]_{k,t}$ and $[C^\psi]_{k,t}$ only differ in the usual or reversed order of numbering concerning the collection of fundamental constants $b_k$, $k = 1, 2, \ldots, n-1$. Finally, we remark that each adapted unanimity game $\langle N, u_T^t\rangle$ is a so-called convex game, whereas each complementary unanimity game $\langle N, z_T^t\rangle$ is a so-called 1-concave game (Driessen et al., 2010).

## 5. Concluding Remarks

The social acceptability properties for the egalitarian, Shapley, and solidarity values may be summarized as follows.

| **Value** | $\psi$ | $b_k$ | **Type** $I$ | **Type** $II$ | **Type** $III$ |
|---|---|---|---|---|---|
| Egalitarian value | $\psi^{EG}$ | $b_k = 0$ | Yes | Yes | Yes |
| Shapley value | $\psi^{Sh}$ | $b_k = 1$ | Yes | No | Yes |
| Solidarity value | $\psi^{Sol}$ | $b_k = \frac{1}{k+1}$ | Yes | Yes | Yes |
| New value | $\psi$ | $b_k = \frac{n}{k\cdot(k+1)}$ | Yes | Yes | No |

The efficient, linear, and symmetric value $\psi$, associated with the collection of constants $b_k = \frac{n}{k\cdot(k+1)}$ is indeed of type $I$ since (13) is met because of (7), to be of type $II$ too since (15) is met as an equality because of a telescoping sum, but this value fails to be of type $III$ since (17) is not met because of (8). In fact, the latter value satisfies the strict reversed inequalities. Remarkably, the solidarity value is socially acceptable of each of these three types.

**Corollary 1.** Let $\psi$ be an efficient, linear, and symmetric value on $\mathcal{G}_N$ of the form (4) with reference to a collection of constants $\mathcal{B} = \{b_k\}_{k=1}^n$ with $b_n = 1$. For every $t = 1, 2, \ldots, n$, define the payoff $p_t^\psi$ to productive players in the $\{0,1\}$-unanimity game $\langle N, u_T\rangle$ by

$$p_t^\psi = \binom{n}{t}^{-1} \cdot \sum_{k=t}^{n} [A^\psi]_{k,t} \qquad \text{where} \qquad [A^\psi]_{k,t} = \binom{k}{t} \cdot \frac{b_k}{k} \qquad \text{for all } t \leq k \leq n$$

If $\psi$ verifies desirability and monotonicity, then $\psi$ is socially acceptable of type $I$ such that the payoffs $(p_t^\psi)_{t=1}^n$ form a decreasing sequence (the more productive players, the less their payoffs), that is

$$\tfrac{1}{n} = p_n^\psi \le p_{n-1}^\psi \le p_{n-2}^\psi \le \ldots\ldots \le p_1^\psi \le 1. \tag{18}$$

*Proof.* Let $t = 1, 2, \ldots, n-1$. Due to some combinatorial calculations, we derive

$$\begin{aligned} p_t^\psi - p_{t+1}^\psi &= \tfrac{[A^\psi]_{t,t}}{\binom{n}{t}} + \sum_{k=t+1}^{n} \left[ \tfrac{[A^\psi]_{k,t}}{\binom{n}{t}} - \tfrac{[A^\psi]_{k,t+1}}{\binom{n}{t+1}} \right] \\ &= \tfrac{[A^\psi]_{t,t}}{\binom{n}{t}} + \sum_{k=t+1}^{n} \left[ \frac{\binom{k}{t}}{\binom{n}{t}} - \frac{\binom{k}{t+1}}{\binom{n}{t+1}} \right] \cdot \frac{b_k}{k} \\ &= \tfrac{[A^\psi]_{t,t}}{\binom{n}{t}} + \sum_{k=t+1}^{n} \frac{n-k}{n-t} \cdot \frac{\binom{k}{t}}{\binom{n}{t}} \cdot \frac{b_k}{k} = \sum_{k=t}^{n} \frac{n-k}{n-t} \cdot \tfrac{[A^\psi]_{k,t}}{\binom{n}{t}} \end{aligned}$$

By Theorem 3 (i), $b_k \ge 0$ for all $k = 1, 2, \ldots, n-1$, and so, $[A^\psi]_{k,t} \ge 0$ for all $t \le k \le n$. It follows immediately that $p_t^\psi \ge p_{t+1}^\psi$ for all $t = 1, 2, \ldots, n-1$. So, (18) holds. □

**Remark 6.** In (Hernández-Lamoneda et al., 2007) the basic representation theory of the group of permutations $\mathcal{S}_n$ has been applied to cooperative $n$-person game theory. Through a specific direct sum decomposition of both the payoff space $\mathbb{R}^n$ and the space $\mathcal{G}_N$ of $n$-person games, it is shown that an efficient, linear, and symmetric value $\psi$ on $\mathcal{G}_N$ is of the following form (cf. Hernández-Lamoneda et al., 2007, Theorem 2, page 411): for all $i \in N$

$$\psi_i(N, v) = \frac{v(N)}{n} + \sum_{\substack{S \subsetneq N, \\ S \ni i}} (n-s) \cdot \Big[ \beta_s \cdot v(S) - \beta_{n-s} \cdot v(N \backslash S) \Big]. \tag{19}$$

Clearly, the above expression agrees with the one on the right hand of (3) by choosing $\beta_k = \frac{\rho_k}{k \cdot (n-k)}$ for all $k = 1, 2, \ldots, n-1$, and hence, (19) and (4) are equivalent by choosing $\beta_k = \frac{b_k}{k \cdot (n-k) \cdot \binom{n}{k}}$ for all $k = 1, 2, \ldots, n-1$. According to (Hernández-Lamoneda et al., 2007, Corollary 5, page 419), an efficient, linear, and symmetric value $\psi$ verifies self-duality (i.e., $\psi(N, v^*) = \psi(N, v)$ for all games $\langle N, v \rangle$) if and only if $\beta_k = \beta_{n-k}$ for all $k = 1, 2, \ldots, , n-1$. The latter condition is equivalent to $b_k = b_{n-k}$ or $[A^\psi]_{k,t} = [C^\psi]_{k,t}$, i.e., coincidence of the two matrices $A^\psi$ and $C^\psi$.

**Remark 7.** In (Joosten, 1994) it is shown that a value is efficient, symmetric, additive, and $\beta$-egalitarian (for some $\beta \in \mathbb{R}$) if and only if the value is the convex combination of the egalitarian value and the Shapley value in that $\psi(N, v) = \beta \cdot \psi^{EG}(N, v) + (1-\beta) \cdot \psi^{Sh}(N, v)$ for all games $\langle N, v \rangle$. Here a value $\psi$ on $\mathcal{G}_N$ is called *$\beta$-egalitarian* if

$$\psi_i(N, v) = \frac{\beta}{n} \cdot \sum_{j \in N} \psi_j(N, v) \qquad \text{for every null player } i \text{ in the game } \langle N, v \rangle.$$

A similar result is shown in (Nowak and Radzik, 1996) concerning an axiomatization of the class of values that are convex combinations of the Shapley value and the solidarity value. Clearly, each value of this class is socially acceptable. In (Dragan et al., 1996) collinearity between the Shapley value and various types of egalitarian values has been treated for a class of zero-normalized games called proportional average worth games.

**Remark 8.** We conclude this paper with the proof of the "if" part of Theorem 3(ii). Let $\psi$ be an efficient, linear, and symmetric value on $\mathcal{G}_N$ of the form (4) with reference to a collection of constants $\mathcal{B} = \{b_s\}_{s=1}^n$ with $b_n = 1$ and $0 \leq b_s \leq 1$ for all $s = 1, 2, \ldots, n-1$ as well. By Theorem 3(i), $\psi$ verifies desirability. It remains to prove that $\psi$ verifies monotonicity too. Let $\langle N, v\rangle$ be a monotonic $n$-person game and $i \in N$. We show $\psi_i(N, v) \geq 0$. Write $b_0 = 0$ and as usual, $\gamma(n, s) = \frac{s!\cdot(n-1-s)!}{n!}$ for all $s = 0, 1, \ldots, n-1$. At this stage, we put forward our claim that the player's payoff satisfies, for all $k = 0, 1, \ldots, n-2$,

$$\psi_i(N, v) \geq f_k(\psi, v, \{i\}) + g_{k+1}(\psi, v, \{i\}) \quad \text{where for all } \ell = 0, 1, \ldots, n-1 \tag{20}$$

$$f_\ell(\psi, v, \{i\}) = \sum_{\substack{S\subseteq N\setminus\{i\},\\ s\leq \ell}} \gamma(n, s)\cdot [b_{s+1} - b_s]\cdot v(S\cup\{i\}) \tag{21}$$

$$g_\ell(\psi, v, \{i\}) = \gamma(n, \ell)\cdot [b_n - b_\ell]\cdot \sum_{\substack{S\subseteq N\setminus\{i\},\\ s=\ell}} v(S\cup\{i\}) \tag{22}$$

The proof of the claim (20) proceeds by backwards induction on $k$, $k = 0, 1, \ldots, n-2$. For $k = n-2$, the claim follows immediately from the representation (4) for $\psi$ by observing that $b_n = 1$ and $b_s \cdot v(S) \leq b_s \cdot v(S\cup\{i\})$ for all $S \subseteq N\setminus\{i\}$ due to the monotonicity of the game $\langle N, v\rangle$ together with $b_s \geq 0$ for all $s = 0, 1, \ldots, n-1$. Suppose that the claim holds for some $k$, $k \in \{1, 2, \ldots, n-2\}$. We verify the claim for $k-1$. For that purpose, note that $s\cdot v(S\cup\{i\}) \geq \sum_{j\in S} v((S\cup\{i\})\setminus\{j\})$ for all $S \subseteq N\setminus\{i\}$ by the monotonicity of the game $\langle N, v\rangle$. By summing up over all coalitions of size $k+1$, not containing player $i$, we obtain

$$\sum_{\substack{S\subseteq N\setminus\{i\},\\ s=k+1}} v(S\cup\{i\}) \geq \frac{1}{k+1}\sum_{\substack{S\subseteq N\setminus\{i\},\\ s=k+1}}\sum_{j\in S} v((S\cup\{i\})\setminus\{j\}) = \frac{n-1-k}{k+1}\sum_{\substack{T\subseteq N\setminus\{i\},\\ t=k}} v(T\cup\{i\}).$$

where the last equality is due to the combinatorial argument that any $T \subseteq N\setminus\{i\}$ of size $k$ arises from $n-k-1$ coalitions $S$ of the form $T\cup\{j\}$, where $j \in N\setminus T$, $j \neq i$. ¿From the latter inequality, together with (22) and $b_{k+1} \leq 1 = b_n$, we derive the following:

$$\begin{aligned} g_{k+1}(\psi, v, \{i\}) &= \gamma(n, k+1)\cdot [b_n - b_{k+1}]\cdot \sum_{\substack{S\subseteq N\setminus\{i\},\\ s=k+1}} v(S\cup\{i\}) \\ &\geq \gamma(n, k+1)\cdot [b_n - b_{k+1}]\cdot \frac{n-1-k}{k+1}\sum_{\substack{T\subseteq N\setminus\{i\},\\ t=k}} v(T\cup\{i\}) \\ &= \gamma(n, k)\cdot [b_n - b_{k+1}]\sum_{\substack{S\subseteq N\setminus\{i\},\\ s=k}} v(S\cup\{i\}) \end{aligned}$$

where the latter equality holds because of $\gamma(n, k+1)\cdot \frac{n-1-k}{k+1} = \gamma(n, k)$. ¿From the latter inequality, together with the induction hypothesis (20), (21), (22) respectively, it follows that

$$\begin{aligned}\psi_i(N,v) &\geq f_k(\psi,v,\{i\}) + g_{k+1}(\psi,v,\{i\}) \\ &= f_{k-1}(\psi,v,\{i\}) + \gamma(n,k)\cdot[b_{k+1}-b_k]\cdot\sum_{\substack{S\subseteq N\setminus\{i\},\\ s=k}} v(S\cup\{i\}) + g_{k+1}(\psi,v,\{i\}) \\ &\geq f_{k-1}(\psi,v,\{i\}) + \gamma(n,k)\cdot[b_n-b_k]\cdot\sum_{\substack{S\subseteq N\setminus\{i\},\\ s=k}} v(S\cup\{i\}) \\ &= f_{k-1}(\psi,v,\{i\}) + g_k(\psi,v,\{i\})\end{aligned}$$

This completes the backwards inductive proof of the claim (20). For $k=0$ the claim yields

$$\begin{aligned}\psi_i(N,v) &\geq f_0(\psi,v,\{i\}) + g_1(\psi,v,\{i\}) \\ &= \gamma(n,0)\cdot[b_1-b_0]\cdot v(\{i\}) + \gamma(n,1)\cdot[b_n-b_1]\cdot\sum_{j\in N\setminus\{i\}} v(\{i,j\}) \\ &= \frac{b_1}{n}\cdot v(\{i\}) + \frac{1-b_1}{n\cdot(n-1)}\cdot\sum_{j\in N\setminus\{i\}} v(\{i,j\}).\end{aligned}$$

Note that $v(S)\geq 0$ for all $S\subseteq N$ by monotonicity of $\langle N,v\rangle$. Together with $0\leq b_1\leq 1$, the latter inequality yields $\psi_i(N,v)\geq 0$. This completes the proof of Theorem 3(ii). □

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