P. Dostál, T. Mach, Progressive projection and log-optimal investment in the frictionless market, *Theory Stoch. Process.*, 2020, Volume 25, Issue 1, 37–77

Theory of Stochastic Processes
Vol. 25 (41), no. 1, 2020, pp. 37–77

P. DOSTÁL AND T. MACH

PROGRESSIVE PROJECTION AND LOG-OPTIMAL INVESTMENT IN THE FRICTIONLESS MARKET

In this paper, we introduce notion of progressive projection, closely related to the extended predictable projection. This notion is flexible enough to help us treat the problem of log-optimal investment without transaction costs almost exhaustively in case when the rate of return is not observed. We prove some results saying that the semimartingale property of a continuous process is preserved when changing the filtration to the one generated by the process under very general conditions. We also had to introduce a very useful and flexible notion of so called enriched filtration.

1. Introduction

The purpose of the paper is (1) to motivate the problem of robust filtering by the notion of *log-optimal investment* and (2) to prove some technical tools that will help us obtain further results in the forthcoming papers. This includes Proposition 3.22 which is the main technical result of this paper. We introduce the notion of *progressive projection*, closely related to the notion of *predictable projection*, which helps us remove classical L_2-assumptions in the problem of filtering.

Proposition 3.22 is essentially an answer to question (Q) described below, after introducing the corresponding notions. Let $\mathcal{F}$ be a given filtration and let $\mathcal{X}(\mathcal{F})$ stand here for the set of all continuous $\mathcal{F}$-semimartingales, say X, with the trend part, say $\mathcal{V}^X$, having locally absolutely continuous trajectories. Let us also mention that $\mathbb{P}$ will stand for the underlying probability measure here and that 1_A stands for the indicator function of a set A in general. The question here is the following.

(Q) If $X \in \mathcal{X}(\mathcal{F})$ and if $\mathcal{G}$ is a subfiltration of $\mathcal{F}$, under which conditions $X \in \mathcal{X}(\mathcal{G})$?

The answer given by Proposition 3.22 is essentially the following. It is enough to require that (i) X is $\mathcal{G}$-adapted, (ii) the following measure on $\mathcal{G}$-progressive sets is σ-finite $\mathbb{G} \mapsto \int\int 1_{\mathbb{G}}(t,\omega)\,\mathrm{d}\mathcal{V}_t^X \mathrm{d}\mathbb{P}(\omega)$, and (iii) the filtrations $\mathcal{F}, \mathcal{G}$ are enriched, which is satisfied for example if $\mathcal{F}_0, \mathcal{G}_0$ contain all null sets from the underlying probability space. The condition (ii) is equivalent to the condition that $|\,\mathrm{d}\mathcal{V}_t^X/\mathrm{d}t\,|$ has a $\mathcal{G}$-predictable projection finite almost surely at almost every $t \in [0,\infty)$, see Lemma 3.33.

We believe that, in the financial markets, it is essentially necessary to put together the process of estimating parameters and the process of making decisions reflected in the applied strategy, and it can be done only by using filtering techniques. In the frictionless financial market, the investor observes the market prices and hence also their volatilities, but one thing, which is crucial, is not observed. It is the rate of return. To observe the rate of return is equivalent to the observation of so called *log-optimal proportion.* If the investor has such information, he/she can use something which we call *log-optimal strategy* and which corresponds to what is in [18] called the *numéraire portfolio.*This strategy simply keeps the vector of proportions of wealth invested in each risky asset equal to the log-optimal proportion and maximizes the long run growth rate of the investor's wealth process in the frictionless market. For details see Definition 3.12 and

2010 *Mathematics Subject Classification.* 60H30, 60G44, 91G80.
Key words and phrases. log-optimal investment, progressive projection, filtering.

Proposition 3.13. Also, if the investor faces small transaction fees, in order to be able to apply some kind of an *almost log-optimal strategy*, one needs to know the log-optimal proportion with respect to the filtration generated by the market (here shown to be the projection of the log-optimal proportion to the corresponding filtration) and also its dynamics which is not covered in this paper due its current length.

For the interested reader we can outline what type of results we obtain in the forthcoming papers based on this one. Here, we consider the most simple example just for illustration. Imagine the Black-Scholes (BS) model of a geometric Brownian motion of the stock market price with zero interest rate, with unknown rate of return μ and volatility $\sigma > 0$. Then the log-optimal proportion $\theta = \sigma^{-2}\mu$ is also the unknown parameter which we are interested in. Let us model it as a random variable independent of the standard Brownian motion driving the (BS) model and let us assume, for simplicity, that it attains values in a countable set. Let $\hat{\mathcal{F}}$ stand for the filtration generated by the stock market price (up to enrichment) and let $\mathcal{W}^{(\xi)}$ be the wealth process of a primary investor who assumes that $\theta = \xi \in \mathbb{R}$ following the log-optimal strategy under this assumption. Then the posterior distribution of θ can be expressed in terms of $\mathcal{W}^{(\xi)}$ as follows

$$\mathbb{P}(\theta = \xi | \hat{\mathcal{F}}_t) = \frac{\mathbb{P}(\theta=\xi)\mathcal{W}_t^{(\xi)}/\mathcal{W}_0^{(\xi)}}{\sum_\varepsilon \mathbb{P}(\theta=\varepsilon)\mathcal{W}_t^{(\varepsilon)}/\mathcal{W}_0^{(\varepsilon)}}, \quad t \in [0,\infty). \tag{1.1}$$

Note that this equation is related to so called *Kallianpur-Striebel* formula and also note that the roots of the formula can be found in Lemma 3.27 and Proposition 3.28 in this paper.

The ideas behind this and also of the forthcoming papers were derived mostly independently of the current literature on filtering and we believe that this circumstance can help the experienced reader look on the already known things from another perspective and the inexperienced one to find a way how to get into the problem which can otherwise be very difficult.

The literature compared with our results comes from the following books [14, 23, 25]. All these sources use L_2-approach to filtering which seems insufficient to us when applied to the model of log-optimal investment, since the corresponding arising restrictions would be unnatural and this is also the reason why we decided to leave the classical L_2-approach and to seek a more flexible one. The results we obtained could not be included in a single paper unless the number of pages and provided information exceeded reasonable boundaries.

The reader interested in older papers on filtering is referred to [11, 15, 16, 17, 21] and the one interested in more recent papers to [6, 7], where further literature can be found, similarly as in the book [2]. We refer the reader interested in log-optimal investment in the discrete time to [1, 3, 4, 5, 20], especially to [1] when interested in the *asymptotic optimality principle*, to [4] in order to see that the log-optimal investment (almost) minimizes the expected time necessary to reach a large amount of money, and to [5] for the Bayesian approach to log-optimal investment in the discrete time. Note that maximizing of the geometric mean in the long run is also called *Kelly criterion*, named after the author of [20].

The paper is organized as follows. In Section 2, we introduce basic notation, used also in appendix, including the notion of enriched filtration, absolute convergence in a metric space. We encourage the reader to start reading from the third section and consult Section 2 as necessary. Section 3 is devoted to trading in frictionless market and especially to the log-optimal trading. As we will show in the forthcoming paper, the wealth process of a log-optimal trading strategy (starting with unit initial value) is closely connected to the density of the posterior distribution of the unobserved random variable, which we are interested in, w.r.t. the prior one. Here, the log-optimal trading serves primarily as the motivation for our task, but later on in another paper, we will show how important the concept of log-optimal trading without transaction costs is, cf. (1.1).

The main result of Section 3 is Proposition 3.22 which gives an answer to question (Q) from the beginning of this section. The main concept of Section 3, especially of Subsection 3.1, is based on a new notion called *progressive projection* which is shown to be related to the notion of *dual predictable projection*, see Example 3.21. In Subsection 3.2, we show that the progressive projection is very closely related to the *(extended) predictable projection* considered in [12].Very interesting is also Proposition 3.28 which tells us how the progressive projection changes if we replace our probability measure by another one that is locally absolutely continuous w.r.t. the original measure.

The last result, used in the forthcoming paper, is Proposition 3.44 based on Proposition 3.22 which among other things says that if our market is regular and log-optimal proportion has a progressive projection to the (enriched) filtration generated by the market, then the market is regular also w.r.t. its own filtration and that the progressive projection of the log-optimal proportion plays the role of log-optimal proportion w.r.t. to the (enriched) filtration generated by the market. Section 4 is devoted to proofs and the last section serves as an appendix.

2. Elementary notation and enriched filtration

Notation 2.1. Let $(\Omega, \mathcal{A})$ be a measurable space. We denote by $\mathbb{L}(\mathcal{A})$ the set of all real-valued $\mathcal{A}$-measurable functions and by $\mathbb{L}(\mathcal{A}, \mathcal{S})$ the set of all measurable maps from $(\Omega, \mathcal{A})$ to a measurable space $(S, \mathcal{S})$. By $\mathbb{C}$ we denote the set of all continuous functions on $\mathbb{R}^+ \overset{\text{def}}{=} [0, \infty)$ endowed with a complete and separable metric

$$\mathbb{r}(x, y) \overset{\text{def}}{=} \sum_{k \in \mathbb{N}} 2^{-k} \wedge |x - y|_k^*, \qquad \text{where} \qquad |x|_t^* \overset{\text{def}}{=} \sup_{s \le t} |x_s|,$$

and by $\mathscr{C} = (\mathscr{C}_t)_{t \ge 0}$ we denote the canonical filtration, i.e.,

$$\mathscr{C}_t \overset{\text{def}}{=} \sigma(\mathbb{p}_s; s \in [0, t]), \qquad \text{where} \qquad \mathbb{p}_t(x) \overset{\text{def}}{=} x_t, \qquad \text{whenever} \quad x \in \mathbb{C}, t \ge 0.$$

Note that $\mathscr{C}_\infty$ is just the Borel σ-algebra on $\mathbb{C}$ w.r.t. the metric $\mathbb{r}$ and that it is usually referred to as the *cylindrical σ-algebra*. Further note that a map $X : \Omega \to \mathbb{C}$ belongs to $\mathbb{L}(\mathcal{A}, \mathscr{C}_\infty)$ if and only if $X_t \overset{\text{def}}{=} \mathbb{p}_t \circ X \in \mathbb{L}(\mathcal{A})$ for every $t \ge 0$, i.e., if $X = (X_t)_{t \ge 0}$ is a (continuous) real valued random process on $(\Omega, \mathcal{A})$. This view leads us to a more intuitive notation $\mathbb{C}(\Omega, \mathcal{A}) \overset{\text{def}}{=} \mathbb{L}(\mathcal{A}, \mathscr{C}_\infty)$ for the *set of all continuous processes* on $(\Omega, \mathcal{A})$. We also denote the set of all positive continuous functions on $\mathbb{R}^+$ equipped with the canonical filtration as follows

$$\tilde{\mathbb{C}} \overset{\text{def}}{=} \mathbb{C} \cap (0, \infty)^{[0, \infty)}, \quad \tilde{\mathscr{C}} \overset{\text{def}}{=} (\tilde{\mathscr{C}_t})_{t \ge 0}, \qquad \text{where} \qquad \tilde{\mathscr{C}_t} \overset{\text{def}}{=} \{C \cap \tilde{\mathbb{C}}; C \in \mathscr{C}_t\}, \; t \in [0, \infty).$$

If $\mathscr{F} = (\mathscr{F}_t)_{t \ge 0}$ and $\mathscr{D} \overset{\text{def}}{=} (\mathscr{D}_t)_{t \ge 0}$ are filtrations, we put

$$\mathscr{F} \otimes \mathscr{D} \overset{\text{def}}{=} (\mathscr{F}_t \otimes \mathscr{D}_t)_{t \ge 0}. \tag{2.1}$$

In this paper, every vector is (by default) assumed to be column and even (x, y) denotes the same column vector as $\binom{x}{y}$ whenever x, y are (column) vectors or just numbers. The corresponding transposition of x is denoted as x^{T} if x is a real vector or a matrix. If A is a set and $n \in \mathbb{N}$, A^n stands for the n-th Cartesian power defined by induction as follows $A^1 \overset{\text{def}}{=} A$, $A^{n+1} \overset{\text{def}}{=} A^n \times A$ in general if not stated otherwise. One exception from this rule is the following one. If $n \in \mathbb{N}$, we put $\mathscr{C}^1 \overset{\text{def}}{=} \mathscr{C}$ and $\mathscr{C}^{n+1} \overset{\text{def}}{=} \mathscr{C}^n \otimes \mathscr{C}$. The same notation will be used also with $\mathscr{C}$ replaced by $\tilde{\mathscr{C}}$. Note that $\tilde{\mathscr{C}}_t^n = \{C \cap \tilde{\mathbb{C}}^n; C \in \mathscr{C}_t^n\}, t \ge 0$, and therefore if π is a $\mathscr{C}^n$-stopping time, its restriction to $\tilde{\mathbb{C}}^n$ is a $\tilde{\mathscr{C}}^n$-stopping time.

Similarly as in (2.1), we denote $\mathscr{F} \otimes \mathcal{D} \overset{\text{def}}{=} (\mathscr{F}_t \otimes \mathcal{D})_{t \ge 0}$ and $\mathcal{D} \otimes \mathscr{F} \overset{\text{def}}{=} (\mathcal{D} \otimes \mathscr{F}_t)_{t \ge 0}$ if $\mathcal{D}$ is a σ-algebra. If A, B are sets and $a : A \to \mathbb{R}, b : B \to \mathbb{R}$, we denote by $a \odot b$ their *tensor* *product* defined on $A \times B$ as follows

$$(a \odot b)(x,y) \stackrel{\text{def}}{=} a(x)b(y), \quad x \in A, y \in B.$$

In the paper, 1_A stands for a function attaining the value 1 just on the set A and the value 0 otherwise (with unspecified domain) and on the other hand $\mathbb{1}_A : A \to A$ stands for the identity map on A. We also use the following symbols $1_m \in \{1\}^m \subseteq \mathbb{R}^m$ and $\mathbb{1}_m \stackrel{\text{def}}{=} \operatorname{diag}(1_m) \in \mathbb{R}^{m\times m}$ if $m \in \mathbb{N}$. Finally, if $\Sigma \in \mathbb{R}^{m\times m}$, by the inequality $\Sigma > 0$ we mean that the matrix Σ is positive definite. If $x \in \mathbb{R}$, we denote $x^+ \stackrel{\text{def}}{=} \max\{x, 0\}$ and $x^- \stackrel{\text{def}}{=} \max\{-x, 0\}$.

Remark 2.2. Let $(x_n)_{n=1}^\infty \in \mathbb{C}^\mathbb{N}$ and $x \in \mathbb{C}$, then

$$\mathbb{r}(x_n, x) \to 0 \qquad \text{if and only if} \qquad \forall\, t \geq 0 \;\; |x_n - x|_t^* \to 0 \tag{2.2}$$

as $n \to \infty$, i.e., $\mathbb{r}$ generates compact-open topology on $\mathbb{C}$ corresponding to the uniform convergence on compact sets. Note that the more difficult implication '$\Leftarrow$' can be easily shown using the Dominated Convergence Theorem while the easier implication '$\Rightarrow$' in (2.2) is left to the reader.

Definition 2.3. If $X, Y \in \mathbb{L}(\mathcal{A}, \mathscr{C}_\infty) = \mathbb{C}(\Omega, \mathcal{A})$, we have that $X_t - Y_t \in \mathbb{L}(\mathcal{A})$ whenever $t \in [0, \infty)$, and therefore also

$$\mathbb{r}(X,Y) = \textstyle\sum_{k=1}^\infty 2^{-k} \wedge \sup\{|X_q - Y_q|; q \in \mathbb{Q} \cap [0,k]\} \in \mathbb{L}(\mathcal{A}).$$

Hence, if $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space, we are allowed to define the distance of X, Y as

$$\mathbb{p}(X,Y) \stackrel{\text{def}}{=} \mathbb{E}[\mathbb{r}(X,Y)]. \tag{2.3}$$

As $\mathbb{r}$ is bounded, it is clear that $\mathbb{p}$ is a bounded pseudometric on $\mathbb{C}(\Omega, \mathcal{A})$. Note that $\mathbb{p}(X,Y) = 0$ if and only if $X \stackrel{\text{as}}{=} Y$ and by a slight abuse of notation we will call $\mathbb{p}$ a metric on $(\Omega, \mathcal{A})$ interpreting the equality almost surely as equality (instead of dealing with a pseudometric or a metric on a factor space) as is usual when dealing with $\mathbb{L}_p$ spaces. We will denote the convergence in $\mathbb{p}$ by $\rightsquigarrow$.

Remark 2.4. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $X^{(n)}, Y \in \mathbb{C}(\Omega, \mathcal{A}), n \in \mathbb{N}$, then

$$\mathbb{p}(X^{(n)}, Y) \to 0 \quad \text{iff} \quad \forall\; t \in [0,\infty) \quad |X^{(n)} - Y|_t^* \to 0 \;\text{ in probability } \mathbb{P}, \quad n \to \infty. \tag{2.4}$$

Notation 2.5. Let $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$ be a filtration on a measurable space $(\Omega, \mathcal{A})$. By $\mathcal{M}(\mathscr{F})$ we denote the σ-algebra of $\mathscr{F}$-progressive sets, i.e.,

$$\mathcal{M}(\mathscr{F}) \stackrel{\text{def}}{=} \{\mathbb{F} \subseteq \Omega_\infty; \forall\; t \geq 0 \;\; \mathbb{F} \cap \Omega_t \in \mathscr{B}_t \otimes \mathscr{F}_t\},$$

where $\Omega_t \stackrel{\text{def}}{=} [0,t] \times \Omega$, $\mathscr{B}_t$ is a Borel σ-algebra on $[0,t]$ if $t \in [0,\infty)$ and $\Omega_\infty \stackrel{\text{def}}{=} [0,\infty) \times \Omega = \cup_{n\in\mathbb{N}} \Omega_n$. By $\mathbb{A}(\mathscr{F})$ we denote the set of all $\mathscr{F}$-adapted processes and by

$$\mathbb{CA}(\mathscr{F}) \stackrel{\text{def}}{=} \mathbb{C}(\Omega, \mathscr{F}_\infty) \cap \mathbb{A}(\mathscr{F}) \subseteq \mathbb{L}(\mathcal{M}(\mathscr{F})),$$
$$\mathbb{CI}_0(\mathscr{F}) \stackrel{\text{def}}{=} \{X \in \mathbb{CA}(\mathscr{F}) \text{ non-decreasing }; X_0 = 0\},$$
$$\mathbb{CFV}(\mathscr{F}) \stackrel{\text{def}}{=} \{X \in \mathbb{CA}(\mathscr{F}) \text{ with locally finite variation }\}$$

the set of all continuous $\mathscr{F}$-adapted processes, and its subsets of non-decreasing processes starting from zero or with locally finite variation, respectively. Whenever $\mathscr{F}, \mathscr{D}$ are filtrations, we put

$$\mathbb{A}(\mathscr{F}, \mathscr{D}) \stackrel{\text{def}}{=} \cap_{t\geq 0} \mathbb{L}(\mathscr{F}_t, \mathscr{D}_t) \subseteq \mathbb{L}(\mathscr{F}_\infty, \mathscr{D}_\infty) \quad \text{so that} \quad \mathbb{A}(\mathscr{F}, \mathscr{C}^k) = \mathbb{CA}(\mathscr{F})^k, \;\; k \in \mathbb{N}.$$

If $(\Omega, \mathcal{A}, P, \mathscr{F})$ is a filtered probability space, we denote by

$$\mathbb{CM}(\mathscr{F}) \stackrel{\text{def}}{=} \mathbb{CM}^P(\mathscr{F}) \stackrel{\text{def}}{=} \{X \in \mathbb{CA}(\mathscr{F}) \text{ is an } \mathscr{F}\text{-martingale under } P\},$$
$$\mathbb{CM}_{loc}(\mathscr{F}) \stackrel{\text{def}}{=} \mathbb{CM}_{loc}^P(\mathscr{F}) \stackrel{\text{def}}{=} \{X \in \mathbb{CA}(\mathscr{F}) \text{ is a local } \mathscr{F}\text{-martingale under } P\},$$
$$\mathbb{CS}(\mathscr{F}) \stackrel{\text{def}}{=} \mathbb{CS}^P(\mathscr{F}) \stackrel{\text{def}}{=} \{X + Y; X \in \mathbb{CM}_{loc}^P(\mathscr{F}), Y \in \mathbb{CFV}(\mathscr{F})\}$$

the set of all continuous $\mathscr{F}$-martingales, continuous local $\mathscr{F}$-martingales and continous $\mathscr{F}$-semimartingales, respectively. If $K \in \mathbb{CI}_0(\mathscr{F})$, put

$$\mathcal{M}_K^p(\mathscr{F}) \stackrel{\text{def}}{=} \left\{G \in \mathbb{L}(\mathcal{M}(\mathscr{F}));\ \forall k \in \mathbb{N}\ \int_0^k G^p \,\mathrm{d}K < \infty\right\}, \quad p \in [1, \infty), \tag{2.5}$$

and similarly, we define $\tilde{\mathcal{M}}_K^p(\mathscr{F})$ but with the integrability w.r.t. K required only almost surely. Further, we omit the lower index K in the notation in case $K_t = t, t \geq 0$.

2.1. **Enriched filtration.** In this and the forthcoming papers, we need a filtration, say $\mathcal{F}$, that admits a continuous adapted version of any process which is a limit in $\mathbb{p}$ of processes from $\mathbb{CA}(\mathcal{F})$. Note that the classical approach, based on ensuring that all null sets from $\mathcal{F}_\infty$ are also in $\mathcal{F}_0$, is insufficient here, since we need to ensure that we can move from one probability measure to another one, which is locally equivalent, without losing the desired property of the filtration.

Definition 2.6. Let $\mathscr{F}$ be a filtration on a measurable space $(\Omega, \mathcal{A})$. Probability measures P, Q on $\mathcal{A}$ are said to be *locally $\mathscr{F}$-equivalent* if $P|\mathscr{F}_t \sim Q|\mathscr{F}_t$ are equivalent measures whenever $t \geq 0$. Note that "$\sim$" is used for equivalence of measures and it means that the measures have the same null sets and domain.

Notation 2.7. If $\mathcal{B}$ and $\mathcal{D}$ are σ-algebras on the same set, we denote $\mathcal{B} \vee \mathcal{D} \stackrel{\text{def}}{=} \sigma(\mathcal{B} \cup \mathcal{D})$. Let $(\Omega, \mathcal{A}, \mathbb{P}, \mathscr{F})$ be a filtered probability space. We introduce a system of negligible sets

$$\mathscr{N}_t^{\mathscr{F},\mathbb{P}} \stackrel{\text{def}}{=} \{N \in \mathscr{F}_\infty; \exists\, F \in \mathscr{F}_t, \mathbb{P}(F) = 0, F \supseteq N\}, \quad t \in [0, \infty), \tag{2.6}$$

$$\mathscr{N}_\infty^{\mathscr{F},\mathbb{P}} \stackrel{\text{def}}{=} \{\cup_{n=0}^\infty N_n; N_n \in \mathscr{N}_n^{\mathscr{F},\mathbb{P}}, n \in \mathbb{N}_0\}. \tag{2.7}$$

We also introduce *enriched* filtration as

$$\mathcal{F}^{\mathbb{P}} \stackrel{\text{def}}{=} (\mathcal{F}_t^{\mathbb{P}})_{t \geq 0}, \quad \text{where} \quad \mathcal{F}_t^{\mathbb{P}} \stackrel{\text{def}}{=} \mathscr{F}_t \vee \sigma(\mathscr{N}_\infty^{\mathscr{F},\mathbb{P}}). \tag{2.8}$$

We will omit upper indices if there is no doubt which filtration or which measure is considered, respectively. If $X \in \mathbb{L}(\mathcal{A}, \mathscr{C}_\infty)^k$, $k \in \mathbb{N}$, we introduce the corresponding *canonical filtration* and its enrichment

(2.9)

$$\mathscr{F}_t^X \stackrel{\text{def}}{=} \sigma(X_s; s \leq t) = \{[X \in C]; C \in \mathscr{C}_t^k\}, \quad \mathcal{F}_t^{X,\mathbb{P}} \stackrel{\text{def}}{=} \mathscr{F}_t^X \vee \sigma(\mathscr{N}_\infty^{X,\mathbb{P}}), \quad t \in [0, \infty),$$

where $\mathscr{N}_\infty^{X,\mathbb{P}} \stackrel{\text{def}}{=} \mathscr{N}_\infty^{\mathscr{F}^X,\mathbb{P}}$. If $\mathscr{G} = (\mathscr{G}_t)_{t \geq 0}$ is a filtration such that $\mathscr{F}_t \subseteq \mathscr{G}_t \subseteq \mathcal{A}$ whenever $t \in [0, \infty)$, we also introduce a filtration

$$\mathcal{F}^{\mathscr{F},\mathscr{G}} \stackrel{\text{def}}{=} \mathcal{F}^{\mathscr{F},\mathscr{G},\mathbb{P}} \quad \text{such that} \quad \mathcal{F}_t^{\mathscr{F},\mathscr{G},\mathbb{P}} \stackrel{\text{def}}{=} \mathscr{F}_t \vee \sigma(\mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}), \quad t \in [0, \infty),$$

and we call it a *relative enrichment* of $\mathscr{F}$ by $\mathscr{G}$ under $\mathbb{P}$ (or a *relatively enriched filtration)*. If the filtration $\mathscr{F}$ is generated by a process Y and $\mathscr{G}$ by X, we will use also the following notation $\mathcal{F}^{Y,X} \stackrel{\text{def}}{=} \mathcal{F}^{\mathscr{F},\mathscr{G}}$. Finally, in the special case when $(\Omega, \mathcal{A}, \mathbb{P}, \mathscr{F}) = (\mathbb{C}^k, \mathscr{C}_\infty^k, \nu, \mathscr{C}^k)$, we put

$$\mathcal{C}_t^\nu \stackrel{\text{def}}{=} \mathscr{N}_t^{\mathscr{C}^k,\nu}, \qquad t \in [0, \infty],$$

whenever ν is a probability measure on $(\mathbb{C}^k, \mathscr{C}_\infty^k), k \in \mathbb{N}$.

Remark 2.8. If the underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is complete and if $\mathscr{G}_t = \mathcal{A}$ holds for every $t \in [0, \infty)$, then the relatively enriched filtration $\mathcal{F}^{\mathscr{F},\mathscr{G}}$ corresponds to the smallest complete extension of $\mathscr{F}$ considered in [13]. It also corresponds to what is called the augmentation in [19, Definition 2.7.2] if $\mathscr{G}_t = \mathscr{F}_\infty = \mathcal{A}, t \in [0, \infty)$, and if the underlying probability space is complete, but the reader have to keep in mind that in that definition it is assumed that the filtration $\mathscr{F}$ is generated by a Brownian motion.

Here, we need a notion that is more subtle, since we want to be able to replace the original probability measure $\mathbb{P}$ by a measure which is not equivalent with $\mathbb{P}$, but only locally $\mathscr{F}$-equivalent. This is the reason why $\mathscr{N}_\infty$ does not have to contain all $\mathbb{P}$-null sets from $\mathscr{G}_\infty$ and for other (technical) reasons, we do not restrict ourselves to the case when $\mathscr{F} = \mathscr{G}$.

Remark 2.9. Let us consider the same context as in Notation 2.7. Then $\mathcal{F}_\infty^{\mathscr{F},\mathscr{G}} \subseteq \mathscr{G}_\infty$ and $\mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}$ is closed under countable unions and it contains $\mathscr{G}_\infty$-measurable subsets of its elements. Further, all its elements are $\mathscr{G}_\infty$-measurable $\mathbb{P}$-null sets. In particular,

$$\mathscr{N}_\infty^{\mathscr{G},\mathbb{P}} = \{G \in \mathscr{G}_\infty; G \subseteq \cup_{n=1}^\infty N_n, N_n \in \mathscr{G}_n, \mathbb{P}(N_n) = 0, n \in \mathbb{N}\}. \tag{2.10}$$

Remark 2.10. Let $\mathcal{F} = (\mathcal{F}_t^{\mathscr{F},\mathscr{G}})_{t\geq 0}$ be as in Notation 2.7. As $\mathcal{F}$ is an extension of $\mathscr{F}$ by the system $\sigma(\mathscr{N}_\infty^{\mathscr{G},\mathbb{P}})$ of $\mathbb{P}$-trivial sets (independent of any subsystem of $\mathcal{A}$), we get from Theorem 1.4.3 in [10, part III] that $\mathbb{CM}_{loc}(\mathscr{F}) \subseteq \mathbb{CM}_{loc}(\mathcal{F})$ and then also $\mathbb{CS}(\mathscr{F}) \subseteq \mathbb{CS}(\mathcal{F})$ as obviously $\mathbb{CVF}(\mathscr{F}) \subseteq \mathbb{CVF}(\mathcal{F})$. Finally, as $\mathscr{F}$ is a subfiltration of $\mathcal{F}$, the same theorem can be used in order to obtain that $\mathbb{CM}_{loc}(\mathcal{F}) \cap \mathbb{CA}(\mathscr{F}) \subseteq \mathbb{CM}_{loc}(\mathscr{F})$, i.e., we have that

$$\mathbb{CM}_{loc}(\mathscr{F}) = \mathbb{CM}_{loc}(\mathcal{F}) \cap \mathbb{CA}(\mathscr{F}), \quad \mathbb{CS}(\mathscr{F}) \subseteq \mathbb{CS}(\mathcal{F}). \tag{2.11}$$

Notation 2.11. Whenever X is a continuous semimartingale, $\langle X\rangle$ will stand for a version of its *quadratic variation* that is non-decreasing, continuous, adapted to $\mathcal{F}^X$ and that starts from $\langle X\rangle_0 = 0$. See Corollary A.17 in order to agree that such process exists. Note that this assumption is assumed in the whole paper except in the proof of the corollary and that we do not deal with the quadratic variation in the appendix until the corollary is stated.

Similarly, we will assume that the result of stochastic integration is a continuous process adapted to a given enriched filtration such that the integrator is a (multidimensional) continuous semimartingale and the integrand is progressively measurable. To justify this assumption, see Lemma A.31 for the case of integration w.r.t. a (multidimensional) local martingale. Similarly, we may and will assume that the stochastic integrals start from 0. Note that in the whole paper, we use the following notation.

Notation 2.12. The stochastic integral is in this paper understood in the Itô sense except for one special case, when the integrator is of locally finite variation. In this case, the integral is understood in Lebesgue-Stieltjes sense, which is consistent with the above-mentioned Itô sense defining the integral uniquely only up to a null set.

(i) Let $M \in \mathbb{CM}_{loc}(\mathcal{F})^k$ and $H \in \mathbb{L}(\mathcal{M}(\mathcal{F}))^k, k \in \mathbb{N}$. We consider the integral

$$\int H^\top \mathrm{d}M \overset{\text{def}}{=} \mathbb{P}\text{-}\lim_{n\to\infty} \int 1_{[H^\top H \leq n]} H^\top \mathrm{d}M \quad \text{if} \quad \int_0^t \mathrm{tr}\{HH^\top \mathrm{d}\langle\!\langle M\rangle\!\rangle\} \overset{\text{as}}{<} \infty, \quad t \in [0,\infty). \tag{2.12}$$

Note that without this extension the integral $\int H^\top \mathrm{d}M$ is well defined if and only if

$$\textstyle\sum_{i=1}^k \int_0^t (H^{(i)})^2 \,\mathrm{d}\langle M^{(i)}\rangle \overset{\text{as}}{<} \infty, \quad t \in [0,\infty), \tag{2.13}$$

and if (2.13) is satisfied, we have equality almost surely in (2.12) on the left. Moreover, if the components of M are uncovariated, then (2.13) is equivalent to the condition in (2.12) on the right. (ii) Generally, let $M = L + I - J \in \mathbb{CS}(\mathcal{F})^k$, where $L \in \mathbb{CM}_{loc}(\mathcal{F})^k$ and $I, J \in \mathbb{CI}_0(\mathcal{F})^k$ be such that

$$H^{(i)} \in \tilde{\mathcal{M}}^1_{I^{(i)}+J^{(i)}}(\mathcal{F}), \quad i \leq k. \tag{2.14}$$

Then $\int H^\top \mathrm{d}M$ is well defined by (2.12) if the corresponding condition on the right is satisfied and obviously $\int H^\top \mathrm{d}M \overset{\text{as}}{=} \int H^\top \mathrm{d}L + \int H^\top \mathrm{d}I - \int H^\top \mathrm{d}J$ in that case.

3. Log-optimal trading in a frictionless market and filtering

We consider an investor that may invest in the money market and also in the stock market with m stocks. Let us consider a fixed filtered probability space $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$, where $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ is an enriched filtration.

Definition 3.1. By a *(frictionless) market* we mean $\hat{\mathcal{S}} = (\mathcal{S}^{(i)})_{i=0}^m$, where $\mathcal{S}^{(i)}$ is a positive continuous $\mathcal{F}$-semimartingale whenever $i \in \{0,\dots,m\}$. In that case, there are continuous $\mathcal{F}$-semimartingales $V^{(i)} \stackrel{\text{def}}{=} \ln(\mathcal{S}^{(i)}/\mathcal{S}_0^{(i)}) + \frac{1}{2}\langle \ln \mathcal{S}^{(i)}\rangle$ adapted to $\mathcal{F}^{\hat{\mathcal{S}}}$, see Notation 2.11, such that

$$\mathrm{d}\mathcal{S}^{(i)} = \mathcal{S}^{(i)}\mathrm{d}V^{(i)}, \quad V_0^{(i)} = 0, \qquad i \in \{0,\dots,m\}. \tag{3.1}$$

The process $\mathcal{S}^{(i)}$ represents the price of the i-th asset in the market. If $i=0$, it is called *bond* and its price describes the value of money in time and this way it represents the money market. If $i = 1,\dots m$, the i-th asset is called the i-th *stock*. Further, we denote $\mathcal{S} \stackrel{\text{def}}{=} (\mathcal{S}^{(i)})_{i=1}^m$.

Definition 3.2. A pair $(\varphi^{(0)},\varphi)$ is called a *trading strategy* in the frictionless market $(\mathcal{S}^{(0)},\mathcal{S})$ if $\varphi = (\varphi^{(i)})_{i=1}^m$ and if $\varphi^{(i)}$ is an $\mathcal{F}$-progressive process for each $i = 0,\dots m$.

In this paper, the process $\varphi_t^{(i)}$ stands for the random variable describing the number of shares of the i-th asset held by the investor at time $t \in [0,\infty)$.

Definition 3.3. Let $(\varphi^{(0)},\varphi)$ be a trading strategy in the market $(\mathcal{S}^{(0)},\mathcal{S})$. The corresponding *wealth process* $\mathcal{W}$ is defined as follows

$$\mathcal{W} \stackrel{\text{def}}{=} \textstyle\sum_{i=0}^m \varphi^{(i)}\mathcal{S}^{(i)} = \varphi^{(0)}\mathcal{S}^{(0)} + \varphi^{\intercal}\mathcal{S}.$$

If the wealth process $\mathcal{W}$ is positive, we are allowed to introduce the i-th *position in the market* for $i = 0,\dots m$ and the position process as follows

$$\pi^{(i)} \stackrel{\text{def}}{=} \varphi^{(i)}\mathcal{S}^{(i)}\mathcal{W}^{-1}, \qquad \pi \stackrel{\text{def}}{=} (\pi^{(i)})_{i=1}^m.$$

Definition 3.4. A trading strategy $(\varphi^{(0)},\varphi)$ in the market $(\mathcal{S}^{(0)},\mathcal{S})$ is called *self-financing* if $\mathcal{W}$ is a continuous $\mathcal{F}$-semimartingale with the differential

$$\mathrm{d}\mathcal{W} = \textstyle\sum_{i=0}^m \varphi^{(i)}\mathrm{d}\mathcal{S}^{(i)} = \sum_{i=0}^m \varphi^{(i)}\mathcal{S}^{(i)}\mathrm{d}V^{(i)}, \tag{3.2}$$

i.e., the changes of the wealth process are simply generated by the changes of the market prices of the assets without any additional income or costs.

Remark 3.5. Let $(\varphi^{(0)},\varphi)$ be a self-financing strategy in the market $(\mathcal{S}^{(0)},\mathcal{S})$ and let $D_t^{(i)}$ be the absolute variation of the drift part of $\mathcal{S}^{(i)}$ from the decomposition on the interval $[0,t]$. As the stochastic integral from the expression in (3.2) on the right has to be defined correctly, we get from (2.12) in Notation 2.12 that the self-financing condition is satisfied only if

$$\textstyle\int_0^t \mathrm{tr}[\varphi\varphi^{\intercal}\mathrm{d}\langle\!\langle\mathcal{S}\rangle\!\rangle] \stackrel{\text{as}}{<} \infty, \quad t \in [0,\infty), \qquad \varphi^{(i)} \in \tilde{\mathcal{M}}^1_{D^{(i)}}(\mathcal{F}), \quad i = 0,\dots,m. \tag{3.3}$$

Definition 3.6. A market $(\mathcal{S}^{(0)},\mathcal{S})$ will be called *regular* if there exist an m-dimensional standard $\mathcal{F}$-Brownian motion B, $\{\alpha^{(i)}\}_{i=0}^m \subseteq \mathcal{M}^1(\mathcal{F})$ and $(\sigma^{(i,j)})_{i,j=1}^m \in \mathcal{M}^2(\mathcal{F})^{m\times m}$ with values within the set of all regular matrices such that

$$\mathcal{S}^{(0)} \stackrel{\text{as}}{=} \mathcal{S}_0^{(0)} + \textstyle\int \alpha_t^{(0)}\mathcal{S}_t^{(0)}\mathrm{d}t, \qquad \mathcal{S}_t^{(i)} \stackrel{\text{as}}{=} \mathcal{S}_0^{(i)} + \int \mathcal{S}_t^{(i)}(\alpha_t^{(i)}\mathrm{d}t + \sum_{j=1}^m \sigma_t^{(i,j)}\mathrm{d}B_t^{(j)}). \tag{3.4}$$

Remark 3.7. The reader interested in the question of existence of an arbitrage in regular markets should look at Example 4.6 in [18] which offers an example of an arbitrage (on the interval $[0,1]$). Example 4.7 in [18] may be also interesting from this point of view.

Notation 3.8. In the context of Definition 3.6, we put

$$\alpha \stackrel{\text{def}}{=} (\alpha^{(i)})_{i=1}^m, \qquad \sigma \stackrel{\text{def}}{=} (\sigma^{(i,j)})_{i,j=1}^m, \qquad \Sigma \stackrel{\text{def}}{=} (\Sigma^{(i,j)})_{i,j=1}^m \stackrel{\text{def}}{=} \sigma\sigma^{\intercal} \in \mathcal{M}^1(\mathcal{F})^{m\times m}, \tag{3.5}$$

and by $\mathrm{diag}(x) \stackrel{\text{def}}{=} \mathbb{x} \in \mathbb{R}^{m\times m}$ we denote a diagonal matrix with $\mathbb{x}^{(i,j)} = x^{(i)}1_{[i=j]}$.

Remark 3.9. Note that the SDE in (3.4) on the right can be equivalently rewritten in the form

$$\mathrm{d}\mathcal{S}_t = \mathrm{diag}(\mathcal{S}_t)[\alpha_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t] = \mathrm{diag}(\mathcal{S}_t)\,\mathrm{d}V_t, \quad \text{where} \quad V \overset{\text{def}}{=} (V^{(i)})_{i=1}^m. \tag{3.6}$$

Further note that we are able to express $(V^{(0)}, V)$ in the regular market as follows

$$V^{(0)} \overset{\text{as}}{=} \textstyle\int \alpha_t^{(0)}\mathrm{d}t, \qquad V \overset{\text{as}}{=} \int \alpha_t \mathrm{d}t + \int \sigma \mathrm{d}B. \tag{3.7}$$

Lemma 3.10. *Let $(\varphi^{(0)}, \varphi)$ be a self-financing strategy in a regular market with the position π. Then*

$$\pi^{\mathsf{T}}\Sigma\pi,\ \pi^{(i)}\alpha^{(i)} \in \tilde{\mathcal{M}}^1(\mathcal{F}), \qquad i = 0, \dots, m.$$

Proof. See Subsection 4.1 in section Proofs. □

Remark 3.11. If $(\varphi^{(0)}, \varphi)$ is a self-financing strategy in the regular market $(\mathcal{S}^{(0)}, \mathcal{S})$ with the wealth process $\mathcal{W}$ and with the position process π, we get from (3.2,3.7) the following SDE for the wealth process in terms of the position process

$$\mathrm{d}\mathcal{W}_t = \sum_{i=0}^m \varphi_t \mathcal{S}_t^{(i)} \mathrm{d}V_t^{(i)} = \mathcal{W}_t \sum_{i=0}^m \pi_t^{(i)} \mathrm{d}V_t^{(i)} = \mathcal{W}_t[(\pi_t^{(0)}\alpha_t^{(0)} + \pi_t^{\mathsf{T}}\alpha_t)\,\mathrm{d}t + \pi_t^{\mathsf{T}}\sigma_t \mathrm{d}B_t]$$

and we get from Lemma A.25 an almost surely unique solution in the form

$$\mathcal{W}_t \overset{\text{as}}{=} \mathcal{W}_0 \exp\{\textstyle\int_0^t \pi^{\mathsf{T}}\sigma\,\mathrm{d}B + \int_0^t (\pi_s^{(0)}\alpha_s^{(0)} + \pi_s^{\mathsf{T}}\alpha_s - \frac{1}{2}\pi_s^{\mathsf{T}}\Sigma_s\pi_s)\,\mathrm{d}s\}.$$

Hence, we have a decomposition of the logarithm $\ln \mathcal{W}$ of the wealth process (up to the initial value) into the local $\mathcal{F}$-martingale part $M_t \overset{\text{def}}{=} \int_0^t \pi^{\mathsf{T}}\sigma\,\mathrm{d}B$ and the drift part almost surely equal to

$$D_t \overset{\text{def}}{=} \textstyle\int_0^t q(\pi_s, \alpha_s^{(0)}, \alpha_s, \Sigma_s)\,\mathrm{d}s, \quad \text{where} \quad q(x, r, a, A) \overset{\text{def}}{=} x^{\mathsf{T}}a - \frac{1}{2}x^{\mathsf{T}}Ax + r(1 - x^{\mathsf{T}}1_m),$$

$t \ge 0$. Here, we have used an obvious equality $\pi_t^{(0)} = 1 - \pi_t^{\mathsf{T}}1_m$. Since the market is regular by assumption, we get that $\Sigma_t = \sigma_t\sigma_t^{\mathsf{T}}$ is always a positive definite matrix, and therefore we obtain the maximum of q in x simply by differentiating which leads to the equation $0 = a - Ax - r1_m$. Hence, the maximum is attained at $x = A^{-1}(a - r1_m)$.

Definition 3.12. In a regular market $(\mathcal{S}^{(0)}, \mathcal{S})$, we define the *log-optimal proportion* θ as

$$\theta_t \overset{\text{def}}{=} \Sigma_t^{-1}(\alpha_t - \alpha_t^{(0)}1_m), \quad t \ge 0. \tag{3.8}$$

A self-financing strategy $(\varphi^{(0)}, \varphi)$ with the position process π is called *log-optimal* if $\pi_t \overset{\text{as}}{=} \theta_t$ holds for almost every $t \ge 0$. This definition is justified by Proposition 3.13, see also Remark 3.15.

Proposition 3.13. *Let $(\mathcal{S}^{(0)}, \mathcal{S})$ be a regular market, let $(\varphi^{(0)}, \varphi)$ be a log-optimal strategy with the wealth process $\mathcal{W}$. If $(\tilde{\varphi}^{(0)}, \tilde{\varphi})$ is a self-financing strategy with a positive wealth process $\tilde{\mathcal{W}}$, then*

$$\limsup_{t\to\infty} \tfrac{1}{t}\ln(\tilde{\mathcal{W}}_t/\mathcal{W}_t) \overset{\text{as}}{\le} 0. \tag{3.9}$$

Proof. See Subsection 4.2 in section Proofs. The proof uses Notation 3.14. □

Notation 3.14. Let us consider a regular market $(\mathcal{S}^{(0)}, \mathcal{S})$. In order to avoid dealing with the value $\pi^{(0)} = 1 - \pi^{\mathsf{T}}1_m$, we introduce notation for the process playing a similar role as V but for the discounted market prices. It is defined as

$$\mathbb{V} \overset{\text{def}}{=} V - \textstyle\int \alpha_t^{(0)}\mathrm{d}t 1_m \in \mathbb{CS}(\mathcal{F})^m. \tag{3.10}$$

Then the wealth process $\mathcal{W}$ of a self-financing strategy with the position π satisfies

$$\mathrm{d}\mathcal{W}_t = \mathcal{W}_t[\alpha_t^{(0)}\mathrm{d}t + \pi_t^{\mathsf{T}}\mathrm{d}\mathbb{V}_t], \text{ i.e., } \mathcal{W}_t \overset{\text{as}}{=} \mathcal{W}_0 \exp\{\textstyle\int_0^t(\alpha_s^{(0)} - \frac{1}{2}\pi_s^{\mathsf{T}}\Sigma_s\pi_s)\mathrm{d}s + \int_0^t \pi^{\mathsf{T}}\mathrm{d}\mathbb{V}\}, \tag{3.11}$$

$t \geq 0$, by Lemma A.25, since $\mathbb{V} \in \mathbb{CS}(\mathcal{F})^m$ is such that $\langle\!\langle \mathbb{V} \rangle\!\rangle_t \overset{\text{as}}{=} \langle\!\langle V \rangle\!\rangle_t \overset{\text{as}}{=} \int_0^t \Sigma_s \mathrm{d}s$. Further,

$$\mathbb{V} \overset{\text{as}}{=} \int \sigma \mathrm{d}B + \int (\alpha_s - \alpha_s^{(0)} 1_m) \mathrm{d}s \overset{\text{as}}{=} \int \sigma \mathrm{d}B + \int \Sigma_s \theta_s \mathrm{d}s, \tag{3.12}$$

where θ is the log-optimal proportion introduced in Definition 3.12.

Remark 3.15. The log-optimal strategy corresponds to what is called the *numéraire portfolio* in [18]. It does not have to exist in a regular market. It is sufficient to consider the case when

$$\alpha_s^{(0)} \overset{\text{def}}{=} 0, \quad \alpha_s \overset{\text{def}}{=} 1, \quad \sigma_s \overset{\text{def}}{=} (1-s)^{1/2} 1_{[s<1]} + 1_{[s \geq 1]}.$$

In this case, the log-optimal proportion is of the form $\theta_s \overset{\text{def}}{=} (1-s)^{-1} 1_{[s<1]} + 1_{[s \geq 1]}$, but no log-optimal strategy exists in this case, since its wealth at time $t = 1$ would have to be ∞ almost surely which is something we do not allow. This example also shows that the condition of *no unbounded profit with bounded risk*, considered in [18], does not hold in regular markets in general.

3.1. **Filtering.** This subsection serves as a technical background for Subsection 3.3, where we will cope with the difficulty that the log-optimal proportion is not observed directly from the market.

Definition 3.16. Let ν be a measure on a measurable space $(T, \mathcal{T})$. Let $f \in \mathbb{L}(\mathcal{T})$ and $\mathcal{H} \subseteq \mathcal{T}$ be a σ-algebra such that $\nu|_{\mathcal{H}}$ and $(\int_H |f| \mathrm{d}\nu)_{H \in \mathcal{H}}$ are σ-finite measures. Put

$$\mu^{(+)} \overset{\text{def}}{=} (\textstyle\int_H f^+ \mathrm{d}\nu)_{H \in \mathcal{H}} \quad \text{and} \quad \mu^{(-)} \overset{\text{def}}{=} (\textstyle\int_H f^- \mathrm{d}\nu)_{H \in \mathcal{H}}.$$

A function $\hat{f} \in \mathbb{L}(\mathcal{H})$ is called a *ν-projection* of f to $\mathcal{H}$ if the following equality holds ν-a.e.

$$\hat{f} = \mathscr{P}^{\nu}_{\mathcal{H}}(f) \overset{\text{def}}{=} \frac{\mathrm{d}\mu^{(+)}}{\mathrm{d}\nu|\mathcal{H}} - \frac{\mathrm{d}\mu^{(-)}}{\mathrm{d}\nu|\mathcal{H}}. \tag{3.13}$$

Note that if ν is a probability measure, then $\hat{f}$ is just the conditional expectation of f given $\mathcal{H}$ under ν, i.e., $\hat{f} \overset{\text{as}}{=} \mathbb{E}_\nu[f|\mathcal{H}]$, where $\mathbb{E}_\nu$ here stands for the (conditional) expectation w.r.t. ν.

Remark 3.17. Note that the fractions in (3.13) on the right stand for the Radon-Nikodym derivatives. The reader interested in an elementary proof of the Radon-Nikodym Theorem (in a finite case) or in the references to its various proofs can look at [22].

The following lemma summarizes the basic properties of the ν-projection.

Lemma 3.18. *Let $(T, \mathcal{T}, \nu), f, \mathcal{H}$, be as in Definition 3.16. (i) Then $\hat{f} \in \mathbb{L}(\mathcal{H})$ is a ν-projection of f to $\mathcal{H}$ if and only if*

$$\textstyle\int_H f \mathrm{d}\nu = \int_H \hat{f} \mathrm{d}\nu \quad \text{holds for every} \quad H \in \mathcal{H}_f \overset{\text{def}}{=} \{H \in \mathcal{H}; \int_H |f| \mathrm{d}\nu < \infty\}, \tag{3.14}$$

and if (3.14) *holds, then*

$$\textstyle\int_H |\hat{f}| \mathrm{d}\nu \leq \int_H |f| \mathrm{d}\nu, \quad H \in \mathcal{H}. \tag{3.15}$$

(ii) If $f, g \in \mathbb{L}(\mathcal{T})$ have ν-projections $\hat{f}$ and $\hat{g}$ to $\mathcal{H}$, respectively, then $f + g$ has a ν-projection $\hat{f} + \hat{g}$ to $\mathcal{H}$, i.e.,

$$\mathscr{P}^{\nu}_{\mathcal{H}}(f+g) = \mathscr{P}^{\nu}_{\mathcal{H}}(f) + \mathscr{P}^{\nu}_{\mathcal{H}}(g) \quad \text{holds } \nu\text{-almost everywhere.}$$

(iii) If $f \in \mathbb{L}(\mathcal{T})$ has a ν-projection $\hat{f}$ to $\mathcal{H}$, and $g \in \mathbb{L}(\mathcal{H})$, then fg has a ν-projection $\hat{f}g$ to $\mathcal{H}$, i.e.,

$$\mathscr{P}^{\nu}_{\mathcal{H}}(fg) = g \mathscr{P}^{\nu}_{\mathcal{H}}(f) \quad \text{holds } \nu\text{-almost everywhere.}$$

(iv) Let $\mathcal{K} \subseteq \mathcal{H}$ be sub-σ-algebras of $\mathcal{T}$. If $f \in \mathbb{L}(\mathcal{T})$ has a ν-projections $\tilde{f} = \mathscr{P}^{\nu}_{\mathcal{K}}(f), \hat{f} = \mathscr{P}^{\nu}_{\mathcal{H}}(f)$ of f to $\mathcal{K}$ and to $\mathcal{H}$, respectively, then $\tilde{f}$ is a ν-projection of $\hat{f}$ to $\mathcal{K}$, i.e.,

$$\mathscr{P}^{\nu}_{\mathcal{K}}(f) = \mathscr{P}^{\nu}_{\mathcal{K}}(\mathscr{P}^{\nu}_{\mathcal{H}}(f)) \quad \text{holds } \nu\text{-almost everywhere.} \tag{3.16}$$

Proof. See Subsection 4.3 in section Proofs. □

Notation 3.19. Let $\mathscr{B}_\infty$ stand for the Borel σ-algebra on $[0,\infty)$ and let $\mathfrak{L}$ stand for the Lebesgue measure restricted to $([0,\infty),\mathscr{B}_\infty)$. Let $\mathcal{G}$ be a filtration on $(\Omega,\mathcal{A})$. Then $\mathbb{v} \overset{\text{def}}{=} \mathfrak{L}\otimes\mathbb{P}$ restricted to $\mathcal{M}(\mathcal{G})$ is a σ-finite measure. A process $H \in \mathbb{L}(\mathscr{B}_\infty\otimes\mathcal{A})$ has a $\mathbb{v}$-projection to $\mathcal{M}(\mathcal{G})$, from here on called the *$\mathbb{v}$-progressive projection* of H to the filtration $\mathcal{G}$, if and only if the measure

$$\mathbb{v}^{|H|}_{\mathcal{M}(\mathcal{G})} \overset{\text{def}}{=} (\textstyle\int_{\mathbb{G}}|H|\,\mathrm{d}\mathbb{v})_{\mathbb{G}\in\mathcal{M}(\mathcal{G})} \quad \text{is } \sigma\text{-finite.} \tag{3.17}$$

This happens for example if $\int|H_s|\mathrm{d}s$ is an integrable process. A $\mathbb{v}$-progressive projection of H to $\mathcal{G}$ is further denoted also as

$$\mathcal{P}^{\mathbb{P}}_{\mathcal{G}}(H) \overset{\text{def}}{=} \mathscr{P}^{\mathbb{v}}_{\mathcal{M}(\mathcal{G})}(H).$$

We will use $\overset{\text{ae}}{=}$ for equality up to a set $\mathbb{M}$ with zero measure $\mathbb{v}(\mathbb{M})=0$, where $\mathbb{v}=\mathfrak{L}\otimes\mathbb{P}$. Note that the meaning of this equality does not change if $\mathbb{P}$ is replaced by an equivalent measure Q. Even if Q is only locally $\mathscr{F}$-equivalent to $\mathbb{P}$ and $\mathbb{M}\in\mathcal{M}(\mathscr{F})$, then $(\mathfrak{L}\otimes\mathbb{P})(\mathbb{M})=0$ has the same meaning as in the case, where $\mathbb{P}$ is replaced by Q. For example, if $(\mathfrak{L}\otimes Q)(\mathbb{M})=0$, then $(\mathfrak{L}\otimes Q)(\mathbb{M}_n)=0$ holds with $\mathbb{M}_n \overset{\text{def}}{=} \mathbb{M}\cap\Omega_n\in\mathscr{B}_n\otimes\mathscr{F}_n$, and as $Q,\mathbb{P}$ are locally $\mathscr{F}$-equivalent, we have that $(\mathfrak{L}\otimes Q)|\mathscr{B}_n\otimes\mathscr{F}_n \sim \mathbb{v}|\mathscr{B}_n\otimes\mathscr{F}_n$ which ensures that $\mathbb{v}(\mathbb{M}_n)=0$, $n\in\mathbb{N}$, and finally that $\mathbb{v}(\mathbb{M})=0$.

Remark 3.20. By Lemma 3.18, if (3.17) holds and if G is a $\mathcal{G}$-progressive process, then also (3.17) holds with H replaced by GH and if $\hat{H}$ is a $\mathbb{v}$-projection of H to $\mathcal{G}$, then $G\hat{H}$ is a $\mathbb{v}$-projection of GH to $\mathcal{G}$. In particular, if $G\in\mathcal{M}(\mathcal{G})$ attains only non-zero values, the $\mathbb{v}$-progressive projection $\hat{H}$ of H to $\mathcal{G}$ exists if and only if HG has a $\mathbb{v}$-progressive projection to $\mathcal{G}$ and in both cases the $\mathbb{v}$-projection of GH to $\mathcal{G}$ is equal to $G\hat{H}$ up to a $\mathbb{v}$-null set

Example 3.21. Let $0\le H\in\mathcal{M}^1(\mathcal{F})$ be such that the process $U \overset{\text{def}}{=} \int H_u\mathrm{d}u$ is integrable. First, as mentioned in Notation 3.19, the condition (3.17) is satisfied in this case. Hence, we have that H has a $\mathbb{v}$-progressive projection $\hat{H}\in\mathcal{M}^1(\mathcal{G})$. Note that we have from the assumption $H\ge 0$, the definition and uniqueness of the progressive projection that also $\hat{H}\ge 0$ holds $\mathbb{v}$-almost everywhere which means that we may assume (without loss of generality) that $\hat{H}=\hat{H}^+\ge 0$ holds. Then we have from (3.15) in Lemma 3.18 that

$$\mathbb{E}[\textstyle\int_0^\infty H1_{\mathbb{G}}\mathrm{d}\mathfrak{L}] \ge \mathbb{E}[\int_0^\infty \hat{H}1_{\mathbb{G}}\mathrm{d}\mathfrak{L}], \quad \mathbb{G}\in\mathcal{M}(\mathcal{G}) \tag{3.18}$$

and that we have the equality in (3.18) if the expression on the left is finite. In particular, if $0\le s\le t<\infty$ and $G\in\mathcal{G}_s$, we get from the choice $\mathbb{G} \overset{\text{def}}{=} [s,t)\times G\in\mathcal{M}(\mathcal{G})$ that

$$\mathbb{E}[U_t-U_s;G] = \mathbb{E}[\hat{U}_t-\hat{U}_s;G], \quad \text{where} \quad \hat{U} \overset{\text{def}}{=} \textstyle\int \hat{H}_u\mathrm{d}u.$$

Further note that if U is even uniformly integrable, i.e., $\infty > \mathbb{E}\int_0^\infty H_u\mathrm{d}u = \mathbb{E}\int_0^\infty \hat{H}_u\mathrm{d}u$, then $\hat{U}$ is also uniformly integrable. In this case, we have that $(\hat{U},U,\mathcal{G})$ satisfy the conditions on $(U^*,U,\mathscr{F})$ in Theorem 3.1.4 in [14], which means that $\hat{U}$ is something which is in [14] later on called the *dual predictable projection* of U. For the relation of the progressive projection and the (extended) predictable projection, see the next subsection.

The main technical result of this paper reads as follows.

Proposition 3.22. *Let $M\in\mathbb{CM}_{loc}(\mathcal{F})$ and let $H\in\mathcal{M}^1(\mathcal{F})$ satisfy (3.17) with an enriched subfiltration $\mathcal{G}$ of $\mathcal{F}$ such that*

$$X \overset{\text{def}}{=} M + \textstyle\int H_u\,\mathrm{d}u \in \mathbb{A}(\mathcal{G}). \tag{3.19}$$

Then there exists a $\mathbb{v}$-progressive projection $\hat{H}\in\mathcal{M}^1(\mathcal{G})$ of H to $\mathcal{G}$ such that

$$\hat{M} \overset{\text{def}}{=} X - \textstyle\int \hat{H}_u\,\mathrm{d}u \in \mathbb{CM}_{loc}(\mathcal{G}). \tag{3.20}$$

Proof. See Subsection 4.4 in section Proofs. □

Remark 3.23. Recall that in the whole section $\mathcal{F}$ is assumed to be an enriched filtration on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Note that this assumption is not essential in Proposition 3.22. It can be seen as follows. Let $\mathscr{F}$ be a filtration, $M \in \mathbb{CM}_{loc}(\mathscr{F})$, $H \in \mathcal{M}^1(\mathscr{F})$ and let (3.17,3.19) hold. Then obviously $H \in \mathcal{M}^1(\mathscr{F}) \subseteq \mathcal{M}^1(\mathcal{F})$ and by (2.10) in Remark 2.9 also $M \in \mathbb{CM}_{loc}(\mathscr{F}) \subseteq \mathbb{CM}_{loc}(\mathcal{F})$, where $\mathcal{F}$ is an enrichment of $\mathscr{F}$. Then it is enough to use Proposition 3.22 in order to get that it holds also with $\mathcal{F}$ replaced by $\mathscr{F}$. We conjecture that also the assumption that the filtration $\mathcal{G}$ is enriched could be removed if certain technical details were generalized in this paper, but the above statement is sufficient for our reason in this form.

The following lemma tells us how to verify that a progressive process is a progressive projection of another process if we face no problems with integrability.

Lemma 3.24. *Let $\mathscr{G}$ be a subfiltration of $\mathscr{F}$ and let $H \in \mathcal{M}^1(\mathscr{F})$ be such that $\int |H_s| \,\mathrm{d}s$ is an integrable process. Further, let $\hat{H} \in \mathbb{L}(\mathcal{M}(\mathscr{G}))$ be such that*

$$\hat{H}_t \overset{\text{as}}{=} \mathbb{E}[H_t|\mathscr{G}_t] \quad \text{holds for a.e.} \quad t \geq 0. \tag{3.21}$$

Then $\hat{H}$ is a ν-progressive projection of H to $\mathscr{G}$.

Proof. See Subsection 4.5 in section Proofs. □

Remark 3.25. Later on, we will use the following obvious property. If $\mathbb{P}, Q$ are equivalent measures, then also $\nu \overset{\text{def}}{=} \mathfrak{L} \otimes \mathbb{P}$ is equivalent to $\mu \overset{\text{def}}{=} \mathfrak{L} \otimes Q$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \overset{\text{ae}}{=} \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}Q}$.

Indeed, the set $\{\mathbb{G} \in \mathscr{B}_\infty \otimes \mathcal{A}; \nu(\mathbb{G}) = \int_{\mathbb{G}} \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}Q} \,\mathrm{d}\mu\}$ is a σ-algebra containing sets of type $B \times F$, where $B \in \mathscr{B}_\infty$ and $F \in \mathcal{A}$, where $\mathcal{A} \overset{\text{def}}{=} \mathrm{dom}(\mathbb{P}) = \mathrm{dom}(Q)$.

The following lemma says that any progressive process playing the role of the current density between two equivalent measures $\mathbb{P}, Q$ is just the density between the two measures introduced in Remark 3.25 w.r.t. the progressive σ-algebra.

Lemma 3.26. *Let $\mathscr{G}$ be a subfiltration of $\mathscr{F}$ and let Q be a probability measure equivalent to $\mathbb{P}$ and let $0 \leq \hat{\mathfrak{D}} \in \mathbb{L}(\mathcal{M}(\mathscr{G}))$ be such that*

$$\hat{\mathfrak{D}}_t \overset{\text{as}}{=} \frac{\mathrm{d}\mathbb{P}|\mathscr{G}_t}{\mathrm{d}Q|\mathscr{G}_t}, \quad t \geq 0. \tag{3.22}$$

Then

$$\hat{\mathfrak{D}} \overset{\text{ae}}{=} \frac{\mathrm{d}(\mathfrak{L}\otimes\mathbb{P})|\mathcal{M}(\mathscr{G})}{\mathrm{d}(\mathfrak{L}\otimes Q)|\mathcal{M}(\mathscr{G})}. \tag{3.23}$$

Proof. See Subsection 4.6 in section Proofs. □

Part (ii) of the following lemma says how the problem of seeking for a progressive projection transforms itself if we switch from one probability measure to an equivalent one.

Lemma 3.27. *Let $\mathscr{G}$ be a subfiltration of $\mathscr{F}$ and let Q be a probability measure equivalent to $\mathbb{P}$ with $\mathfrak{D} = \frac{\mathrm{d}\mathbb{P}|\mathscr{F}_\infty}{\mathrm{d}Q|\mathscr{F}_\infty}$. Put $\nu \overset{\text{def}}{=} \mathfrak{L} \otimes \mathbb{P}$ and $\mu \overset{\text{def}}{=} \mathfrak{L} \otimes Q$. (i) Then $\mathfrak{D}$ also plays the role of $\frac{\mathrm{d}\nu|\mathscr{B}_\infty\otimes\mathscr{F}_\infty}{\mathrm{d}\mu|\mathscr{B}_\infty\otimes\mathscr{F}_\infty}$ and*

$$\hat{\mathfrak{D}} \overset{\text{def}}{=} \frac{\mathrm{d}\nu|\mathcal{M}(\mathscr{G})}{\mathrm{d}\mu|\mathcal{M}(\mathscr{G})} = \frac{\mathrm{d}(\mathfrak{L}\otimes\mathbb{P})|\mathcal{M}(\mathscr{G})}{\mathrm{d}(\mathfrak{L}\otimes Q)|\mathcal{M}(\mathscr{G})} \tag{3.24}$$

is a μ-progressive projection of $\mathfrak{D}$ to $\mathscr{G}$.
(ii) If $H \in \mathcal{M}^1(\mathscr{F})$, then there exists its ν-progressive projection $\mathcal{P}^{\mathbb{P}}_{\mathscr{G}}(H)$ to $\mathscr{G}$ if and only if there exists μ-progressive projection $\mathcal{P}^{Q}_{\mathscr{G}}(\mathfrak{D}H)$ of $\mathfrak{D}H$ to $\mathscr{G}$, and if they exist, then

$$\hat{\mathfrak{D}}\,\mathcal{P}^{\mathbb{P}}_{\mathscr{G}}(H) \overset{\text{ae}}{=} \mathcal{P}^{Q}_{\mathscr{G}}(\mathfrak{D}H).$$

In particular, as $Q \sim \mathbb{P}$, we have that $\mathbb{\mu}|\mathcal{M}(\mathscr{G}) \sim \mathbb{\nu}|\mathcal{M}(\mathscr{G})$ ensuring that there exists a positive version of $\hat{\mathfrak{D}}$, and then we get that $\mathcal{P}^{\mathbb{P}}_{\mathscr{G}}(H) = \mathcal{P}^{Q}_{\mathscr{G}}(\mathfrak{D}H)/\hat{\mathfrak{D}}$ holds $\mathbb{\mu}$-a.e.

Proof. See Subsection 4.7 in section Proofs. □

The following proposition says how the progressive projection can be actually computed in a special model.

Proposition 3.28. *Let $(\mathsf{D}, \mathcal{D})$ be a measurable space, $(\Omega, \mathcal{A}, \mathbb{P}, \mathscr{F})$ be a filtered probability space. Let $\mathscr{G}$ be a subfiltration of $\mathscr{F}$ and Q be a probability measure equivalent with $\mathbb{P}$. Let $Y \in \mathbb{L}(\mathscr{F}_0, \mathcal{D})$ be Q-independent of $\mathscr{G}_\infty$ and $0 < \mathscr{E} \in \mathbb{L}(\mathcal{M}(\mathscr{G}) \otimes \mathcal{D})$ be such that*

$$\mathscr{E}^{(Y)}_t \overset{\text{as}}{=} \frac{\mathrm{d}\mathbb{P}|\mathscr{F}_t}{\mathrm{d}Q|\mathscr{F}_t}, \quad t \geq 0. \tag{3.25}$$

Then there exists a $\mathfrak{L} \otimes Q$-progressive projection $\hat{\mathscr{E}} > 0$ of $\mathscr{E}^{(Y)}$ to $\mathscr{G}$ such that

$$\hat{\mathscr{E}} \overset{\mathbb{ae}}{=} \int \mathscr{E}^{(y)} \mathrm{d}Q_Y(y).$$

(i) If $\mathbb{h} \in \mathbb{L}(\mathcal{M}(\mathscr{G}) \otimes \mathcal{D})$ is such that $H \overset{\text{def}}{=} \mathbb{h}(Y) \in \mathcal{M}^1(\mathscr{F})$ has a $\mathbb{\nu}$-progressive projection $\hat{H}$ to $\mathscr{G}$, where $\mathbb{\nu} \overset{\text{def}}{=} \mathfrak{L} \otimes \mathbb{P}$, then

$$\hat{H} \overset{\mathbb{ae}}{=} \int \mathbb{h}(y) \mathscr{E}^{(y)} \mathrm{d}Q_Y(y)/\hat{\mathscr{E}}. \tag{3.26}$$

In particular, the integral in (3.26) on the right is well defined and finite $\mathbb{\nu}$-a.e.

Proof. See Subsection 4.8 in section Proofs. □

3.2. **Relation between the progressive and the predictable projection.** First of all, we consider the definition of the (extended) predictable projection from [12, Theorem I.2.28]. We have to point out that the point (b) of the theorem simply does not hold as the process X there does not have to be adapted to the considered filtration. Moreover, the proof of the theorem contains a gap which has to be fixed. Finally, the corresponding filtration is assumed to be right-continous there, which is something that we do not assume here. These several reasons lead us to the conclusion that we have to re-prove its essential part.

Definition 3.29. Let $\mathscr{F}$ be a filtration on a measurable space $(\Omega, \mathcal{A})$, then every element of $\mathbb{L}(\mathfrak{P}, \mathcal{B}[-\infty, \infty])$ will be called an *$\mathscr{F}$-predictable process*, where $\mathfrak{P}$ is the smallest σ-algebra on Ω_∞ such that $Y \in \mathbb{L}(\mathfrak{P})$ holds whenever Y is a left-continous $\mathscr{F}$-adapted process and where $\mathcal{B}[-\infty, \infty]$ stands for the Borel σ-algebra on $[-\infty, \infty]$, similarly as $\mathcal{B}(-\infty, \infty]$ will stand for the Borel σ-algebra on $(-\infty, \infty]$ later on. A random *time* $\tau : \Omega \to [0, \infty]$ is called *$\mathscr{F}$-predictable* if $(1_{[\tau \leq t]})_{t \geq 0}$ is an $\mathscr{F}$-predictable process.

Theorem 3.30. *Let $(\Omega, \mathcal{A}, \mathbb{P}, \mathscr{F})$ be a probability space and $X \in \mathbb{L}(\mathscr{B}_\infty \otimes \mathcal{A}, \mathcal{B}[-\infty, \infty])$. Then there exists ${}^pX \in \mathbb{L}(\mathscr{B}_\infty \otimes \mathcal{A}, \mathcal{B}[-\infty, \infty])$, called the $\mathscr{F}$-predictable projection of X, such that*
(i) ${}^pX$ is $\mathscr{F}$-predictable,
(ii) for all $\mathscr{F}$-predictable times τ

$$({}^pX)_\tau 1_{[\tau < \infty]} \overset{\text{as}}{=} \lim_{m \to -\infty} \lim_{n \to \infty} \mathbb{E}[n \wedge X_\tau \vee m; \tau < \infty | \mathscr{F}_{\tau-}], \tag{3.27}$$

where $\mathscr{F}_{\tau-} \overset{\text{def}}{=} \mathscr{F}_0 \vee \sigma\{A \cap [t < \tau]; A \in \mathscr{F}_t, t \in [0, \infty)\} \subseteq \mathscr{F}_\tau$.

Proof. See Subsection 4.9 in the Section Proofs. □

We also have to re-prove the uniqueness of the predictable projection, see Remark 3.32, and for this purpose, we need the point (iii) of the following lemma which uses the notation $\mathscr{F}^+ \overset{\text{def}}{=} (\mathscr{F}_{t+})_{t \geq 0}$ if $\mathscr{F}$ is a filtration. Note that then the filtration $\mathscr{F}^+$ is already right-continous.

Lemma 3.31. *(i) If τ is an $\mathscr{F}^+$-predictable time, then $\tilde{\tau} \overset{\text{def}}{=} \tau + \infty 1_{[\tau=0]}$ is an $\mathscr{F}$-predictable time.*
(ii) Let X be an $\mathscr{F}^+$-predictable process, then $(X_t 1_{[t>0]})_{t\geq 0}$ is an $\mathscr{F}$-predictable process.
(iii) Let X be an $\mathscr{F}$-predictable process such that $X_\tau 1_{[\tau<\infty]} \overset{\text{as}}{=} 0$ holds whenever τ is an $\mathscr{F}$-predictable time. Then $X \overset{\text{as}}{=} 0$.

Proof. See Subsection 4.10 in the Section Proofs. □

Remark 3.32. If ${}^pX, {}^p\bar{X}$ are two $\mathscr{F}$-predictable projections of X, then $Y^{(n)} \overset{\text{def}}{=} (-n) \vee {}^pX \wedge n - ((-n) \vee {}^p\bar{X} \wedge n)$ are $\mathscr{F}$-predictable processes such that $Y^{(n)}_\tau 1_{[\tau<\infty]} \overset{\text{as}}{=} 0$ whenever τ is an $\mathscr{F}$-predictable time, and it gives us by Lemma 3.31 (iii) that $Y^{(n)} \overset{\text{as}}{=} 0, n \in \mathbb{N}$. Hence, we have that ${}^pX \overset{\text{as}}{=} {}^p\bar{X}$.

Finally, we can decide whether a process has a progressive projection, depending on the predictable projection of its absolute value, and in the positive case, the progressive projection can be easily obtained from the predictable projection.

Lemma 3.33. *Let $(\Omega, \mathcal{A}, \mathbb{P}, \mathscr{G})$ be a filtered probability space. A process $H \in \mathbb{L}(\mathscr{B}_\infty \otimes \mathcal{A})$ has a $\mathscr{G}$-progressive projection if and only if the $\mathscr{G}$-predictable projection of $|H|$ is finite ν-a.e., and in that case any $\mathscr{G}$-predictable projection $\widehat{H}$ of H is finite ν-a.e. and $\Re(\widehat{H})$ plays the role of the $\mathscr{G}$-progressive projection, where $\Re(x) \overset{\text{def}}{=} x 1_{[x\in\mathbb{R}]}$ if $x \in [-\infty, \infty]$.*

Proof. See Subsection 4.11 in the Section Proofs. □

Corollary 3.34. *Let $(\Omega, \mathcal{A}, \mathbb{P}, \mathscr{G})$ be a filtered probability space and let $H \in \mathbb{L}(\mathscr{B}_\infty \otimes \mathcal{A})$ have a $\mathscr{G}$-progressive projection. If H has a finite $\mathscr{G}$-predictable projection $\widehat{H}$, then $\widehat{H}$ is also a $\mathscr{G}$-progressive projection of H.*

Example 3.35. There exists a filtered probability space $(\Omega, \mathcal{A}, \mathbb{P}, \mathscr{G})$ and $H \in \mathbb{L}(\mathscr{B}_\infty \otimes \mathcal{A})$, which has a $\mathscr{G}$-progressive projection, such that $\widehat{H} \neq \hat{H}$ almost surely whenever $\widehat{H}$ and $\hat{H}$ are a $\mathscr{G}$-predictable projection and a $\mathscr{G}$-progressive projection of H, respectively.

Let Y be the canonical random variable on $\Omega \overset{\text{def}}{=} \mathbb{N}$ and consider $H \overset{\text{def}}{=} (Y 1_{[t=0]})_{t\geq 0}$,

$$\mathbb{P}(A) \overset{\text{def}}{=} \textstyle\sum_{n\in A} \frac{1}{n(n+1)} \quad \text{if} \quad A \subseteq \mathbb{N}, \quad \text{and} \quad \mathcal{A} \overset{\text{def}}{=} \operatorname{dom}(\mathbb{P}), \quad \mathscr{G}_t \overset{\text{def}}{=} \{\varnothing, \Omega\},\ t \in [0, \infty).$$

The process H has a $\mathscr{G}$-progressive projection 0 and a $\mathscr{G}$-predictable projection of the form $\widehat{H} \overset{\text{def}}{=} (\infty 1_{[t=0]})_{t\geq 0}$. Then if $\hat{H}$ is any of its $\mathscr{G}$-progressive projections, then $\widehat{H}_0 = \infty \neq \hat{H}_0 \in \mathbb{R}$ almost surely, hence $\mathbb{P}(\widehat{H} \neq \hat{H}) = 1$ and the same conclusion holds if $\widehat{H}$ is any $\mathscr{G}$-predictable projection by Remark 3.32.

3.3. **Trading in a filtered market.** At the beginning of this section, we considered an investor who wants to maximize the long run growth rate of the wealth process, which means to keep the proportion of the wealth process invested in the i-th asset equal to $\theta^{(i)}$ if $i = 1, \ldots, m$, where θ is the log-optimal proportion introduced in Definition 3.12.

The practical problem is that the we do not know the value of the log-optimal proportion. So, we assume that we have some a priori information and that we obtain additional information from the market while also assuming that we have access to no information from any other source.

Let $\hat{\mathcal{S}} = (\mathcal{S}^{(0)}, \mathcal{S})$ be a regular market on a filtered probability space $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$. Put

$$\hat{\mathscr{F}} \overset{\text{def}}{=} \mathscr{F}^{\hat{\mathcal{S}}}, \quad \hat{\mathcal{F}} \overset{\text{def}}{=} \mathcal{F}^{\hat{\mathcal{S}}} = \mathcal{F}^{\hat{\mathscr{F}}, \hat{\mathscr{F}}}, \quad \widehat{\mathcal{F}} \overset{\text{def}}{=} \mathcal{F}^{\hat{\mathscr{F}}, \mathcal{F}} = \mathcal{F}^{\hat{\mathscr{F}}, \widehat{\mathcal{F}}} = \mathcal{F}^{\hat{\mathcal{F}}, \widehat{\mathcal{F}}}. \tag{3.28}$$

Note that the last two equalities in (3.28) can be verified as follows. The first of them follows from the point (2) of Lemma A.15 which says that $\mathscr{N}^{\mathcal{F},\mathbb{P}}_\infty = \mathscr{N}^{\widehat{\mathcal{F}},\mathbb{P}}_\infty$, and as $\hat{\mathscr{F}}$ is a subfiltration of $\widehat{\mathcal{F}}$, we get first that $\mathscr{N}_\infty^{\hat{\mathscr{F}},\mathbb{P}} \subseteq \mathscr{N}_\infty^{\widehat{\mathcal{F}},\mathbb{P}}$ and then also that

$$\mathcal{F}_t^{\hat{\mathcal{F}},\widehat{\mathcal{F}}} = \hat{\mathscr{F}}_t \vee \sigma(\mathscr{N}_\infty^{\hat{\mathscr{F}},\mathbb{P}}) \vee \sigma(\mathscr{N}_\infty^{\widehat{\mathcal{F}},\mathbb{P}}) = \hat{\mathscr{F}}_t \vee \sigma(\mathscr{N}_\infty^{\widehat{\mathcal{F}},\mathbb{P}}) = \mathcal{F}_t^{\hat{\mathscr{F}},\widehat{\mathcal{F}}}, \quad t \in [0,\infty). \tag{3.29}$$

Remark 3.36. From (3.28) and from (2.11) in Remark 2.10, we get that

$$\mathbb{CM}_{loc}(\hat{\mathcal{F}}) = \mathbb{CM}_{loc}(\widehat{\mathcal{F}}) \cap \mathbb{CA}(\hat{\mathcal{F}}), \quad \mathbb{CS}(\hat{\mathcal{F}}) \subseteq \mathbb{CS}(\widehat{\mathcal{F}}).$$

The following lemma clarifies the relationship between $\mathbb{L}(\mathcal{M}(\hat{\mathcal{F}}))$ and $\mathbb{L}(\mathcal{M}(\widehat{\mathcal{F}}))$, and between $\mathcal{X}(\hat{\mathcal{F}})$ and $\mathcal{X}(\widehat{\mathcal{F}})$ if we use the notation introduced just above (Q) at the beginning of the paper.

Lemma 3.37. *(i) If* $\mathbb{CS}(\hat{\mathcal{F}}) \ni \hat{X} \overset{\text{as}}{=} \widehat{X} \in \mathbb{CA}(\mathcal{F})$, *then* $\widehat{X} \in \mathbb{CS}(\widehat{\mathcal{F}})$.
(ii) If $X \in \mathbb{CS}(\widehat{\mathcal{F}})$ *is such that*

$$\mathbb{CA}(\hat{\mathcal{F}}) \ni X = M + \int H_u \mathrm{d}u, \quad \textit{where} \quad M \in \mathbb{CM}_{loc}(\widehat{\mathcal{F}}), H \in \mathcal{M}^1(\widehat{\mathcal{F}}),$$

then there are $\hat{M} \in \mathbb{CM}_{loc}(\hat{\mathcal{F}})$ *and* $\hat{H} \in \mathcal{M}^1(\hat{\mathcal{F}})$ *such that* $X = \hat{M} + \int \hat{H}_u \mathrm{d}u \in \mathbb{CS}(\hat{\mathcal{F}})$.
(iii) Let $(\mathsf{S}, \mathfrak{S})$ *be a measurable space and* $H \in \mathbb{L}(\mathcal{M}(\widehat{\mathcal{F}}) \otimes \mathfrak{S})$, *then there exist*

$$\mathbb{H} \in \mathbb{L}(\mathcal{M}(\hat{\mathscr{F}}) \otimes \mathfrak{S}), \ N \in \mathscr{N}_\infty^{\mathcal{F},\mathbb{P}} \quad \textit{such that} \quad 0 = 1_{\Omega \setminus N} \int_0^\infty 1_{[\mathbb{H}_t \neq H_t]} \,\mathrm{d}t. \tag{3.30}$$

(iv) The point (iii) holds also if we omit "$\otimes \mathfrak{S}$*" in both its occurrences there. In particular, if* $\widehat{\mathbb{G}} \in \mathcal{M}(\widehat{\mathcal{F}})$, *then there exists* $\mathbb{G} \in \mathcal{M}(\hat{\mathscr{F}})$ *such that* $1_{\mathbb{G}} \overset{\text{ae}}{=} 1_{\widehat{\mathbb{G}}}$.

Proof. See Subsection 4.12 in section Proofs. □

The next lemma clarifies the relationship between progressive projections to $\hat{\mathcal{F}}$ and to $\widehat{\mathcal{F}}$.

Lemma 3.38. *(i) If* $H \in \mathbb{L}(\mathcal{M}(\mathcal{F}))$ *has a* ν*-progressive projection* $\hat{H}$ *to* $\hat{\mathcal{F}}$, *then* $\hat{H}$ *is also a* ν*-progressive projection of* H *to* $\widehat{\mathcal{F}}$.
(ii) If $H \in \mathbb{L}(\mathcal{M}(\mathcal{F}))$ *has a* ν*-progressive projection* $\widehat{H}$ *to* $\widehat{\mathcal{F}}$, *then* H *has also a* ν*-progressive projection* $\hat{H}$ *to* $\hat{\mathcal{F}}$ *such that* $\hat{H} \overset{\text{ae}}{=} \widehat{H}$.

Proof. See Subsection 4.13 in section Proofs. □

The following lemma says that we may more or less assume that certain coefficients of the regular market are progressive w.r.t. $\hat{\mathcal{F}}$ or $\widehat{\mathcal{F}}$ depending on what we may afford to neglect. They are the coefficients that are essentially observed in the market, namely the interest rate and $\sigma\sigma^\intercal$, where σ is the volatility matrix. Note that if the volatility matrix were assumed to be positive definite, we could (more or less) assume that it is progressive w.r.t. $\hat{\mathcal{F}}$ or $\widehat{\mathcal{F}}$ as well. In the statement and the proof of the following lemma, we write $\overset{\text{ae}}{=}$ for equality almost everywhere w.r.t. the Lebesgue measure on $[0,\infty)$.

Lemma 3.39. *Let* $\hat{\mathcal{S}} = (\mathcal{S}^{(0)}, \mathcal{S})$ *be a regular market on a filtered probability space* $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$. *Then there are* $\tilde{\alpha}^{(0)} \in \tilde{\mathcal{M}}^1(\hat{\mathcal{F}}), \widehat{\alpha}^{(0)} \in \mathcal{M}^1(\widehat{\mathcal{F}})$ *and* $0 < \tilde{\Sigma} \in \tilde{\mathcal{M}}^1(\hat{\mathcal{F}})^{m\times m}$, $0 < \widehat{\Sigma} \in \mathcal{M}^1(\widehat{\mathcal{F}})^{m\times m}$ *such that*

$$\tilde{\alpha}^{(0)} \overset{\text{as}}{=} \widehat{\alpha}^{(0)} \overset{ae}{=} \alpha^{(0)} \quad \textit{and} \quad \tilde{\Sigma} \overset{\text{as}}{=} \widehat{\Sigma} \overset{ae}{=} \Sigma \overset{\text{def}}{=\!=} \sigma\sigma^\intercal.$$

In particular, there exist $\hat{\alpha}^{(0)} \in \mathcal{M}^1(\hat{\mathcal{F}})$ *and* $0 < \hat{\Sigma} \in \mathcal{M}^1(\hat{\mathcal{F}})^{m\times m}$ *such that* $\hat{\alpha}^{(0)} \overset{\text{ae}}{=} \alpha^{(0)}$ *and* $\hat{\Sigma} \overset{\text{ae}}{=} \Sigma$.

Proof. See Subsection 4.14 in section Proofs. □

In order to be able to follow the log-optimal strategy corresponding to the information coming just from the market, we need to study the progressive projection of the rates or return of the risky assets that are not observed directly from the market.

Definition 3.40. A regular market $(\mathcal{S}^{(0)}, \mathcal{S})$ from Definition 3.6 is called *filterable* if

$$(\textstyle\int_{\mathbb{F}} \sum_{i=1}^{m} |\alpha^{(i)}| \, \mathrm{d}\nu)_{\mathbb{F} \in \mathcal{M}(\hat{\mathcal{F}})} \quad \text{is a } \sigma\text{-finite measure, where} \quad \nu \overset{\text{def}}{=} \mathfrak{L} \otimes \mathbb{P}. \tag{3.31}$$

It can be easily shown with the help of Lemma 3.33 that that (3.31) holds if and only if the process $\|\alpha\|$ has an $\hat{\mathcal{F}}$-predictable projection finite ν-almost everywhere.

Remark 3.41. A regular market $(\mathcal{S}^{(0)}, \mathcal{S})$ is filterable for example if there exists $0 \le a \in \mathbb{L}(\mathcal{M}(\hat{\mathcal{F}}))$ dominating $\sum_i |\alpha^{(i)}|$ on $\mathcal{M}(\hat{\mathcal{F}})$ in the following way

$$\textstyle\int_{\mathbb{F}} \sum_{i=1}^{m} |\alpha^{(i)}| \, \mathrm{d}\nu \le \int_{\mathbb{F}} a \, \mathrm{d}\nu, \quad \mathbb{F} \in \mathcal{M}(\hat{\mathcal{F}}), \tag{3.32}$$

since then the measure from (3.31) of $\mathbb{F}_n \overset{\text{def}}{=} \{(t, \omega) \in \Omega_n; |a_t(\omega)| \le n\} \in \mathcal{M}(\hat{\mathcal{F}})$ is not greater than $n^2 < \infty$ if $n \in \mathbb{N}$ and these sets unite to $\cup_n \mathbb{F}_n = \Omega_\infty$. The market is also filterable, for example, if $\sum_{i=1}^{m} |\alpha^{(i)}|$ is an integrable process, cf. the text just below (3.17) in Notation 3.19.

Definition 3.42. Let $(\mathcal{S}^{(0)}, \mathcal{S})$ be a filterable market and let $\hat{\alpha}^{(0)}, \hat{\Sigma}$ be as in Lemma 3.39. Then for each $i = 1, \ldots, m$ there exists a ν-progressive projection $\hat{\alpha}^{(i)}$ to $\hat{\mathcal{F}}$ and the corresponding m-dimensional process

$$\hat{\alpha} \overset{\text{def}}{=} (\hat{\alpha}^{(i)})_{i=1}^{m} \in \mathbb{L}(\mathcal{M}(\hat{\mathcal{F}}))^m \tag{3.33}$$

will be called a *ν-progressive projection* of α to $\hat{\mathcal{F}}$ and the m-dimensional process

$$\hat{\theta} \overset{\text{def}}{=} \hat{\Sigma}^{-1}(\hat{\alpha} - \hat{\alpha}^{(0)} 1_m) \in \mathbb{L}(\mathcal{M}(\hat{\mathcal{F}}))^m, \tag{3.34}$$

cf. (3.8), will be called an *$\hat{\mathcal{F}}$-log-optimal proportion* in the market $\hat{\mathcal{S}}$. Note that

$$\hat{\alpha} \overset{\text{ae}}{=} \alpha^{(0)} 1_m + \Sigma \hat{\theta}.$$

Definition 3.43. A trading strategy $(\varphi^{(0)}, \varphi)$ is said to be *$\hat{\mathcal{F}}$-progressive* if $\{\varphi^{(i)}\}_{i=0}^{m} \subseteq \mathbb{L}(\mathcal{M}(\hat{\mathcal{F}}))$. An $\hat{\mathcal{F}}$-progressive self-financing trading strategy with a positive wealth process is called *$\hat{\mathcal{F}}$-log-optimal* if its position process π is equal to the $\hat{\mathcal{F}}$-log-optimal proportion $\hat{\theta}$ from (3.34) ν-almost everywhere.

In the following proposition, we show that the self-financing strategy keeping the position on the projection of the log-optimal proportion is asymptotically optimal within the set of $\hat{\mathcal{F}}$-progressive self-financing strategies. The reason behind this is that the projection of the log-optimal proportion plays the role of the log-optimal proportion w.r.t. a new model obtained by filtering the original one w.r.t. the filtration generated by the market, driven by so called innovation process introduced in Definition 3.45.

Proposition 3.44. *Let $(\mathcal{S}^{(0)}, \mathcal{S})$ be a filterable market. (i) Then $\mathcal{S}, V \in \mathbb{CS}(\hat{\mathcal{F}})^m$ and*

$$\mathbb{CS}(\widehat{\mathcal{F}})^m \ni \mathbb{V} \overset{\text{as}}{=} \hat{\mathbb{V}} \overset{\text{def}}{=} V - \textstyle\int \hat{\alpha}_t^{(0)} \mathrm{d}t 1_m \in \mathbb{CS}(\hat{\mathcal{F}})^m$$

where $\hat{\alpha}^{(0)} \in \mathcal{M}^1(\hat{\mathcal{F}})^m$ is as in Lemma 3.39.
(ii) Let $(\hat{\varphi}^{(0)}, \hat{\varphi})$ be an $\hat{\mathcal{F}}$-log-optimal trading strategy with the wealth process $\hat{\mathcal{W}}$, let $(\widehat{\varphi}^{(0)}, \widehat{\varphi})$ be another self-financing trading strategy adapted to the filtration $\hat{\mathcal{F}}$ with wealth process $\widehat{\mathcal{W}} > 0$. Then

$$\limsup_{t \to \infty} \tfrac{1}{t} \ln(\widehat{\mathcal{W}}_t / \hat{\mathcal{W}}_t) \overset{\text{as}}{\le} 0. \tag{3.35}$$

(iii) If $\sigma \in \mathcal{M}^2(\hat{\mathcal{F}})^{m \times m}$, there exists a standard m-dimensional $\hat{\mathcal{F}}$-Brownian motion $\hat{B}$ s.t.

$$V \overset{\text{as}}{=} \textstyle\int \hat{\alpha}_t \mathrm{d}t + \int \sigma \mathrm{d}\hat{B}, \tag{3.36}$$

cf. (3.7), where $\hat{\alpha} \in \mathcal{M}^1(\hat{\mathcal{F}})^m$ is a ν-progressive projection of α to $\hat{\mathcal{F}}$.

Proof. See Subsection 4.15 in section Proofs. □

Definition 3.45. The m-dimensional standard $\hat{\mathcal{F}}$-Brownian motion $\hat{B}$ from Proposition 3.44 point (iii) is called an *innovation process* (or *innovation standard $\hat{\mathcal{F}}$-Brownian motion*) in the market $(\mathcal{S}^{(0)}, \mathcal{S})$, cf. Definition 8.1.1 and Theorem 8.1.4 in [14].

Note that the results of this paper are going to serve primarily as the technical background for the forthcoming papers on robust filtering based on Bayesian approach.

4. Proofs

4.1. **Proof of Lemma 3.10.**

Proof. 1. As the position π is well defined by assumption, the wealth process $\mathcal{W}$ can attain only positive values. Then $\mathcal{W}^{-1} \in \mathbb{CA}(\mathcal{F}) \subseteq \mathbb{L}(\mathcal{M}(\mathcal{F}))$ has locally bounded trajectories. Therefore it is enough to verify that $\mathcal{W}\pi^{(i)}\alpha^{(i)}$ and $\mathcal{W}^2\pi^{\intercal}\Sigma\pi$ belong to $\tilde{\mathcal{M}}^1(\mathcal{F})$ whenever $i \in \{0, \dots, m\}$.

2. See (3.4) in order to agree that $D^{(i)}$ from Remark 3.5 is of the form

$$D_t^{(i)} \overset{\text{as}}{=} \int_0^t |\alpha_s^{(i)}| \mathcal{S}_s^{(i)} \, \mathrm{d}s, \quad t \geq 0, \; i = 0, \dots, m. \tag{4.1}$$

Since $\mathcal{W}\pi^{(i)} = \varphi^{(i)}\mathcal{S}^{(i)}$, we get from (3.3, 4.1) that

$$\int_0^t |\mathcal{W}_s \pi_s^{(i)} \alpha_s^{(i)}| \, \mathrm{d}s \overset{\text{as}}{=} \int_0^t |\alpha_s^{(i)} \varphi_s^{(i)}| \mathcal{S}_s^{(i)} \mathrm{d}s \overset{\text{as}}{=} \int_0^t |\varphi^{(i)}| \, \mathrm{d}D^{(i)} \overset{\text{as}}{<} \infty, \qquad t \in [0, \infty). \tag{4.2}$$

From (3.6) in Remark 3.9, we get that the tensor quadratic variation of $\mathcal{S}$ is of the form

$$\langle\!\langle \mathcal{S} \rangle\!\rangle \overset{\text{def}}{=} \big(\langle \mathcal{S}^{(i)}, \mathcal{S}^{(j)} \rangle \big)_{ij=1}^m \overset{\text{as}}{=} \int \operatorname{diag}(\mathcal{S}_t) \, \Sigma_t \operatorname{diag}(\mathcal{S}_t) \mathrm{d}t. \tag{4.3}$$

As $\mathcal{W}\pi^{(i)} = \varphi^{(i)}\mathcal{S}^{(i)}$, we get from (4.3) and from (3.3) in Remark 3.5 that

$$\int_0^t \mathcal{W}_s^2 \pi_s^{\intercal} \Sigma \pi_s \, \mathrm{d}s = \int_0^t \sum_{ij=1}^m \varphi^{(i)} \varphi^{(j)} \mathcal{S}_s^{(i)} \mathcal{S}_s^{(j)} \Sigma_s^{(i,j)} \mathrm{d}s \overset{\text{as}}{=} \int_0^t \operatorname{tr}[\varphi \varphi^{\intercal} \, \mathrm{d}\langle\!\langle \mathcal{S} \rangle\!\rangle] \overset{\text{as}}{<} \infty, \quad t \in [0, \infty). \tag{4.4}$$

Since the desired processes from the statement are $\mathcal{F}$-progressive and satisfy (4.2, 4.4), the rest of the proof follows by step 1. ☐

4.2. **Proof of Proposition 3.13.**

Proof. Let $\tilde{\pi}$ be the position of $(\tilde{\varphi}^{(0)}, \tilde{\varphi})$. Then the equalities in (3.11, 3.12) give that

$$\ln \frac{\tilde{\mathcal{W}}_t}{\mathcal{W}_t} - \ln \frac{\tilde{\mathcal{W}}_0}{\mathcal{W}_0} \overset{\text{as}}{=} \int_0^t (\tilde{\pi} - \theta)^{\intercal} \mathrm{d}\mathbb{V} - \frac{1}{2} \int_0^t (\tilde{\pi}_s^{\intercal} \Sigma_s \tilde{\pi}_s - \theta_s^{\intercal} \Sigma_s \theta_s) \, \mathrm{d}s \overset{\text{as}}{=} L_t - \frac{1}{2} \langle L \rangle_t, \tag{4.5}$$

where $L \overset{\text{def}}{=} \int (\tilde{\pi} - \theta)^{\intercal} \sigma \, \mathrm{d}B \in \mathbb{CM}_{loc}(\mathcal{F})$ starts from $L_0 = 0$. By Lemma 2.27 in [9],

$$\limsup_{t \to \infty} \frac{1}{t} (L_t - \frac{1}{2} \langle L \rangle_t) \overset{\text{as}}{\leq} 0 \tag{4.6}$$

and then (3.9) follows immediately from (4.5, 4.6). ☐

4.3. **Proof of Lemma 3.18.**

Proof. (i) First, let $\hat{f} \in \mathbb{L}(\mathcal{H})$ be a ν-projection of f to $\mathcal{H}$. Then, by definition, (3.13) holds ν-almost everywhere. If $H \in \mathcal{H}_f$, then we get that

$$\int_H \hat{f} \mathrm{d}\nu = \mu^{(+)}(H) - \mu^{(-)}(H) = \int_H f^+ \mathrm{d}\nu - \int_H f^- \mathrm{d}\nu = \int_H f \mathrm{d}\nu.$$

Let $\hat{f} \in \mathbb{L}(\mathcal{H})$ be such that (3.14) holds and let $\tilde{f}$ stand for the expression in (3.13) on the right. We are going to show that $\hat{f} = \tilde{f}$ holds ν-a.e., i.e., that

$$\nu(H^{(i)}) = 0, \quad i \in \{-1, 1\}, \quad \text{where} \quad H^{(i)} \overset{\text{def}}{=} \{x \in H; \operatorname{sign}\big[\hat{f}(x) - \tilde{f}(x)\big] = i\} \in \mathcal{H}. \tag{4.7}$$

By Definition 3.16 there exists $(H_n)_{n=1}^{\infty} \in \mathcal{H}_f^{\mathbb{N}}$ with $\cup_n H_n = T$ and without loss of generality, we may assume that $H_n \subseteq H_{n+1}$ holds if $n \in \mathbb{N}$. Then also $H_n^{(i)} \overset{\text{def}}{=} H_n \cap H^{(i)} \in \mathcal{H}_f$ and then we get from (3.14) and the Monotone Convergence Theorem that

$$\int_{H^{(i)}} (\hat{f} - \tilde{f})\, \mathrm{d}\nu = \lim_{n\to\infty} \left[\int_{H_n^{(i)}} \hat{f} \mathrm{d}\nu - \int_{H_n^{(i)}} \tilde{f} \mathrm{d}\nu\right] = 0. \tag{4.8}$$

Then we get from the definition of $H^{(i)}$ in (4.7) that $\nu(H^{(i)}) = 0$ holds if $i \in \{-1, 1\}$. Further in this point, we assume that (3.13) holds ν-almost everywhere. Then we get (3.15) as follows

$$\int_H |\hat{f}|\, \mathrm{d}\nu \le \int_H \left(\tfrac{\mathrm{d}\mu^{(+)}}{\mathrm{d}\nu|\mathcal{H}} + \tfrac{\mathrm{d}\mu^{(-)}}{\mathrm{d}\nu|\mathcal{H}}\right) \mathrm{d}\nu = \int_H f^+ \mathrm{d}\nu + \int_H f^- \mathrm{d}\nu = \int_H |f|\, \mathrm{d}\nu.$$

(ii) Since f, g have ν-projections to $\mathcal{H}$ by assumption, we immediately obtain from the triangle inequality that $(\int_H |f+g|\, \mathrm{d}\nu)_{H\in\mathcal{H}}$ is dominated by a sum of two σ-finite measures, and hence it is also a σ-finite measure. Then as $\hat{f} + \hat{g} \in \mathbb{L}(\mathcal{H})$, we get from (i) that (ii) holds since

$$\int_H (f+g) \mathrm{d}\nu = \int_H f\, \mathrm{d}\nu + \int_H g\, \mathrm{d}\nu = \int_H \hat{f}\, \mathrm{d}\nu + \int_H \hat{g}\, \mathrm{d}\nu = \int_H (\hat{f} + \hat{g})\, \mathrm{d}\nu, \quad H \in \mathcal{H}_f \cap \mathcal{H}_g.$$

The equality between both sides can be easily extended with the help of the Dominated and the Monotone Convergence Theorems to those $H \in \mathcal{H}_{f+g}$ such that $\operatorname{sign}(\hat{f} + \hat{g}) = i \in \{-1, 0, 1\}$ holds on H and then this last condition can be easily removed.

(iii) Similarly as in the point (i), consider $(H_n)_{n=1}^{\infty} \in (\mathcal{H}_f \cap \mathcal{H}_1)^{\mathbb{N}}$ with $\cup_n H_n = T$ s.t. $H_n \subseteq H_{n+1}$ holds if $n \in \mathbb{N}$. Then as $g \in \mathbb{L}(\mathcal{H})$ holds by assumption, we get that

$$G_{n+1} \supseteq G_n \overset{\text{def}}{=} \{x \in H_n; |g(x)| \le n\} \in \mathcal{H}_{fg} \cap \mathcal{H}_f \cap \mathcal{H}_1, \quad \cup_n G_n = \cup_n H_n = T. \tag{4.9}$$

In particular, $(\int_H |fg|\, \mathrm{d}\nu)_{H\in\mathcal{H}}$ is a σ-finite measure. If $n \in \mathbb{N}$, then there exists a probability measure $\mathbb{P}_n$ such that $\nu(H \cap G_n) = \nu(G_n)\mathbb{P}_n(H), H \in \mathcal{H}$. If $n \in \mathbb{N}$, we have by (4.9) that $G_n \in \mathcal{H}_f$ and hence we also have that $G_n \cap G \in \mathcal{H}_f$ holds if $G \in \mathcal{H}$. Then we get from (3.14) that

$$\nu(G_n)\, \mathbb{E}_n[\hat{f}; G] = \int_{G\cap G_n} \hat{f}\, \mathrm{d}\nu = \int_{G\cap G_n} f\, \mathrm{d}\nu = \nu(G_n)\, \mathbb{E}_n[f; G] \quad \text{if } G \in \mathcal{H}.$$

Hence, if $\nu(G_n) > 0$, then $f \in \mathbb{L}_1(\mathbb{P}_n)$ and $\hat{f} = \mathbb{E}_n[f|\mathcal{H}]$ holds $\mathbb{P}_n$-a.e. Then

$$\int_{G_n\cap H} \hat{f} g\, \mathrm{d}\nu = \nu(G_n)\mathbb{E}_n[g\hat{f}; H] = \nu(G_n)\mathbb{E}_n[g\, \mathbb{E}_n(f|\mathcal{H}); H] = \int_{H\cap G_n} fg\, \mathrm{d}\nu, \quad H \in \mathcal{H},$$

as $g \in \mathbb{L}(\mathcal{H})$ holds by assumption, and then the Monotone Convergence Theorem yields

$$\int_H fg\, \mathrm{d}\nu = \lim_{n\to\infty} \int_{H\cap G_n} fg\, \mathrm{d}\nu = \lim_{n\to\infty} \int_{H\cap G_n} \hat{f} g\, \mathrm{d}\nu = \int_H \hat{f} g\, \mathrm{d}\nu, \quad H \in \mathcal{H}^{(i)}, \tag{4.10}$$

holds if $i \in \{-1, 0, 1\}$, where $\mathcal{H}^{(i)} \overset{\text{def}}{=} \{H \in \mathcal{H}; \forall x \in H \operatorname{sign}[\hat{f}(x)g(x)] = i\}$. Since $\hat{f}g \in \mathbb{L}(\mathcal{H})$, we get that (4.10) holds with H replaced by $H^{(i)} \overset{\text{def}}{=} H \cap [\operatorname{sign}(\hat{f}g) = i] \in \mathcal{H}^{(i)}$ whenever $H \in \mathcal{H}_{fg}$. If we sum those equalities over $i \in \{-1, 0, 1\}$ we obtain (4.10) for any $H \in \mathcal{H}_{fg}$, and then it is sufficient to use point (i) in order to show that (iii) holds.

(iv) Since $\mathcal{K} \subseteq \mathcal{H}$, we have that $\mathcal{K}_f \overset{\text{def}}{=} \{K \in \mathcal{K}; \int_K |f|\, \mathrm{d}\nu < \infty\} \subseteq \mathcal{H}_f$. Further, as $\hat{f}$ is a ν-projection of f to $\mathcal{H}$, we get from (3.15) that $\mathcal{K}_f \subseteq \mathcal{K}_{\hat{f}}$. Since f has a ν-projection to $\mathcal{K}$, there exist $K_n \in \mathcal{K}_f \subseteq \mathcal{H}_f \cap \mathcal{K}_{\hat{f}}, n \in \mathbb{N}$, such that $\cup_{n=1}^{\infty} K_n = T$ and that $K_n \subseteq K_{n+1}$. Then if $\hat{K} \in \mathcal{K}_{\hat{f}}$, we have by (i), used twice, that

$$\int_{\hat{K}_n} \tilde{f} \mathrm{d}\nu = \int_{\hat{K}_n} \mathscr{P}^{\nu}_{\mathcal{K}}(f)\, \mathrm{d}\nu = \int_{\hat{K}_n} f \mathrm{d}\nu = \int_{\hat{K}_n} \mathscr{P}^{\nu}_{\mathcal{H}}(f)\, \mathrm{d}\nu = \int_{\hat{K}_n} \hat{f} \mathrm{d}\nu, \qquad \hat{K}_n \overset{\text{def}}{=} \hat{K} \cap K_n \in \mathcal{K}_f, \tag{4.11}$$

$n \in \mathbb{N}$. If we pass $n \to \infty$ in (4.11), we get that $\int_{\hat{K}} \tilde{f} \mathrm{d}\nu = \int_{\hat{K}} \hat{f} \mathrm{d}\nu, \hat{K} \in \mathcal{K}_{\hat{f}}$, by the Dominated Convergence Theorem and then we get by (i) that $\tilde{f}$ is a ν-projection of f to $\mathcal{K}$. This is correct once we have that $\mathcal{K}_{\hat{f}} \subseteq \mathcal{K}_{\tilde{f}}$ and it is what remains to show. Let $\hat{K} \in \mathcal{K}_{\hat{f}}$. If

$\hat{K} \subseteq [\tilde{f} \le 0]$ or $\hat{K} \subseteq [\tilde{f} \ge 0]$, then we may use the Monotone Convergence Theorem and (4.11) to get that $\int_{\hat{K}} \tilde{f} \mathrm{d}\nu = \int_{\hat{K}} \hat{f} \mathrm{d}\nu$. In general, we have this equality with $\hat{K}$ replaced by $\hat{K}^+ \overset{\text{def}}{=} \hat{K} \cap [\tilde{f} \ge 0] \in \mathcal{K}_{\hat{f}}$ and by $\hat{K}^- \overset{\text{def}}{=} \hat{K} \cap [\tilde{f} < 0] \in \mathcal{K}_{\hat{f}}$, respectively, which gives that $\int_{\hat{K}} |\tilde{f}| \mathrm{d}\nu = \int_{\hat{K}^+} \tilde{f} \mathrm{d}\nu - \int_{\hat{K}^-} \tilde{f} \mathrm{d}\nu = \int_{\hat{K}^+} \hat{f} \mathrm{d}\nu - \int_{\hat{K}^-} \hat{f} \mathrm{d}\nu \le 2 \int_{\hat{K}} |\hat{f}| \mathrm{d}\nu < \infty$, i.e., $\hat{K} \in \mathcal{K}_{\tilde{f}}$. □

4.4. Proof of Proposition 3.22.

Lemma 4.1. *Let $\mathcal{F}$ be an enriched filtration and let $(X^{(n)})_{n=1}^\infty \in \mathbb{CA}(\mathcal{F})^{\mathbb{N}}$ satisfy*

$$\lim_{i,n\to\infty} \mathbb{E} \sup_{s\le t} |X^{(n)}_{s\wedge\tau_m} - X^{(i)}_{s\wedge\tau_m}|^2 = 0 \tag{4.12}$$

for some non-decreasing sequence of random times $(\tau_m)_{m=1}^\infty \in [0,\infty]^{\mathbb{N}\times\Omega}$ tending to ∞. Then there exists $X \in \mathbb{CA}(\mathcal{F})$ such that (4.12) *holds with $X^{(i)}$ replaced by X.*

Proof. First, we will show that $(X^{(n)})_{n=1}^\infty$ is a $\mathbb{p}$-Cauchy sequence. Let $\varepsilon > 0$ be given and consider $k_0 \in \mathbb{N}$ such that $2^{-k_0} < \varepsilon$. Since $\tau_m \to \infty$ as $m \to \infty$, we get that there exists $m_0 \in \mathbb{N}$ such that $\mathbb{P}(\tau_m \le k_0) < \varepsilon$ holds whenever $m \ge m_0$. Then

$$\mathbb{E}[2^{-k} \wedge |X^{(n)} - X^{(i)}|^*_k] \le \mathbb{E}[2^{-k} \wedge \sup_{s\le k} |X^{(n)}_{s\wedge\tau_m} - X^{(i)}_{s\wedge\tau_m}|] + 2^{-k}\varepsilon, \quad i,n \in \mathbb{N},\ m \ge m_0,\ k \le k_0.$$

Then we obtain from the Dominated Convergence Theorem and (4.12) that

$$\mathbb{p}(X^{(n)}, X^{(i)}) \le 2\varepsilon + \sum_{k\in\mathbb{N}} 2^{-k} \wedge \sqrt{\mathbb{E} \sup_{s\le k} |X^{(n)}_{s\wedge\tau_{m_0}} - X^{(i)}_{s\wedge\tau_{m_0}}|^2} \to 2\varepsilon, \quad i,n \to \infty,$$

as $\sum_{k>k_0} 2^{-k} = 2^{-k_0} < \varepsilon$ and $\sum_{k=1}^{k_0} 2^{-k} < 1$. Since $\varepsilon > 0$ was arbitrary, we have that $(X^{(n)})_{n=1}^\infty$ is a $\mathbb{p}$-Cauchy sequence and hence $\mathbb{p}$-convergent as $\mathbb{p}$ is a complete metric by Proposition A.10. Then we get from Lemma A.16 (iv) that there exists $X \in \mathbb{CA}(\mathcal{F})$ such that $\lim_n \mathbb{p}(X^{(n)}, X) = 0$. By Remark A.3 we can select an absolutely $\mathbb{p}$-convergent subsequence of $(X^{(n)})_{n=1}^\infty$, say $(Y^{(n)})_{n=1}^\infty$, see Definition A.2. Then $\lim_n \mathbb{r}(Y^{(n)}, X) \overset{\text{as}}{=} 0$ holds by Corollary A.9 and then by Fatou Lemma

$$\mathbb{E} \sup_{s\le t} |X^{(n)}_{s\wedge\tau_m} - X_{s\wedge\tau_m}|^2 \le \liminf_{i\to\infty} \mathbb{E} \sup_{s\le t} |X^{(n)}_{s\wedge\tau_m} - Y^{(i)}_{s\wedge\tau_m}|^2 \to 0$$

as $n \to \infty$ whenever $m \in \mathbb{N}$. □

Remark 4.2. If $\mathcal{G}$ is a filtration and τ is a $\mathcal{G}$-stopping time, then $(1_{[t\le\tau]})_{t\ge0} \in \mathbb{A}(\mathcal{G})$ is a $\mathcal{G}$-progressive process, as it is left-continuous and adapted to $\mathcal{G}$. Hence, we have that

$$\{(t,\omega) \in \Omega_\infty; t \le \tau(\omega)\} \in \mathcal{M}(\mathcal{G}).$$

In the following proof, we use Notation 2.11 saying that the result of stochastic integration is assumed to be adapted to the considered enriched filtration without further remarks. In particular, we have from Lemma A.28 that if we integrate with respect to a continous local martingale, the result of the integration is again a continous local martingale.

Proof of Proposition 3.22. 1. First, we will assume that there exists a non-negative $\mathbb{v}$-progressive projection $\tilde{H}$ of H to $\mathcal{G}$ and we will show that there exists $\tilde{M} \in \mathbb{CM}_{loc}(\mathcal{G})$ s.t.

$$X \overset{\text{as}}{=} \tilde{M} + \int \tilde{H}_u \, \mathrm{d}u, \tag{4.13}$$

where the right-hand side is allowed to attain also the infinite value $+\infty$ at any $(t,\omega) \in \Omega_\infty$. As $X \in \mathbb{CS}(\mathcal{F}) \cap \mathbb{A}(\mathcal{G})$ and as $\mathcal{G}$ is an enriched filtration, we get that by Corollary A.17 that the corresponding quadratic variation has a $\mathcal{G}$-adapted version $\langle X\rangle \in \mathbb{CA}(\mathcal{G})$, and as $\langle X\rangle_0 \overset{\text{as}}{=} 0$, there exists a sequence of $\mathcal{G}$-stopping times $(\tau_m)_{m=1}^\infty$ tending to ∞ such that

$$\langle X\rangle_{t\wedge\tau_m} \overset{\text{as}}{\le} m, \quad t \in [0,\infty), \quad m \in \mathbb{N}.$$

We have that (3.17) holds by assumption and hence there exists a non-decreasing sequence $\mathbb{G}_n$, $n \in \mathbb{N}$, from $\mathcal{M}(\mathcal{G})_H \overset{\text{def}}{=} \{\mathbb{G} \in \mathcal{M}(\mathcal{G});\ \int_{\mathbb{G}} |H| \mathrm{d}\nu < \infty\}$ with $\cup_n \mathbb{G}_n = \Omega_\infty$. By (3.15) in Lemma 3.18 (i)

$$\int_{\mathbb{G}_n} |\tilde{H}|\,\mathrm{d}\nu \le \int_{\mathbb{G}_n} |H|\,\mathrm{d}\nu < \infty, \quad n \in \mathbb{N}. \tag{4.14}$$

Since $M \in \mathbb{CM}_{loc}(\mathcal{F})$ holds by assumption and as $\mathbb{G}_n \in \mathcal{M}(\mathcal{G}) \subseteq \mathcal{M}(\mathcal{F})$, we obtain from Corollary A.17 that

$$M^{(n)} \overset{\text{def}}{=} \int 1_{\mathbb{G}_n} \mathrm{d}M \in \mathbb{CM}_{loc}(\mathcal{F}), \qquad \text{and} \qquad \langle M^{(n)} \rangle_{t\wedge\tau_m} \overset{\text{as}}{\le} \langle X \rangle_{t\wedge\tau_m} \overset{\text{as}}{\le} m, \tag{4.15}$$

$$X^{(n)} \overset{\text{def}}{=} \int 1_{\mathbb{G}_n} \mathrm{d}X \in \mathbb{CS}(\mathcal{F}) \cap \mathbb{A}(\mathcal{G}), \quad \text{and} \quad \tilde{M}^{(n)} \overset{\text{def}}{=} X^{(n)} - \int 1_{\mathbb{G}_n} \tilde{H}\,\mathrm{d}\tilde{\mathfrak{L}} \in \mathbb{CA}(\mathcal{G}), \tag{4.16}$$

where $\tilde{\mathfrak{L}}_t \overset{\text{def}}{=} t, t \in [0,\infty)$. By Lemma A.27 $(M^{(n)}_{t\wedge\tau_m})_{t\ge0} \in \mathbb{CM}(\mathcal{F}), n, m \in \mathbb{N}$. We are going to show that also $(\tilde{M}^{(n)}_{t\wedge\tau_m})_{t\ge0} \in \mathbb{CM}(\mathcal{G})$. From the above stated martingale property, we immediately get that $M^{(n)}_{t\wedge\tau_m} \in \mathbb{L}_1$ and if we use it again together with (3.19, 4.15, 4.16), we obtain the equality in

$$\mathbb{E}|\tilde{M}^{(n)}_{t\wedge\tau_m} - M^{(n)}_{t\wedge\tau_m}| = \mathbb{E}|\int_0^{t\wedge\tau_m} 1_{\mathbb{G}_n}(H - \tilde{H})\,\mathrm{d}\mathfrak{L}| \le 2\int_{\mathbb{G}_n} |H|\,\mathrm{d}\nu < \infty,$$

while the inequalities follow from (4.14). Then we conclude that also $\tilde{M}^{(n)}_{t\wedge\tau_m} \in \mathbb{L}_1$ holds whenever $t \in [0,\infty)$ and $m, n \in \mathbb{N}$. Further, let $0 \le s < t < \infty$ and $G \in \mathcal{G}_s$. Then $\mathbb{G}^{s,t} \overset{\text{def}}{=} (s,t] \times G \in \mathcal{M}(\mathcal{G})$, and we obtain by Remark 4.2 that

$$\mathbb{G}^{s,t}_{n,m} \overset{\text{def}}{=} \mathbb{G}^{s,t} \cap \mathbb{G}_n \cap \{(u,\omega) \in \Omega_\infty; u \le \tau_m(\omega)\} \in \mathcal{M}(\mathcal{G}), \qquad \int_{\mathbb{G}^{s,t}_{n,m}} |H|\,\mathrm{d}\nu \le \int_{\mathbb{G}_n} |H|\,\mathrm{d}\nu < \infty.$$

As $G \in \mathcal{G}_s \subseteq \mathcal{F}_s$ and $(M^{(n)}_{u\wedge\tau_m})_{u\ge0} \in \mathbb{CM}(\mathcal{F})$, Lemma 3.18 (i) and (3.19, 4.16) give that

$$\mathbb{E}[\tilde{M}^{(n)}_{t\wedge\tau_m} - \tilde{M}^{(n)}_{s\wedge\tau_m}; G] = \mathbb{E}[\int_{s\wedge\tau_m}^{t\wedge\tau_m} 1_{\mathbb{G}_n}(H - \tilde{H})\,\mathrm{d}\mathfrak{L}; G] = \int_{\mathbb{G}^{s,t}_n} H\,\mathrm{d}\nu - \int_{\mathbb{G}^{s,t}_n} \tilde{H}\,\mathrm{d}\nu = 0,$$

which verifies that $(\tilde{M}^{(n)}_{t\wedge\tau_m})_{t\ge0} \in \mathbb{CM}(\mathcal{G})$ if $m, n \in \mathbb{N}$. As the system $\{\mathbb{G}_n\}_{n=1}^\infty$ is here considered to be non-decreasing, i.e., $\mathbb{G}_i \subseteq \mathbb{G}_n$ if $i \le n$, we get from the Doob inequality that

$$\mathbb{E}\sup_{s\le t} |\tilde{M}^{(n)}_{s\wedge\tau_m} - \tilde{M}^{(i)}_{s\wedge\tau_m}|^2 \le 4\mathbb{E}\langle \tilde{M}^{(n)} - \tilde{M}^{(i)} \rangle_{t\wedge\tau_m} \le 4\mathbb{E}\int_0^{t\wedge\tau_m} 1_{\Omega_\infty \setminus \mathbb{G}_i}\,\mathrm{d}\langle X \rangle, \ n \ge i \in \mathbb{N}, \tag{4.17}$$

holds if $m, n \in \mathbb{N}$ and $t \ge 0$. Then we get by the Dominated Convergence Theorem and (4.15) that the expression in (4.17) on the right tends to zero as $i \to \infty$ whenever $m \in \mathbb{N}$ and $t \ge 0$. Further, we obtain from (4.17) and from Lemma 4.1 that there exists a process $\tilde{M} \in \mathbb{CA}(\mathcal{G})$ such that

$$\lim_{n\to\infty} \mathbb{E}\sup_{s\le t} |\tilde{M}^{(n)}_{s\wedge\tau_m} - \tilde{M}_{s\wedge\tau_m}|^2 = 0, \qquad m \in \mathbb{N}, \ t \in [0,\infty). \tag{4.18}$$

Since the process $(\tilde{M}_{t\wedge\tau_m})_{t\ge0} \in \mathbb{CA}(\mathcal{G})$ is a pointwise $\mathbb{L}_1$-limit of martingales $(\tilde{M}^{(n)}_{t\wedge\tau_m})_{t\ge0} \in \mathbb{CM}(\mathcal{G})$, it is also a $\mathcal{G}$-martingale, and we get that $\tilde{M} \in \mathbb{CM}_{loc}(\mathcal{G})$. Further, since we assume that $\tilde{H} \ge 0$ in this step of the proof, we get from the Monotone Convergence Theorem the first equality in

$$\int_0^{t\wedge\tau_m} \tilde{H}\,\mathrm{d}\mathfrak{L} = \lim_{n\to\infty} \int_0^{t\wedge\tau_m} 1_{\mathbb{G}_n} \tilde{H}\,\mathrm{d}\mathfrak{L} \overset{\text{as}}{=} \mathbb{P}\text{-}\lim_{n\to\infty} (X^{(n)}_{t\wedge\tau_m} - \tilde{M}^{(n)}_{t\wedge\tau_m}) \overset{\text{as}}{=} X_{t\wedge\tau_m} - \tilde{M}_{t\wedge\tau_m}, \tag{4.19}$$

$m \in \mathbb{N}$. The last equality follows from the properties of continuous integration and from (4.16,4.18). The middle equality in (4.19) follows from the definition of $\tilde{M}^{(n)}$ in (4.16). If we let $m \to \infty$ in (4.19), we obtain the equality (4.13).

2. In general, the assumptions of the proposition ensure that there exists a ν-progressive projection $\tilde{H}$ of H to $\mathcal{G}$. We will show that $\tilde{H}^+, \tilde{H}^- \ge 0$ are ν-progressive projections of

$$H^{(+)} \overset{\text{def}}{=} H 1_{[\tilde{H}>0]} \qquad \text{and} \qquad H^{(-)} \overset{\text{def}}{=} -H 1_{[\tilde{H}\le0]},$$

respectively. As $\tilde{H}$ is a ν-progressive projection of H to $\mathcal{G}$, Lemma 3.18 (i) first gives that

$$\int_{\mathbb{G}} \tilde{H}^+ \mathrm{d}\nu = \int_{\mathbb{G}^+} \tilde{H} \mathrm{d}\nu = \int_{\mathbb{G}^+} H \mathrm{d}\nu = \int_{\mathbb{G}} H^{(+)} \mathrm{d}\nu, \quad \mathbb{G} \in \mathcal{M}(\mathcal{G})_H, \tag{4.20}$$

holds with $\mathbb{G}^+ \overset{\text{def}}{=} \{\omega \in \mathbb{G}; \tilde{H}(\omega) > 0\}$ and further that $\tilde{H}^+$ is a ν-progressive projection of $H^{(+)}$ to $\mathcal{G}$ as the measure $(\int_{\mathbb{G}} |H^{(+)}| \,\mathrm{d}\nu)_{\mathbb{G}\in\mathcal{M}(\mathcal{G})}$ is dominated by a σ-finite measure

$$(\int_{\mathbb{G}} |H| \,\mathrm{d}\nu)_{\mathbb{G}\in\mathcal{M}(\mathcal{G})}.$$

As the progressive projection is linear by Lemma 3.18 (ii,iii), we get that also $\tilde{H}^- = \tilde{H}^+ - \tilde{H} \in \mathbb{L}(\mathcal{M}(\mathcal{G}))$ is a ν-progressive projection of $H^{(-)} = H^{(+)} - H$ to $\mathcal{G}$ as follows

$$\mathcal{P}^{\mathbb{P}}_{\mathcal{G}}(H^{(-)}) \overset{\text{ae}}{=} \mathcal{P}^{\mathbb{P}}_{\mathcal{G}}(H^{(+)} - H) \overset{\text{ae}}{=} \mathcal{P}^{\mathbb{P}}_{\mathcal{G}}(H^{(+)}) - \mathcal{P}^{\mathbb{P}}_{\mathcal{G}}(H) \overset{\text{ae}}{=} \tilde{H}^+ - \tilde{H} = \tilde{H}^-.$$

Hence, the assumptions of the first part of the proof are satisfied with $(X, M, H, \tilde{H})$ replaced by $(X^{(+)}, M^{(+)}, H^{(+)}, \tilde{H}^+)$ and the same holds with '+' replaced by '−', where

$$X^{(+)} \overset{\text{def}}{=} \int 1_{[\tilde{H}>0]} \,\mathrm{d}X \in \mathbb{CA}(\mathcal{G}), \quad M^{(+)} \overset{\text{def}}{=} \int 1_{[\tilde{H}>0]} \,\mathrm{d}M \in \mathbb{CM}_{loc}(\mathcal{F}),$$

and where $X^{(-)} \overset{\text{def}}{=} X^{(+)} - X \in \mathbb{CA}(\mathcal{G}), M^{(-)} \overset{\text{def}}{=} M^{(+)} - M \in \mathbb{CM}_{loc}(\mathcal{F})$. So, by the first part of the proof we get that there exist $\tilde{M}^{(+)}, \tilde{M}^{(-)} \in \mathbb{CM}_{loc}(\mathcal{G})$ such that

$$X^{(+)} \overset{\text{as}}{=} \tilde{M}^{(+)} + \int \tilde{H}^+_u \mathrm{d}u, \quad X^{(-)} \overset{\text{as}}{=} \tilde{M}^{(-)} + \int \tilde{H}^-_u \mathrm{d}u. \tag{4.21}$$

Then $\tilde{H} = \tilde{H}^+ - \tilde{H}^- \in \tilde{\mathcal{M}}^1(\mathcal{G})$, $\tilde{M} \overset{\text{def}}{=} \tilde{M}^{(+)} - \tilde{M}^{(-)} \in \mathbb{CM}_{loc}(\mathcal{G})$ and (4.13) holds. As $\mathcal{G}$ is assumed to be an enriched filtration, we get from Lemma A.15 (2) that $\mathscr{N}^{\mathcal{G}}_\infty \subseteq \mathcal{G}_0$. Then we have that $\tilde{H} \overset{\text{ae}}{=} \hat{H} \overset{\text{def}}{=} (\tilde{H}_t 1_{\Omega\setminus N})_{t\geq 0} \in \mathcal{M}^1(\mathcal{G})$, where

$$N \overset{\text{def}}{=} \{\omega \in \Omega; \exists\, n \in \mathbb{N} \int_0^n |\tilde{H}_s(\omega)| \,\mathrm{d}s = \infty\} \in \mathscr{N}^{\mathcal{G}}_\infty \subseteq \mathcal{G}_0.$$

Thus, we have that the process $\hat{H}$ is (similarly as $\tilde{H}$) also a ν-progressive projection of H to $\mathcal{G}$, the process $\hat{M} \in \mathbb{CA}(\mathcal{G})$ is well-defined by (3.20) and equal to $\tilde{M} \in \mathbb{CM}_{loc}(\mathcal{G})$ up to a null set, which means that $\hat{M} \in \mathbb{CM}_{loc}(\mathcal{G})$ holds. □

4.5. **Proof of Lemma 3.24.**

Proof. Denote by $b\mathcal{M}(\mathscr{G})$ the set of all bounded $\mathscr{G}$-progressive processes. We will show that

$$\mathcal{X} \overset{\text{def}}{=} \{X \in b\mathcal{M}(\mathscr{G});\ \forall\ t \in [0, \infty)\ \ \mathbb{E}[\int_0^t X_u H_u \mathrm{d}u] = \mathbb{E}[\int_0^t X_u \hat{H}_u \mathrm{d}u]\} = b\mathcal{M}(\mathscr{G}), \tag{4.22}$$

where $\hat{H} \in \mathbb{L}(\mathcal{M}(\mathscr{G}))$ is a process satisfying (3.21).

1. First, we will show that $\mathcal{X}$ contains processes of the form

$$X = 1_{(s,\infty)\times G}, \quad \text{where} \quad G \in \mathscr{G}_s,\ s \in [0, \infty). \tag{4.23}$$

Let $t \in [s, \infty)$. If $u \in [s, t]$, then $G \in \mathscr{G}_s \subseteq \mathscr{G}_u$ and by (3.21) we have that $\mathbb{E}[\hat{H}_u; G] = \mathbb{E}[H_u; G]$ holds, which together with the Fubini Theorem gives

$$\mathbb{E}[\int_0^t X_u H_u \,\mathrm{d}u] = \int_s^t \mathbb{E}[H_u; G] \,\mathrm{d}u = \int_s^t \mathbb{E}[\hat{H}_u; G] \,\mathrm{d}u = \mathbb{E}[\int_0^t X_u \hat{H}_u \,\mathrm{d}u].$$

If $t \in [0, s)$, the desired equality holds as both sides are equal to zero regardless of $H, \hat{H}$.

2. If X is bounded and so called $\mathscr{G}$-simple process according to Definition A.12, then $X 1_{\Omega_t}$ is a linear combination of processes of type (4.23) multiplied by 1_{Ω_t}. Hence, it is enough to use simple calculations in order to obtain from the step 1 that $X \in \mathcal{X}$ holds also in this case.

3. As $\mathbb{E}[\int_0^t |H_u| \,\mathrm{d}u] < \infty$ holds by assumption if $t \in [0, \infty)$, we get from the Dominated Convergence Theorem that $\mathcal{X}$ is closed under bounded pointwise convergence (and bounded convergence ν-a.e.). If $X \in \mathbb{CA}(\mathscr{G})$ is bounded, then the following processes are $\mathscr{G}$-simple, equally bounded and converging to X almost everywhere

$$X^{(n)} \overset{\text{def}}{=} \left(\sum_{k=0}^\infty X_{k/n} 1_{[k<tn\leq k+1]}\right)_{t\geq 0}, \quad n \in \mathbb{N}, \tag{4.24}$$

and since $\{X^{(n)}\}_{n=1}^{\infty} \subseteq \mathcal{X}$ holds by step 2, we get that also $X \in \mathcal{X}$. Finally, if $X \in b\mathcal{M}(\mathscr{G})$, similar arguments can be used in order to get that $X \in \mathcal{X}$ holds, this time with

$$X^{(n)} \overset{\text{def}}{=} n(\int_{(t-1/n)^+}^{t} X_s \, \mathrm{d}s)_{t\geq 0} \in \mathbb{CA}(\mathscr{G}), \quad n \in \mathbb{N},$$

equally bounded and converging to X $\mathbb{v}$-almost everywhere. Hence, we have (4.22).

4. If $\mathbb{H} \in \mathcal{M}(\mathscr{G})$ and $\int_{\mathbb{H}} |H| \, \mathrm{d}\mathbb{v} < \infty$, then by the Monotone Convergence Theorem and step 3, we have that

$$\int_{\mathbb{H}} |\hat{H}| \, \mathrm{d}\mathbb{v} = \lim_{n\to\infty} \int_{\mathbb{H}\cap\Omega_n} |\hat{H}| \, \mathrm{d}\mathbb{v} = \lim_{n\to\infty} \int_{\mathbb{H}\cap\Omega_n} \mathrm{sign}(\hat{H}) H \, \mathrm{d}\mathbb{v} \leq \int |H| \, \mathrm{d}\mathbb{v} < \infty.$$

as $\mathrm{sign}(\hat{H}) \in b\mathcal{M}(\mathcal{G})$, and then we get from step 3 and the Dominated Convergence Theorem that

$$\int_{\mathbb{H}} H \, \mathrm{d}\mathbb{v} = \lim_{n\to\infty} \int X^{(n)} H \, \mathrm{d}\mathbb{v} = \lim_{n\to\infty} \int X^{(n)} \hat{H} \, \mathrm{d}\mathbb{v} = \int_{\mathbb{H}} \hat{H} \, \mathrm{d}\mathbb{v}, \quad X^{(n)} \overset{\text{def}}{=} 1_{\mathbb{H}\cap\Omega_n} \in b\mathcal{M}(\mathscr{G}).$$

Now, it is enough to use Lemma 3.18 (i) in order to get that $\hat{H} \in \mathbb{L}(\mathcal{M}(\mathscr{G}))$ is a $\mathbb{v}$-progressive projection of H to $\mathscr{G}$, since the measure $(\int_{\mathbb{H}} |H| \, \mathrm{d}\nu)_{\mathbb{H}\in\mathcal{M}(\mathscr{G})}$ is obviously σ-finite as it is finite on sets $\Omega_n \in \mathcal{M}(\mathscr{G}), n \in \mathbb{N}$, by assumption that $\int |H_s| \, \mathrm{d}s$ is an integrable process. □

4.6. **Proof of Lemma 3.26.**

Proof. Let $\mathcal{A} \overset{\text{def}}{=} \mathrm{dom}(\mathbb{P}) = \mathrm{dom}(Q)$ and put $H \overset{\text{def}}{=} \mathrm{d}\mathbb{P}/\mathrm{d}Q \in \mathbb{L}(\mathcal{A}) \subseteq \mathcal{M}^1(\mathscr{A})$, where $\mathscr{A}_t \overset{\text{def}}{=} \mathcal{A}$, $t \in [0, \infty)$. Then $\hat{H} \overset{\text{def}}{=} \hat{\mathfrak{D}} \in \mathbb{L}(\mathcal{M}(\mathscr{G}))$, where

$$\hat{\mathfrak{D}}_t \overset{\text{as}}{=} \frac{\mathrm{d}\mathbb{P}|\mathscr{G}_t}{\mathrm{d}Q|\mathscr{G}_t} \overset{\text{as}}{=} \mathbb{E}_Q[\tfrac{\mathrm{d}\mathbb{P}}{\mathrm{d}Q}|\mathscr{G}_t] \overset{\text{as}}{=} \mathbb{E}_Q[H_t|\mathscr{G}_t], \quad \mathbb{E}_Q \int_0^t |H_s| \, \mathrm{d}s = \int_0^t \mathbb{E}_Q \tfrac{\mathrm{d}\mathbb{P}}{\mathrm{d}Q} \, \mathrm{d}\mathfrak{L} = t < \infty, \quad t \in [0, \infty).$$

Then we get from Lemma 3.24 that $\hat{\mathfrak{D}}$ is a $\mathbb{µ}$-progressive projection of $\mathrm{d}\mathbb{P}/\mathrm{d}Q$ to $\mathscr{G}$, where $\mathbb{µ} \overset{\text{def}}{=} \mathfrak{L} \otimes Q$, and we get from Lemma 3.18 and the Monotone Convergence Theorem that

$$\begin{aligned} \mathbb{v}(\mathbb{G}) &= \lim_{n\to\infty} \mathbb{v}(\mathbb{G} \cap \Omega_n) = \lim_{n\to\infty} \int_{\mathbb{G}\cap\Omega_n} \tfrac{\mathrm{d}\mathbb{v}}{\mathrm{d}\mathbb{µ}} \, \mathrm{d}\mathbb{µ} \\ &= \lim_{n\to\infty} \int_{\mathbb{G}\cap\Omega_n} \tfrac{\mathrm{d}\mathbb{P}}{\mathrm{d}Q} \, \mathrm{d}\mathbb{µ} = \lim_{n\to\infty} \int_{\mathbb{G}\cap\Omega_n} \hat{\mathfrak{D}} \, \mathrm{d}\mathbb{µ} = \int_{\mathbb{G}} \hat{\mathfrak{D}} \, \mathrm{d}\mathbb{µ} \end{aligned}$$

holds whenever $\mathbb{G} \in \mathcal{M}(\mathscr{G})$, since $\frac{\mathrm{d}\mathbb{v}}{\mathrm{d}\mathbb{µ}} \overset{\text{ae}}{=} \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}Q}$, where $\mathbb{v} \overset{\text{def}}{=} \mathfrak{L} \otimes \mathbb{P}$, see Remark 3.25. □

4.7. **Proof of Lemma 3.27.**

Proof. (i) See Remark 3.25 in order to agree that it is enough to verify that $\hat{\mathfrak{D}}$ is a $\mathbb{µ}$-progressive projection of $\mathfrak{D}$ to $\mathscr{G}$. Obviously, $\mathbb{v}, \mathbb{µ}$ are σ-finite measures on $\mathscr{B}_\infty \otimes \mathscr{F}_\infty$ as their measure of Ω_n is a finite value n whenever $n \in \mathbb{N}$, and similarly we get that their restrictions to $\mathcal{M}(\mathscr{G})$ are also σ-finite. Then since $\mathcal{M}(\mathscr{G}) \subseteq \mathscr{B}_\infty \otimes \mathscr{F}_\infty$, we obtain from the definition of $\hat{\mathfrak{D}}$ and from Lemma 3.18 (i) that $\hat{\mathfrak{D}}$ is equal to the $\mathbb{µ}$-progressive projection of $\mathfrak{D}$ to $\mathscr{G}$ up to a $\mathbb{µ}$-null set.

(ii) First, we get from (i) that

$$\mu^{(+)}(\mathbb{G}) \overset{\text{def}}{=} \int_{\mathbb{G}} H^+ \, \mathrm{d}\mathbb{v} = \int_{\mathbb{G}} \mathfrak{D} H^+ \, \mathrm{d}\mathbb{µ}, \quad \mathbb{G} \in \mathcal{M}(\mathscr{G}), \tag{4.25}$$

and that the same holds with + replaced by −. If we sum these two equalities, we get that $\mathcal{P}^{\mathbb{P}}_{\mathscr{G}}(H)$ exists if and only if $\mathcal{P}^{Q}_{\mathscr{G}}(\mathfrak{D}H)$ exists, since both statements are equivalent to the statement that $\mu^{(+)} + \mu^{(-)}$ is a σ-finite measure on $\mathcal{M}(\mathscr{G})$. Then if these statements hold, we get that

$$\mathcal{P}^{Q}_{\mathscr{G}}(\mathfrak{D}H) \overset{\text{ae}}{=} \frac{\mathrm{d}\mu^{(+)}}{\mathrm{d}\mathbb{µ}|\mathcal{M}(\mathscr{G})} - \frac{\mathrm{d}\mu^{(-)}}{\mathrm{d}\mathbb{µ}|\mathcal{M}(\mathscr{G})} \overset{\text{ae}}{=} \Big(\frac{\mathrm{d}\mu^{(+)}}{\mathrm{d}\mathbb{v}|\mathcal{M}(\mathscr{G})} - \frac{\mathrm{d}\mu^{(-)}}{\mathrm{d}\mathbb{v}|\mathcal{M}(\mathscr{G})}\Big) \frac{\mathrm{d}\mathbb{v}|\mathcal{M}(\mathscr{G})}{\mathrm{d}\mathbb{µ}|\mathcal{M}(\mathscr{G})} \overset{\text{ae}}{=} \mathcal{P}^{\mathbb{P}}_{\mathscr{G}}(H)\, \hat{\mathfrak{D}}.$$ □

4.8. **Proof of Proposition 3.28.**

Proof. As $\mathscr{E} \in \mathbb{L}(\mathcal{M}(\mathscr{G}) \otimes \mathcal{D}) \subseteq \mathbb{L}(\mathcal{M}(\mathscr{G} \otimes \mathcal{D})) \subseteq \mathbb{A}(\mathscr{G} \otimes \mathcal{D})$ holds by Lemma A.21, we get that $\mathscr{E}_t \in \mathbb{L}(\mathscr{G}_t \otimes \mathcal{D})$ holds if $t \in [0,\infty)$ and since Y is Q-independent of $\mathscr{G}_\infty \supseteq \mathscr{G}_t$, we have by (3.25) that

$$0 < \tilde{\mathscr{E}}_t \overset{\text{def}}{=\!=} \int \mathscr{E}_t^{(y)} \mathrm{d}Q_Y(y) \overset{\text{as}}{=} \mathbb{E}_Q[\mathscr{E}_t^{(Y)} | \mathscr{G}_t] \overset{\text{as}}{=} \mathbb{E}_Q[\tfrac{\mathrm{d}\mathbb{P}|\mathscr{F}_t}{\mathrm{d}Q|\mathscr{F}_t} | \mathscr{G}_t] \overset{\text{as}}{=} \tfrac{\mathrm{d}\mathbb{P}|\mathscr{G}_t}{\mathrm{d}Q|\mathscr{G}_t}, \quad t \geq 0, \tag{4.26}$$

as $\mathbb{E}_Q[\mathscr{E}_t^{(Y)}] = 1 < \infty$. Then we get from Lemmas 3.26 and 3.27 and from (3.25, 4.26) that

$$\mathscr{E}^{(Y)} \overset{\text{ae}}{=} \tfrac{\mathrm{d}\nu|\mathcal{M}(\mathscr{F})}{\mathrm{d}\mu|\mathcal{M}(\mathscr{F})}, \qquad \mathbb{L}(\mathcal{M}(\mathscr{G})) \ni \hat{\mathscr{E}} \overset{\text{def}}{=\!=} \tilde{\mathscr{E}} 1_{[\tilde{\mathscr{E}} < \infty]} + 1_{[\tilde{\mathscr{E}} = \infty]} \overset{\text{ae}}{=} \tilde{\mathscr{E}} \overset{\text{ae}}{=} \tfrac{\mathrm{d}\nu|\mathcal{M}(\mathscr{G})}{\mathrm{d}\mu|\mathcal{M}(\mathscr{G})} \tag{4.27}$$

are μ-progressive projections of the following process to $\mathscr{F}$ and to $\mathscr{G}$, respectively,

$$\mathfrak{D} \overset{\text{def}}{=\!=} \tfrac{\mathrm{d}\mathbb{P}|\mathscr{F}_\infty}{\mathrm{d}Q|\mathscr{F}_\infty}, \quad \text{where} \quad \mu \overset{\text{def}}{=\!=} \mathfrak{L} \otimes Q.$$

Then we have from Definition 3.16 and from (4.27) that also $\hat{\mathscr{E}}$ is a μ-progressive projection of $\mathfrak{D}$ to $\mathscr{G}$. Then as $\mathcal{M}(\mathscr{G}) \subseteq \mathcal{M}(\mathscr{F})$, we get from Lemma 3.18 (iv) that $\hat{\mathscr{E}}$ is a μ-progressive projection of $\mathscr{E}^{(Y)}$ to $\mathscr{G}$.

(i) Note that from the previous step of the proof we have that

$$\mathcal{P}^Q_{\mathscr{F}}(\mathfrak{D}) \overset{\text{ae}}{=} \mathscr{E}^{(Y)}, \quad \hat{\mathfrak{D}} \overset{\text{def}}{=\!=} \tfrac{\mathrm{d}\nu|\mathcal{M}(\mathscr{G})}{\mathrm{d}\mu|\mathcal{M}(\mathscr{G})} \overset{\text{ae}}{=} \mathcal{P}^Q_{\mathscr{G}}(\mathfrak{D}) \overset{\text{ae}}{=} \hat{\mathscr{E}}. \tag{4.28}$$

By assumption H has a ν-progressive projection $\hat{H}$ to $\mathscr{G}$, which by Lemma 3.27 (ii) means that there exists a μ-progressive projection of $H\mathfrak{D}$ to $\mathscr{G}$ and that it is of the form $\mathcal{P}^Q_{\mathscr{G}}(H\mathfrak{D}) \overset{\text{ae}}{=} \hat{H}\hat{\mathfrak{D}}$. Hence, in order to verify (3.26), we just have to show that

$$\int \mathbb{h}(y) \mathscr{E}^{(y)} \mathrm{d}Q_Y(y) \overset{\text{ae}}{=} \mathcal{P}^Q_{\mathscr{G}}(H\mathfrak{D}). \tag{4.29}$$

As $H\mathfrak{D}$ has a μ-progressive projection to $\mathscr{G}$ and as $H \in \mathcal{M}^1(\mathscr{F})$ holds by assumption, we get from Lemma 3.18 (iv) and Remark 3.20 and from (4.28) that

$$\mathcal{P}^Q_{\mathscr{G}}(H\mathfrak{D}) \overset{\text{ae}}{=} \mathcal{P}^Q_{\mathscr{G}} \mathcal{P}^Q_{\mathscr{F}}(H\mathfrak{D}) \overset{\text{ae}}{=} \mathcal{P}^Q_{\mathscr{G}}[H \mathcal{P}^Q_{\mathscr{F}}(\mathfrak{D})] \overset{\text{ae}}{=} \mathcal{P}^Q_{\mathscr{G}}[H \mathscr{E}^{(Y)}]. \tag{4.30}$$

By Lemma A.21 (i), $\mathcal{M}(\mathscr{G}) \otimes \mathcal{D} \subseteq \mathcal{M}(\mathscr{G} \otimes \mathcal{D})$. Since Y is Q-independent of $\mathscr{G}_\infty$ and $1_{\mathbb{G}\times\mathsf{D}} \mathbb{h} \mathscr{E} \in \mathbb{L}(\mathcal{M}(\mathscr{G}) \otimes \mathcal{D}) \subseteq \mathbb{L}(\mathcal{M}(\mathscr{G} \otimes \mathcal{D})) \subseteq \mathbb{A}(\mathscr{G} \otimes \mathcal{D})$ holds if $\mathbb{G} \in \mathcal{M}(\mathscr{G})$, we get from the Fubini Theorem (similarly as at the beginning of the proof) that

$$\begin{aligned} \int_{\mathbb{G}} H \mathscr{E}^{(Y)} \mathrm{d}\mu &= \int_0^\infty \int_\Omega 1_{\mathbb{G}}(t)\, \mathbb{h}_t(Y)\, \mathscr{E}_t^{(Y)} \mathrm{d}Q\, \mathrm{d}t = \int_0^\infty \int_\Omega 1_{\mathbb{G}}(t) \int_{\mathsf{D}} \mathbb{h}_t(y)\, \mathscr{E}_t^{(y)} \mathrm{d}Q_Y(y)\, \mathrm{d}Q\, \mathrm{d}t \\ &= \int_{\mathbb{G}} \int_{\mathsf{D}} \mathbb{h}(y)\, \mathscr{E}^{(y)} \mathrm{d}Q_Y(y)\, \mathrm{d}\mu \quad \text{if} \quad \mathbb{G} \in \mathcal{M}(\mathscr{G}) \quad \text{is s.t.} \quad \int_{\mathbb{G}} |H| \mathscr{E}^{(Y)} \mathrm{d}\mu < \infty. \end{aligned}$$

Then we get from Lemma 3.18 and from (4.30) that (4.29) really holds. □

4.9. **Proof of Theorem 3.30.**

Remark 4.3. If X is an $\mathscr{F}$-predictable $\mathbb{R}$-valued process and τ an $\mathscr{F}$-stopping time, then $X_\tau 1_{[\tau<\infty]} \in \mathbb{L}(\mathscr{F}_{\tau-})$. If the process X is left-continuous, it can be seen as follows

$$X_\tau 1_{[\tau<\infty]} = X_0 1_{[\tau=0]} + \lim_{n\to\infty} \textstyle\sum_{k=0}^\infty X_{k/n} 1_{[k<n\tau\leq k+1]} \in \mathbb{L}(\mathscr{F}_{\tau-}),$$

and then it can be easily extended to all $\mathscr{F}$-predictable real-valued processes X.

Proof of Theorem 3.30. 1. First, assume that $X_t(\omega) = 1_A(\omega) 1_{[u\leq t<v]}$, where $A \in \mathcal{A}$ and $0 \leq u < v$. From Theorem 6.27 in [13], we get that there exists a $\mathbb{P}$-null set $N \in \mathcal{A}$ such that the process $Z \overset{\text{def}}{=\!=} 1_{\Omega\setminus N} Y_+$ is right-continuous with (finite) left-hand limits (rcll), where $Y \overset{\text{def}}{=\!=} (\mathbb{P}[A|\mathscr{F}_q])_{\mathbb{Q}\ni q\geq 0}$ and where $Y_+(\omega)$ is defined as $Y_{t+}(\omega)$ at time $t \in [0,\infty)$ if the corresponding limit exists. Then, see Notation A.5,

$$\tilde{U} \overset{\text{def}}{=\!=} \big(\exists \lim_{n\to\infty} U^{(n+1)}_{t+2^{-n}}\big)_{t\geq 0} = Y_+ = Z \quad \text{on} \quad \Omega\setminus N, \quad \text{where} \tag{4.31}$$

$$U^{(n)} \overset{\text{def}}{=\!=} \textstyle\sum_{k=0}^\infty Y_{k2^{-n}} 1_{(k2^{-n},(k+1)2^{-n}]} \tag{4.32}$$

are $\mathscr{F}$-predictable processes as well as $(U^{(n)}_{(t-a)^+})_{t\geq 0}$ if $a \in [0,\infty), n \in \mathbb{N}$, since these processes are left-continuous and $\mathscr{F}$-adapted. Then we get, with the help of Proposition A.7, that also the following processes are $\mathscr{F}$-predictable

$$(4.33)\quad {}^pX \overset{\text{def}}{=} M1_{[u,v)},\ M \overset{\text{def}}{=} \big(\exists \lim_{m\to\infty} \exists \lim_{m\leq n\to\infty} U^{(n+1)}_{(t+2^{-n}-2^{-m})^+}\big)_{t\geq 0} = \big(\exists \lim_{m\to\infty} \tilde{U}_{(t-2^{-m})^+}\big)_{t\geq 0}.$$

Hence, (i) is satisfied and it remains to show (ii) which is in this case of the form

$$(4.34)\qquad \forall \tau\ \mathscr{F}\text{-predictable time}\quad ({}^pX)_\tau 1_{[\tau<\infty]} \overset{\text{as}}{=} \mathbb{P}(A|\mathscr{F}_{\tau-})1_{[u\leq\tau<v]}.$$

In order to show (4.34), we may assume that the probability space $(\Omega,\mathcal{A},\mathbb{P})$ is complete. Let $\mathcal{F}$ stand for the smallest complete and right-continuous extension of $\mathscr{F}$, called augmentation in [13]. From the above mentioned Theorem 6.27 in [13] and its proof, we have that the process Z is an $\mathcal{F}$-martingale, and then from Theorems 6.23 and 6.29 in [13] we obtain that

$$(4.35)\qquad \mathbb{P}(A|\mathcal{F}_\tau)1_{[\tau<\infty]} \overset{\text{as}}{=} \lim_{\mathbb{N}\ni n\to\infty} \mathbb{E}[Z_n|\mathcal{F}_\tau]1_{[\tau<\infty]} \overset{\text{as}}{=} \lim_{\mathbb{N}\ni n\to\infty} Z_{n\wedge\tau}1_{[\tau<\infty]} \overset{\text{as}}{=} Z_\tau 1_{[\tau<\infty]}$$

if τ is an $\mathcal{F}$-stopping time. From (4.31,4.33) we get that ${}^pX = Z_-1_{[u,v)}$ on $\Omega\backslash N$, where

$$Z_- \overset{\text{def}}{=} (Z_{t-}1_{[t>0]} + Z_0 1_{[t=0]})_{t\geq 0}.$$

As in the proof of Theorem I.2.28 in [12], we obtain from Lemma I.2.27 in [12] and from (4.35) that

$$(4.36)\qquad ({}^pX)_\tau 1_{[\tau<\infty]} \overset{\text{as}}{=} (Z_-)_\tau 1_{[u\leq\tau<v]} \overset{\text{as}}{=} \mathbb{E}[Z_\tau|\mathcal{F}_{\tau-}]1_{[u\leq\tau<v]}$$

$$(4.37)\qquad \overset{\text{as}}{=} \mathbb{E}[\mathbb{P}(A|\mathcal{F}_\tau)|\mathcal{F}_{\tau-}]1_{[u\leq\tau<v]} \overset{\text{as}}{=} \mathbb{E}[X_\tau|\mathcal{F}_{\tau-}]$$

if τ is an $\mathcal{F}$-predictable time. If τ is an $\mathscr{F}$-predictable time, it is also an $\mathcal{F}$-predictable time, and then as $({}^pX)_\tau \in \mathbb{L}(\mathscr{F}_{\tau-})$ holds by Remark 4.3, we get from (4.36,4.37) that

$$({}^pX)_\tau 1_{[\tau<\infty]} \overset{\text{as}}{=} \mathbb{E}[({}^pX)_\tau|\mathscr{F}_{\tau-}]1_{[\tau<\infty]} \overset{\text{as}}{=} \mathbb{E}[\mathbb{E}[X_\tau|\mathcal{F}_{\tau-}]|\mathscr{F}_{\tau-}]1_{[\tau<\infty]} \overset{\text{as}}{=} \mathbb{E}[X_\tau|\mathscr{F}_{\tau-}]1_{[\tau<\infty]},$$

which corresponds to (4.34).

2. In general, see the steps 2, 3, 4 in the proof of Theorem I.2.28 in [12] in order to get to know how to obtain the corresponding results first for bounded processes, then for non-negative ones and finally for any $[-\infty,\infty]$-valued process X. □

4.10. **Proof of Lemma 3.31.**

Proof. (ii) Without loss of generality, we may assume that the process X attains real values. If it is left-continuous, then also $\tilde{X} \overset{\text{def}}{=} (X_t 1_{[t>0]})_{t\geq 0}$ is a left-continuous process, and hence to verify that $\tilde{X}$ is $\mathscr{F}$-predictable, it is enough to show that it $\mathscr{F}$-adapted, which can be seen as follows

$$\tilde{X}_t = \lim_{n\to\infty} \textstyle\sum_{k=0}^\infty X_{k/n} 1_{[k<nt\leq k+1]} \in \mathbb{L}(\mathscr{F}_t), \quad t\in[0,\infty),$$

since $X_s \in \mathbb{L}(\mathscr{F}^+_s) = \mathbb{L}(\mathscr{F}_{s+}) \subseteq \mathbb{L}(\mathscr{F}_t)$ if $0\leq s<t<\infty$. Hence, the set

$$\mathfrak{X} \overset{\text{def}}{=} \{X \in \mathbb{A}(\mathscr{F}^+); (X_t 1_{[t>0]})_{t\geq 0} \text{ is } \mathscr{F}\text{-predictable}\}$$

contains all left-continuous $\mathscr{F}^+$-adapted processes and as $\mathfrak{X}$ is closed under the pointwise convergence, it contains all $\mathscr{F}^+$-predictable real-valued processes.

(i) Since τ is an $\mathscr{F}^+$-predictable time, the process $X \overset{\text{def}}{=} (1_{[\tau\leq t]})_{t\geq 0}$ is $\mathscr{F}^+$-predictable, which by (ii) means that $\tilde{X} \overset{\text{def}}{=} (1_{[\tau\leq t, t>0]})_{t\geq 0}$ is $\mathscr{F}$-predictable. Further, the process $Y \overset{\text{def}}{=} (1_{[\tau>0,t>0]})_{t\geq 0}$ is left-continuous and $\mathscr{F}$-adapted, hence it is also $\mathscr{F}$-predictable. Here, we have used that $Y_0 = 0 \in \mathbb{L}(\mathscr{F}_0)$ and that $Y_t \in \mathbb{L}(\mathscr{F}^+_0) \subseteq \mathbb{L}(\mathscr{F}_t)$ whenever $t\in(0,\infty)$. Thus, we have that also the product

$$Z \overset{\text{def}}{=} \tilde{X}Y = (1_{[0<\tau\leq t]})_{t\geq 0} = (1_{[\tilde{\tau}\leq t]})_{t\geq 0}$$

is also an $\mathscr{F}$-predictable process which means that $\tilde{\tau}$ is an $\mathscr{F}$-predictable time.

(iii) 1. If the filtration $\mathscr{F}$ is right-continuous, the statement follows from [12, Proposition I.2.18].

2. In general, since the process X is $\mathscr{F}$-predictable, it is also $\mathscr{F}^+$-predictable and to get that $X \overset{\text{as}}{=} 0$, it is enough by the first step of this part of the proof to verify that

$$0 \overset{\text{as}}{=} X_\tau 1_{[\tau<\infty]} \tag{4.38}$$

holds whenever τ is an $\mathscr{F}^+$-predictable time.

Let τ be an $\mathscr{F}^+$-predictable time. By (i) we have that $\tilde{\tau}$ is an $\mathscr{F}$-predictable time. Then $0 \overset{\text{as}}{=} X_{\tilde{\tau}} 1_{[\tilde{\tau}<\infty]} = X_\tau 1_{[0<\tau<\infty]}$ holds by assumption on X. Hence, to get (4.38) it remains to show that $0 \overset{\text{as}}{=} X_\tau 1_{[\tau=0]}$ and it follows from $X_0 \overset{\text{as}}{=} 0$, which holds by assumption as 0 is an $\mathscr{F}$-predictable time. □

4.11. **Proof of Lemma 3.33.**

Remark 4.4. Let $(\Omega, \mathcal{A}, \mathbb{P}, \mathscr{F})$ be a filtered probability space and let $X, \bar{X} \in \mathbb{L}(\mathscr{B}_\infty \otimes \mathcal{A})$ be such that $X \le \bar{X}$, then the corresponding $\mathscr{F}$-predictable projections satisfy the inequality ${}^pX \le {}^p\bar{X}$ almost surely. It can be seen as follows. If τ is a $\mathscr{F}$-predictable time, by (3.27) we have that

$$\begin{aligned}({}^pX)_\tau 1_{[\tau<\infty]} &\overset{\text{as}}{=} \lim_{m\to-\infty} \lim_{n\to\infty} \mathbb{E}[n \wedge X_\tau \vee m; \tau < \infty | \mathscr{F}_{\tau-}] \\ &\overset{\text{as}}{\le} \lim_{m\to-\infty} \lim_{n\to\infty} \mathbb{E}[n \wedge \bar{X}_\tau \vee m; \tau < \infty | \mathscr{F}_{\tau-}] \overset{\text{as}}{=} ({}^p\bar{X})_\tau 1_{[\tau<\infty]},\end{aligned}$$

which gives us that also ${}^pX \vee {}^p\bar{X}$ plays the role of an $\mathscr{F}$-predictable projection of $\bar{X}$, and then we get from its uniqueness, see Remark 3.32, that ${}^pX \vee {}^p\bar{X} \overset{\text{as}}{=} {}^p\bar{X}$, i.e., ${}^pX \le {}^p\bar{X}$ almost surely.

Definition 4.5. Let $\mathscr{G}$ be a filtration, a set $\mathbb{G} \subseteq \Omega_\infty$ is called *$\mathscr{G}$-predictable* if $1_\mathbb{G}$ is a $\mathscr{G}$-predictable process. Note that the set of all $\mathscr{G}$-predictable sets is a σ-algebra, cf. Definition 3.29.

Remark 4.6. If $\mathbb{G} \in \mathcal{M}(\mathscr{G})$, then there exists a $\mathscr{G}$-predictable set $\hat{\mathbb{G}}$ such that $1_\mathbb{G} \overset{\text{ae}}{=} 1_{\hat{\mathbb{G}}}$. It can be seen as follows. Put $0 \le H \overset{\text{def}}{=} 1_\mathbb{G}$, then $\mathscr{H} \overset{\text{def}}{=} \int H_t \,\mathrm{d}t \in \mathbb{Cl}(\mathscr{G})$ is a $\mathscr{G}$-predictable process as well as

$$\hat{H} \overset{\text{def}}{=} (\exists \lim_{n\to\infty} n[\mathscr{H}_t - \mathscr{H}_{(t-1/n)^+}])_{t\ge0} \overset{\text{ae}}{=} \mathscr{H}' \overset{\text{ae}}{=} H = 1_\mathbb{G},$$

see Proposition A.7. Then $\hat{\mathbb{G}} \overset{\text{def}}{=} \hat{H}^{-1}\{1\}$ is a $\mathscr{G}$-predictable set such that $1_{\hat{\mathbb{G}}} \overset{\text{ae}}{=} 1_\mathbb{G}$.

Proof of Lemma 3.33. 1. As every left-continuous $\mathscr{G}$-adapted process is $\mathscr{G}$-progressive, we get that any $\mathscr{G}$-predictable real-valued process is also $\mathscr{G}$-progressive. In particular, any $\mathscr{G}$-predictable set is in $\mathcal{M}(\mathscr{G})$. Hence, as the $\mathscr{G}$-predictable projection of $|H|$, say $\widehat{|H|}$, is $\mathscr{G}$-predictable, we have that

$$\mathbb{G}^{(n)} \overset{\text{def}}{=} \Omega_n \cap \widehat{|H|}^{-1}[0,n] \in \mathcal{M}(\mathscr{G}), n \in \mathbb{N}, \quad \mathbb{G}^{(0)} \overset{\text{def}}{=} \Omega_\infty \setminus \cup_{n=1}^\infty \mathbb{G}^{(n)} \in \mathcal{M}(\mathscr{G}),$$

and from Remark 4.3 we have that

$$\mathbb{G}_t^{(n)} \overset{\text{def}}{=} \{\omega \in \Omega; (t,\omega) \in \mathbb{G}^{(n)}\} = [\widehat{|H|}_t \le n] \cap [t \le n] \in \mathscr{G}_{t-}, \quad t \in [0,\infty), \quad n \in \mathbb{N},$$

and then as $\widehat{|H|}$ is a $\mathscr{G}$-predictable projection of $|H|$, we get that

$$\int_{\mathbb{G}^{(n)}} |H| \,\mathrm{d}\nu = \int_0^\infty \mathbb{E}[|H_t|; \mathbb{G}_t^{(n)}] \,\mathrm{d}t = \int_0^n \mathbb{E}[\widehat{|H|}_t; \widehat{|H|}_t \le n] \,\mathrm{d}t \le n^2 < \infty, \quad n \in \mathbb{N}.$$

Hence, if $\widehat{|H|} \overset{\text{ae}}{<} \infty$, $\mathbb{G}^{(0)}$ is a ν-null set, and it gives us that H has a $\mathscr{G}$-progressive projection.

2. On the other hand, let us assume that H has a $\mathscr{G}$-progressive projection. Then there are $\mathbb{G}^{[n]} \in \mathcal{M}(\mathscr{G}), n \in \mathbb{N}$, that unite to Ω_∞ such that we have the first inequality in

(4.39). Moreover, we have from Remark 4.6 that there are $\mathscr{G}$-predictable sets $\hat{\mathbb{G}}^{[n]}$ such that $1_{\hat{\mathbb{G}}^{[n]}} \overset{\text{ae}}{=} 1_{\mathbb{G}^{[n]}}, n \in \mathbb{N}$, which gives us the second equality in

$$\infty > \textstyle\int_{\mathbb{G}^{[n]}} |H| \, \mathrm{d}\nu = \int_0^\infty \mathbb{E}[|H_t|; \mathbb{G}_t^{[n]}] \, \mathrm{d}t = \int_0^\infty \mathbb{E}[|H_t|; \hat{\mathbb{G}}_t^{[n]}] \, \mathrm{d}t \tag{4.39}$$

$$= \textstyle\int_0^\infty \mathbb{E}[\widehat{|H|}_t; \hat{\mathbb{G}}_t^{[n]}] \, \mathrm{d}t = \int_{\hat{\mathbb{G}}^{[n]}} \widehat{|H|} \, \mathrm{d}\nu, \tag{4.40}$$

while the third one follows as $\widehat{|H|}$ is a $\mathscr{G}$-predictable projection of $|H|$ and as

$$\hat{\mathbb{G}}_t^{[n]} \overset{\text{def}}{=} \{\omega \in \Omega; (t,\omega) \in \hat{\mathbb{G}}^{[n]}\} \in \mathscr{G}_{t-}, \quad t \in [0,\infty),$$

holds by Remark 4.3. From (4.39,4.40), we get that $\widehat{|H|}$ is finite ν-almost everywhere on $\cup_n \hat{\mathbb{G}}^{[n]}$ which differs from $\cup_{n=1}^\infty \mathbb{G}^{[n]} = \Omega_\infty$ only about a ν-null set. Hence, $\widehat{|H|} \overset{\text{ae}}{<} \infty$.

3. Let $\widehat{|H|} \overset{\text{ae}}{<} \infty$ and let $\widehat{H}$ be a $\mathscr{G}$-predictable projection of H. Since $|H| \geq 0$, we obtain from the corresponding definition that the $\mathscr{G}$-predictable projection of $-|H|$ equals to $-\widehat{|H|}$ almost surely, and then we have from Remark 4.4 that

$$-\infty \overset{\text{ae}}{<} -\widehat{|H|} \overset{\text{as}}{\leq} \widehat{H} \overset{\text{as}}{\leq} \widehat{|H|} \overset{\text{ae}}{<} \infty, \quad \text{i.e.,} \quad \mathfrak{R}(\widehat{H}) \overset{\text{ae}}{=} \widehat{H}.$$

Later on, we will need the following property

$$\forall\, t \in (0,\infty)\ \forall G_t \in \mathscr{G}_{t-} \ \big(\mathbb{E}[|H_t|; G_t] < \infty \quad \Rightarrow \quad \mathbb{E}[H_t; G_t] = \mathbb{E}[\widehat{H}_t; G_t] \big). \tag{4.41}$$

If the condition in (4.41) is satisfied, then $H_t 1_{G_t} \overset{\text{as}}{=} \mathbb{L}_1 \lim_{m\to-\infty} \mathbb{L}_1 \lim_{n\to\infty} (n \wedge H_t \vee m) 1_{G_t}$ and as the conditional expectation preserves the convergence in $\mathbb{L}_1$, we also have that

$$\mathbb{E}[H_t; G_t | \mathscr{G}_{t-}] \overset{\text{as}}{=} \mathbb{L}_1 \lim_{m\to-\infty} \mathbb{L}_1 \lim_{n\to\infty} \mathbb{E}[n \wedge H_t \vee m; G_t | \mathscr{G}_{t-}]$$
$$\overset{\text{as}}{=} \lim_{m\to-\infty} \lim_{n\to\infty} \mathbb{E}[n \wedge H_t \vee m | \mathscr{G}_{t-}] 1_{G_t} \overset{\text{as}}{=} \widehat{H}_t 1_{G_t}$$

as $\widehat{H}$ is a $\mathscr{G}$-predictable projection of H and as t is a $\mathscr{G}$-predictable time. Then (4.41) follows immediately.

Let $\mathbb{G} \in \mathcal{M}(\mathscr{G})$ be s.t. $\int_{\mathbb{G}} |H| \, \mathrm{d}\nu < \infty$. As in the previous step, we get from Remark 4.6 a $\mathscr{G}$-predictable set $\hat{\mathbb{G}}$, which differs from $\mathbb{G}$ about a ν-null set. Then $\hat{\mathbb{G}}_t \overset{\text{def}}{=} \{\omega \in \Omega; (t,\omega) \in \hat{\mathbb{G}}\} \in \mathbb{L}(\mathscr{G}_{t-})$ by Remark 4.3 and $\infty > \int_{\mathbb{G}} |H| \, \mathrm{d}\nu = \int_{\hat{\mathbb{G}}} |H| \, \mathrm{d}\nu = \int_0^\infty \mathbb{E}[|H_t|; \hat{\mathbb{G}}_t] \, \mathrm{d}t$. In particular, $\mathbb{E}[|H_t|; \hat{\mathbb{G}}_t] < \infty$ holds for almost every $t \in (0,\infty)$ and then we get from (4.41) that

$$\textstyle\int_{\mathbb{G}} H \, \mathrm{d}\nu = \int_{\hat{\mathbb{G}}} H \, \mathrm{d}\nu = \int_0^\infty \mathbb{E}[H_t; \hat{\mathbb{G}}_t] \, \mathrm{d}t = \int_0^\infty \mathbb{E}[\widehat{H}_t; \hat{\mathbb{G}}_t] \, \mathrm{d}t = \int_{\hat{\mathbb{G}}} \widehat{H} \, \mathrm{d}\nu = \int_{\mathbb{G}} \mathfrak{R}(\widehat{H}) \, \mathrm{d}\nu. \tag{4.42}$$

Then we get from Lemma 3.18 that $\mathfrak{R}(\widehat{H})$ is really a $\mathscr{G}$-progressive projection of H. □

4.12. **Proof of Lemma 3.37.**

Proof. (i) First, we get from Lemma A.15 (3) that $\widehat{X} \in \mathbb{CA}(\widehat{\mathcal{F}})$. As $0 \overset{\text{as}}{=} D \overset{\text{def}}{=} \widehat{X} - \hat{X} \in \mathbb{CA}(\widehat{\mathcal{F}})$, we get immediately from the definition that $D \in \mathbb{CM}_{loc}(\widehat{\mathcal{F}})$. Let $\hat{A} \in \mathbb{CFV}(\hat{\mathcal{F}}) \subseteq \mathbb{CFV}(\widehat{\mathcal{F}})$ and $\hat{M} \in \mathbb{CM}_{loc}(\hat{\mathcal{F}}) \subseteq \mathbb{CM}_{loc}(\widehat{\mathcal{F}})$ be such that $\hat{X} = \hat{A} + \hat{M}$. Then $\widehat{X} = \widehat{M} + \hat{A}$, where $\widehat{M} \overset{\text{def}}{=} \hat{M} + D \in \mathbb{CM}_{loc}(\widehat{\mathcal{F}})$.

(ii) It follows from Proposition 3.22 once we verify that

$$\hat{\nu}^H \overset{\text{def}}{=} (\textstyle\int_{\mathbb{G}} |H| \mathrm{d}\nu)_{\mathbb{G} \in \mathcal{M}(\hat{\mathcal{F}})} \quad \text{is a } \sigma\text{-finite measure.} \tag{4.43}$$

First, (4.43) holds with $\hat{\mathcal{F}}$ is replaced by $\widehat{\mathcal{F}}$, since

$$\widehat{\mathbb{G}}_n \overset{\text{def}}{=} \{\omega \in \Omega_n; |H(\omega)| \leq n\} \in \mathcal{M}(\widehat{\mathcal{F}}), \quad n \in \mathbb{N},$$

are such that $\int_{\widehat{\mathbb{G}}_n} |H| \mathrm{d}\nu < \infty$ and they unite to Ω. Then as $\widehat{\mathcal{F}} = \mathcal{F}^{\hat{\mathcal{S}},\mathcal{F}}$, we get from Lemmas A.23 and A.24 (applied to 1_{C_n} obtained from Lemma A.23) that there exist $C_n \in \mathcal{M}(\mathscr{C}^k), n \in \mathbb{N}$, such that

$$\textstyle\int |1_{\widehat{\mathbb{G}}_n} - 1_{\hat{\mathbb{G}}_n}| \, \mathrm{d}\nu = 0 \quad \text{holds with} \quad \hat{\mathbb{G}}_n \overset{\text{def}}{=} \hat{\mathcal{S}}^{-1} C_n \in \mathcal{M}(\hat{\mathscr{F}}) \subseteq \mathcal{M}(\hat{\mathcal{F}}), \quad n \in \mathbb{N}.$$

Then we get that (4.43) holds as we have a sequence $(\hat{\mathbb{G}}_n)_{n=0}^{\infty} \in \mathcal{M}(\hat{\mathcal{F}})^{\mathbb{N}_0}$ of $\hat{\nu}^H$-finite sets that unite to Ω_∞, where $\mathbb{G}_0 \overset{\text{def}}{=} \Omega_\infty \setminus \cup_{n=1}^{\infty} \hat{\mathbb{G}}_n$ is a ν-null (and also $\hat{\nu}^H$-null) set.

(iii) By Lemma A.23 there exist $N \in \mathscr{N}_\infty^{\mathcal{F},\mathbb{P}}$ and $\mathbb{h} \in \mathbb{L}(\mathcal{M}(\mathscr{C}^{m+1}) \otimes \mathfrak{S})$ such that (A.24) holds with $(X, Z, \mathbb{z})$ replaced by $(\hat{\mathcal{S}}, H, \mathbb{h})$. Then we get that the equality in (3.30) on the right holds with $\mathbb{H} \overset{\text{def}}{=} \big(\mathbb{h}(\hat{\mathcal{S}}, \mathbb{s})\big)_{\mathbb{s}\in\mathbb{S}}$ and from Lemma A.24 that $\mathbb{H} \in \mathbb{L}(\mathcal{M}(\hat{\mathscr{F}}) \otimes \mathfrak{S})$ as obviously $\hat{\mathcal{S}} \in \mathbb{CA}(\hat{\mathscr{F}})^{m+1}$.

(iv) It is enough to omit "$\otimes\mathfrak{S}$" in both its occurrences in the proof of (iii). □

4.13. **Proof of Lemma 3.38.**

Proof. (ii) By assumption, the measure $(\int_{\mathbb{G}} |H| \, d\nu)_{\mathbb{G}\in\mathcal{M}(\widehat{\mathcal{F}})}$ is σ-finite and our first goal is to show that the same holds with $\widehat{\mathcal{F}}$ replaced by $\hat{\mathcal{F}}$. Let $\widehat{\mathbb{G}}_n \in \mathcal{M}(\widehat{\mathcal{F}})$ be such that $\int_{\widehat{\mathbb{G}}_n} |H| \, d\nu < \infty, n \in \mathbb{N}$, with $\cup_{n=1}^{\infty} \widehat{\mathbb{G}}_n = \Omega_\infty$. From Lemma 3.37 (iv), we obtain that there are $\hat{\mathbb{G}}_n \in \mathcal{M}(\hat{\mathscr{F}}) \subseteq \mathcal{M}(\hat{\mathcal{F}})$ such that $1_{\hat{\mathbb{G}}_n} \overset{\text{ae}}{=} 1_{\widehat{\mathbb{G}}_n}, n \in \mathbb{N}$. Then $\int_{\hat{\mathbb{G}}_n} |H| \, d\nu < \infty, n \in \mathbb{N}_0$, where $\hat{\mathbb{G}}_0 \overset{\text{def}}{=} \Omega_\infty \setminus \cup_{n=1}^{\infty} \hat{\mathbb{G}}_n \in \mathcal{M}(\hat{\mathcal{F}})$ is a ν-null set.

Let $\widehat{H}$ be a ν-progressive projection of H to $\widehat{\mathcal{F}}$. By Lemma 3.37 (iv) there exists $\hat{H} \in \mathbb{L}(\mathcal{M}(\hat{\mathscr{F}})) \subseteq \mathbb{L}(\mathcal{M}(\hat{\mathcal{F}}))$ such that $\hat{H} \overset{\text{ae}}{=} \widehat{H}$ and then we get from Lemma 3.18 (i) that

$$\forall\, \mathbb{G} \in \mathcal{M}(\hat{\mathcal{F}}) \subseteq \mathcal{M}(\widehat{\mathcal{F}}) \qquad \Big(\int_{\mathbb{G}} |H| \, d\nu < \infty \quad \Rightarrow \quad \int_{\mathbb{G}} \hat{H} \, d\nu = \int_{\mathbb{G}} \widehat{H} \, d\nu = \int_{\mathbb{G}} H \, d\nu \Big),$$

and by the second use of Lemma 3.18 (i), $\hat{H}$ is a ν-progressive projection of H to $\hat{\mathcal{F}}$.

(i) If H has a ν-progressive projection $\hat{H}$ to $\hat{\mathcal{F}}$, then H has also a ν-progressive projection to $\widehat{\mathcal{F}}$, say $\widehat{H}$, as a measure extending a σ-finite measure is also σ-finite. From (ii) we get that H has a ν-progressive projection $\tilde{H}$ to $\hat{\mathcal{F}}$ such that $\tilde{H} \overset{\text{ae}}{=} \widehat{H}$ and from the uniqueness of the projection, we have that $\widehat{H} \overset{\text{ae}}{=} \tilde{H} \overset{\text{ae}}{=} \hat{H} \in \mathbb{L}(\mathcal{M}(\hat{\mathcal{F}})) \subseteq \mathbb{L}(\mathcal{M}(\widehat{\mathcal{F}}))$, which according to the just mentioned uniqueness of the projection shows that $\hat{H}$ is really a ν-progressive projection of H to $\widehat{\mathcal{F}}$. □

4.14. **Proof of Lemma 3.39.**

Proof. Since $\alpha^{(0)} \in \mathcal{M}^1(\mathcal{F})$, we get from (3.4) that

$$\mathbb{CA}(\mathcal{F}) \ni X \overset{\text{def}}{=} \int \alpha_s^{(0)} ds \overset{\text{as}}{=} \hat{X} \overset{\text{def}}{=} \ln(\mathcal{S}^{(0)}/\mathcal{S}_0^{(0)}) \in \mathbb{CA}(\hat{\mathcal{F}}) \subseteq \mathbb{CA}(\widehat{\mathcal{F}}),$$

and then from Lemma A.15 (3) that also $X \in \mathbb{CA}(\widehat{\mathcal{F}})$. As X is a process with locally absolutely continuous trajectories equal to $\hat{X}$ almost surely, we get from Proposition A.7 that

$$\alpha^{(0)} \overset{\text{ae}}{=} \widehat{\alpha}^{(0)} \overset{\text{def}}{=} \big(\exists \lim_{n\to\infty} n[X_t - X_{(t-1/n)^+}]\big)_{t\geq 0} \in \mathbb{L}(\mathcal{M}(\widehat{\mathcal{F}})),$$
$$\widehat{\alpha}^{(0)} \overset{\text{as}}{=} \tilde{\alpha}^{(0)} \overset{\text{def}}{=} \big(\exists \lim_{n\to\infty} n[\hat{X}_t - \hat{X}_{(t-1/n)^+}]\big)_{t\geq 0} \in \mathbb{L}(\mathcal{M}(\hat{\mathcal{F}})).$$

Since $\alpha^{(0)} \in \mathcal{M}^1(\mathcal{F})$, we get from the definition of $\widehat{\alpha}^{(0)}$ that $\int_0^t |\tilde{\alpha}_s^{(0)}| ds \overset{\text{as}}{=} \int_0^t |\widehat{\alpha}_s^{(0)}| ds = \int_0^t |\alpha_s^{(0)}| ds < \infty, t \in [0, \infty)$. Further, we get from (3.6, 3.7) that $\int \Sigma_s ds \overset{\text{as}}{=} \langle\!\langle V \rangle\!\rangle \overset{\text{as}}{=} \langle\!\langle (\ln \mathcal{S}^{(i)})_{i=1}^m \rangle\!\rangle$. As $\hat{\mathcal{F}}$ is defined as an enriched filtration and $\mathcal{S} \in \mathbb{CS}(\hat{\mathcal{F}})^m$, we are able to obtain from Corollary A.17 that there exists

$$\text{(4.44)} \quad \mathbb{CFV}(\widehat{\mathcal{F}})^{m\times m} \supseteq \mathbb{CFV}(\hat{\mathcal{F}})^{m\times m} \ni \mathscr{K} \overset{\text{as}}{=} \big(\langle \ln \mathcal{S}^{(i)}, \ln \mathcal{S}^{(j)} \rangle\big)_{ij=1}^m \overset{\text{as}}{=} \int \Sigma_s \, ds \in \mathbb{CA}(\mathcal{F})^{m\times m},$$

where the last relation comes from the assumption that $\Sigma \in \mathcal{M}^2(\mathcal{F})^{m\times m}$. From Lemma A.15 (3), we also get that $\int \Sigma_s \, ds \in \mathbb{CA}(\widehat{\mathcal{F}})^{m\times m}$. Then Proposition A.7 gives that

$$\text{(4.45)} \qquad \widetilde{\Sigma} \overset{\text{def}}{=} \Big(\exists \lim_{n\to\infty} n \int_{(t-1/n)^+}^t \Sigma_s ds\Big)_{t\geq 0} \in \mathbb{L}(\mathcal{M}(\widehat{\mathcal{F}}))^{m\times m}.$$

Here, $\widetilde{\Sigma}_t(\omega) \in \mathbb{R}^{m\times m}$ does not have to be positive semidefinite for every $t \geq 0$ and $\omega \in \Omega$, but as the considered market is assumed to be regular, we get from (4.45) that

$$(4.46) \qquad 0 < \sigma\sigma^{\top} = \Sigma \overset{\text{ae}}{=} \widetilde{\Sigma}, \quad \text{and hence} \quad 0 < \widehat{\Sigma} \overset{\text{def}}{=} \mathbb{1}_m + (\widetilde{\Sigma} - \mathbb{1}_m)1_{[\widetilde{\Sigma}>0]} \overset{\text{ae}}{=} \widetilde{\Sigma}.$$

Then $0 < \widehat{\Sigma} \in \mathbb{L}(\mathcal{M}(\widehat{\mathcal{F}}))^{m\times m}$ and $\int_0^t \mathrm{tr}\{\widehat{\Sigma}_s\}\mathrm{d}s = \int_0^t \mathrm{tr}\{\Sigma_s\}\mathrm{d}s < \infty, t \in [0,\infty)$, and this verifies that $\widehat{\Sigma} \in \mathcal{M}^1(\widehat{\mathcal{F}})^{m\times m}$ since $\widehat{\Sigma} > 0$ holds by (4.46).

If we replace $\int_{(t-1/n)^+}^t \Sigma_s \mathrm{d}s$ in (4.45) by $\mathscr{K}_t - \mathscr{K}_{(t-1/n)^+}$ and if we consider a similar modification as in (4.46), we obtain with the help of (4.44) that there exists $0 < \tilde{\Sigma} \in \mathbb{L}(\mathcal{M}(\hat{\mathcal{F}}))^{m\times m}$ such that $\tilde{\Sigma} \overset{\text{as}}{=} \hat{\Sigma}$. Finally, it is enough to put

$$\hat{\alpha}^{(0)} \overset{\text{def}}{=} 1_A \tilde{\alpha}^{(0)} \in \mathcal{M}^1(\hat{\mathcal{F}}), \quad 0 < \hat{\Sigma} \overset{\text{def}}{=} \mathbb{1}_m + 1_A(\tilde{\Sigma} - \mathbb{1}_m) \in \mathcal{M}^1(\hat{\mathcal{F}})^{m\times m},$$

where $A \overset{\text{def}}{=} [\forall n \in \mathbb{N} \int_0^n (|\tilde{\alpha}_s^{(0)}| + \mathrm{tr}\{\tilde{\Sigma}_s\})\,\mathrm{d}s < \infty] \in \sigma(\mathscr{N}_\infty^{\hat{\mathcal{F}}}) \subseteq \hat{\mathcal{F}}_0$ holds by Lemma A.15 (2). □

4.15. **Proof of Proposition 3.44.**

Proof. (i) As mentioned in Definition 3.1, $V \in \mathbb{CA}(\hat{\mathcal{F}})^m$. Since $\alpha \in \mathcal{M}^1(\mathcal{F})^m$, we get from (3.7) that

$$\mathbb{CA}(\mathcal{F})^m \ni M \overset{\text{def}}{=} V - \int \alpha_t \mathrm{d}t \overset{\text{as}}{=} \int \sigma \mathrm{d}B \in \mathbb{CM}_{loc}(\mathcal{F})^m,$$

which ensures that also $M \in \mathbb{CM}_{loc}(\mathcal{F})^m$. Then since the considered market is assumed to be filterable, we are allowed to use Proposition 3.22 in order to get that there exists a ν-progressive projection $\hat{\alpha} \in \mathcal{M}^1(\hat{\mathcal{F}})^m$ of α to $\hat{\mathcal{F}}$ such that

$$(4.47) \qquad \hat{M} \overset{\text{def}}{=} V - \int \hat{\alpha}_u \,\mathrm{d}u \in \mathbb{CM}_{loc}(\hat{\mathcal{F}})^m, \quad \text{i.e.,} \quad V \in \mathbb{CS}(\hat{\mathcal{F}})^m,$$

and then obviously also $\hat{\mathbb{V}} \in \mathbb{CS}(\hat{\mathcal{F}})^m$. A similar usage of Proposition 3.22 would give us with the help of (3.1) that also $\mathcal{S} \in \mathbb{CS}(\hat{\mathcal{F}})^m$. Just in order to make clear that all assumptions are satisfied, it is helpful to realize first that $\mathcal{S} \in \mathbb{CA}(\mathcal{F})^m$ and $\alpha \in \mathcal{M}^1(\mathcal{F})^m$ ensure that $(\mathcal{S}^{(i)}\alpha^{(i)})_{i=1}^m \in \mathcal{M}^1(\mathcal{F})^m$, (which is the vector of drift coefficients of $\mathcal{S}$) and second, that this process has a ν-progressive projection $(\mathcal{S}^{(i)}\hat{\alpha}^{(i)})_{i=1}^m$ which follows for example by Lemma 3.18 (iii). Finally, we get from (3.10) that $\mathbb{V} \overset{\text{as}}{=} \hat{\mathbb{V}}$ holds as $\hat{\alpha}^{(0)} \overset{\text{ae}}{=} \alpha^{(0)}$ and since $\mathbb{V} \in \mathbb{CA}(\mathcal{F})^m$, we get from Lemma 3.37 (i) that $\mathbb{V} \in \mathbb{CS}(\widehat{\mathcal{F}})^m$ holds.

(ii) Let $\widehat{\pi}$ be the position of $(\widehat{\varphi}^{(0)}, \widehat{\varphi})$, $\hat{\pi} \overset{\text{ae}}{=} \hat{\theta}$ be the position of $(\hat{\varphi}^{(0)}, \hat{\varphi})$ and let $\hat{\alpha}$ be as in (3.33). From (3.10) and (4.47), we get that

$$\mathbb{V} \overset{\text{as}}{=} \hat{M} + \int (\hat{\alpha}_s - \alpha_s^{(0)} 1_m)\,\mathrm{d}s \overset{\text{as}}{=} \hat{M} + \int \Sigma_s \hat{\theta}_s \mathrm{d}s$$

since $\hat{\alpha} \overset{\text{ae}}{=} \alpha^{(0)}1_m + \Sigma\hat{\theta}$ as stated in Definition 3.42. From (3.11) we get that

$$\ln\big(\widehat{\mathcal{W}}_t / \widehat{\mathcal{W}}_0\big) \overset{\text{as}}{=} \int_0^t \widehat{\pi}^{\top} \mathrm{d}\mathbb{V} + \int_0^t (\alpha_s^{(0)} - \tfrac{1}{2}\widehat{\pi}_s^{\top}\Sigma_s\widehat{\pi}_s)\,\mathrm{d}s$$

and that the same holds with $(\widehat{\mathcal{W}}, \widehat{\pi})$ replaced by $(\hat{\mathcal{W}}, \hat{\theta})$. Then we get that

$$\ln\Big(\frac{\widehat{\mathcal{W}}_t/\widehat{\mathcal{W}}_0}{\hat{\mathcal{W}}_t/\hat{\mathcal{W}}_0}\Big) \overset{\text{as}}{=} \int_0^t [(\widehat{\pi} - \hat{\theta})^{\top}\mathrm{d}\mathbb{V} - \tfrac{1}{2}\int_0^t (\widehat{\pi}_s^{\top}\Sigma_s\widehat{\pi}_s - \hat{\theta}_s^{\top}\Sigma_s\hat{\theta}_s)\,\mathrm{d}s] \overset{\text{as}}{=} L_t - \tfrac{1}{2}\langle L\rangle_t,$$

where $L \in \mathbb{CM}_{loc}(\hat{\mathcal{F}})$ is from Corollary A.17 such that $L \overset{\text{as}}{=} \int (\tilde{\pi} - \hat{\theta})^{\top}\mathrm{d}\hat{M}$. Then similarly as in the end of the proof of Proposition 3.13, we obtain by Lemma 2.27 in [9] that (3.35) holds.

(iii) As $\sigma \in \mathcal{M}^2(\hat{\mathcal{F}})^{m\times m}$ has regular values by assumption, we have that $H \overset{\text{def}}{=} \sigma^{-1} \in \mathbb{L}(\mathcal{M}(\hat{\mathcal{F}}))^{m\times m}$. Further note that $\langle\langle \hat{M}\rangle\rangle \overset{\text{as}}{=} \langle\langle V\rangle\rangle \overset{\text{as}}{=} \int \sigma_t\sigma_t^{\top}\mathrm{d}t$ and that $\int_0^t \mathrm{tr}\{H^{\top}H\mathrm{d}\langle\langle\hat{M}\rangle\rangle\} \overset{\text{as}}{=} tm < \infty, t \in [0,\infty)$, and since $\hat{M} \in \mathbb{CM}_{loc}(\hat{\mathcal{F}})^m$ by (4.47), we get that from Corollary A.32 that there exists

$$(4.48) \qquad \mathbb{CM}_{loc}(\hat{\mathcal{F}})^m \ni \hat{B} \overset{\text{as}}{=} \int H\mathrm{d}\hat{M}, \quad \text{s.t.} \quad \hat{M} \overset{\text{as}}{=} \int \sigma \mathrm{d}\hat{B}, \quad \langle\langle\hat{B}\rangle\rangle_t \overset{\text{as}}{=} t\mathbb{1}_m, \quad t \in [0,\infty).$$

Then we get by the Lévy Characterization Theorem, see Proposition A.26, that $\hat{B}$ is really an m-dimensional standard $\hat{\mathcal{F}}$-Brownian motion. Finally, we get that (3.36) follows from (4.47,4.48). ☐

Appendix A.

Lemma A.1. *Let* $X : \mathsf{U} \to \mathsf{D}, Y : \mathsf{V} \to \mathsf{T}$ *be maps and* $(\mathsf{D}, \mathcal{D}), (\mathsf{T}, \mathcal{T})$ *measurable spaces. Then*

$$\sigma(X \odot Y) \overset{\text{def}}{=} \{(X \odot Y)^{-1}C; C \in \mathcal{D} \otimes \mathcal{T}\} = \sigma(X) \otimes \sigma(Y),$$

where $\sigma(X) \overset{\text{def}}{=} \{X^{-1}D; D \in \mathcal{D}\}, \sigma(Y) \overset{\text{def}}{=} \{Y^{-1}T; T \in \mathcal{T}\}$.

Proof. If $D \in \mathcal{D}, T \in \mathcal{T}$, then $C \overset{\text{def}}{=} D \times T \in \mathcal{D} \otimes \mathcal{T}$ and $X^{-1}D \times Y^{-1}T = (X \odot Y)^{-1}C \in \sigma(X \odot Y)$. Hence, we get that

$$\sigma(X) \otimes \sigma(Y) = \sigma\{X^{-1}D \times Y^{-1}T; D \in \mathcal{D}, T \in \mathcal{T}\} \subseteq \sigma(X \odot Y).$$

On the other hand, $\mathcal{H} \overset{\text{def}}{=} \{C \in \mathcal{D} \otimes \mathcal{T}; (X \odot Y)^{-1}C \in \sigma(X) \otimes \sigma(Y)\}$ is a σ-algebra containing sets of type $D \times T, D \in \mathcal{D}, T \in \mathcal{T}$. Hence, $\mathcal{H} = \mathcal{D} \otimes \mathcal{T}$ which means that also $\sigma(X \odot Y) \subseteq \sigma(X) \otimes \sigma(Y)$. ☐

A.1. **Continuous processes and their convergence.** We will now introduce the *absolute Cauchy property* and *absolute convergence* in metric spaces, which will soon come in handy.

Definition A.2. Let (E, d) be a metric space. We say that $(x_n)_{n=1}^{\infty} \in E^{\mathbb{N}}$ is an *absolute d-Cauchy sequence* in E if

$$\sum_{n \in \mathbb{N}} d(x_n, x_{n+1}) < \infty.$$

If such a sequence also converges to $x \in E$, i.e., if $d(x_n, x) \to 0$ as $n \to \infty$, we say that $(x_n)_{n=1}^{\infty}$ *converges in d to x absolutely* and we write $x_n \xrightarrow{abs} x, n \to \infty$. If it is clear from the context, we will omit the link to the considered metric d in the notation introduced above.

Remark A.3. (i) Every absolutely Cauchy sequence is a Cauchy sequence by the triangle inequality.
(ii) Every Cauchy sequence has an absolutely Cauchy subsequence.

Proposition A.4. *A metric space (E, d) is complete if and only if every absolute Cauchy sequence is convergent.*

Proof. If the metric space (E, d) is complete, use (i) of Remark A.3 in order to verify that every absolute Cauchy sequence is convergent by the definition of a complete metric space. On the other hand, in order to show that a metric space is complete, it is enough to show that every Cauchy sequence has a convergent subsequence.

Now, let us assume that every absolute Cauchy sequence is convergent and consider a Cauchy sequence. By Remark A.3 (ii), the sequence has an absolutely Cauchy subsequence, and we get from assumption that this subsequence is convergent. ☐

Notation A.5. Let B be a topological vector space with the zero element denoted as $\mathbb{0}$, with the topology generated by a metric d and with the corresponding Borel σ-algebra denoted as $\mathcal{B}$. If $\{b^{(n)}\}_{n=1}^{\infty} \subseteq B$, we put

$$(\exists\ d\text{-}\lim_{n\to\infty} b^{(n)}) \overset{\text{def}}{=} \begin{cases} d\text{-}\lim_n b^{(n)} & \text{if the limit in } d \text{ exists as } n \to \infty, \\ \mathbb{0} & \text{otherwise.} \end{cases}$$

When it is clear from the context, we will suppress the dependence of the limit on the corresponding metric in the notation.

Remark A.6. The notation introduced above will be used only in the two following cases. First, if $B = \mathbb{R}^k$ is equipped with the Euclidean metric and second, if $(B, \mathcal{B}) = (\mathbb{C}^k, \mathscr{C}^k_\infty)$, where $k \in \mathbb{N}$.

Note that if $\{b^{(n)}\}_{n=1}^\infty \subseteq \mathbb{C}$, then $(\exists \lim_n b^{(n)}) \in \mathbb{C}$, but $(\exists \lim_n b_t^{(n)})_{t\in[0,\infty)}$ does not have to be a continuous function. In particular, these two functions do not have to coincide in general.

Proposition A.7. *Let $(\Omega, \mathcal{A})$ be a measurable space. (i) Let $(I^{(n)})_{n=1}^\infty \in \mathbb{L}(\mathcal{A}, \mathscr{C}_\infty)^{\mathbb{N}}$. Then*

$$I \overset{\text{def}}{=} \exists \lim_{n\to\infty} I^{(n)} \in \mathbb{L}(\mathcal{A}, \mathscr{C}_\infty).$$

(ii) Part (i) of the statement holds also with $\mathbb{L}(\mathcal{A}, \mathscr{C}_\infty)$ replaced by $\mathbb{L}(\mathcal{A})$.

Proof. (i) First, we will consider the set of all $\omega \in \Omega$ such that $(I^{(n)}(\omega))_{n=1}^\infty$ is a convergent sequence in $\mathbb{C}$. As $\mathbb{r}$ is a complete metric, the corresponding set can be formally defined as

$$A \overset{\text{def}}{=} \{\omega \in \Omega;\ \forall k \in \mathbb{N}\ \ \exists n_0 \in \mathbb{N}\ \ \forall m, n \geq n_0\ \ \mathbb{r}(I^{(n)}(\omega), I^{(m)}(\omega)) < 1/k\}. \tag{A.1}$$

Since $\mathbb{r}(X, Y) \in \mathbb{L}(\mathcal{A})$ whenever $X, Y \in \mathbb{C}(\Omega, \mathcal{A})$, as already mentioned at the beginning of Definition 2.3, we immediately get that $A \in \mathcal{A}$. Note that $I\,1_A = I$ holds by the definition of A and the definition of I. Further, we obtain from (A.1) and the definition of $\mathbb{r}$ that

$$\mathbb{r}(I^{(n)}1_A, I\,1_A) = \mathbb{r}(I^{(n)}, I)\,1_A \to 0 \quad \text{as} \quad n \to \infty.$$

As every projection $\mathbb{p}_t : \mathbb{C} \to \mathbb{R}$ is a continuous function, belonging to $\mathbb{L}(\mathscr{C}_\infty)$, we get that

$$I_t \overset{\text{def}}{=} \mathbb{p}_t \circ I = \mathbb{p}_t \circ (I\,1_A) = \lim_{n\to\infty} \mathbb{p}_t \circ (I^{(n)}\,1_A) \in \mathbb{L}(\mathcal{A}), \quad t \geq 0. \tag{A.2}$$

Further note that $I \in \mathbb{C}^\Omega$ holds by definition, i.e., $I(\omega) \in \mathbb{C}$ whenever $\omega \in \Omega$. Then we get from (A.2) that $I \in \mathbb{C}(\Omega, \mathcal{A}) = \mathbb{L}(\mathcal{A}, \mathscr{C}_\infty)$, and then we get that the part (i) is verified.

(ii) The part (ii) can be shown similarly. It is more or less enough to replace $\mathbb{L}(\mathcal{A}, \mathscr{C}_\infty)$ by $\mathbb{L}(\mathcal{A})$, $\mathbb{r}$ by Euclidean metric on $\mathbb{R}$ and to omit the part around (A.2). □

Lemma A.8. *Let $(X^{(n)})_{n=1}^\infty \in \mathbb{C}(\Omega, \mathcal{A})^{\mathbb{N}}$ be an absolutely $\mathbb{p}$-Cauchy sequence. Then there exists $X \in \mathbb{C}(\Omega, \mathcal{A})$ such that $\mathbb{r}(X^{(n)}, X) \to 0$ as $n \to \infty$ almost surely.*

Proof. By Proposition A.7 $X \overset{\text{def}}{=} \exists \lim_n X^{(n)} \in \mathbb{C}(\Omega, \mathcal{A})$. By assumption,

$$\infty > \sum_{n\in\mathbb{N}} \mathbb{p}(X^{(n)}, X^{(n+1)}) = \mathbb{E} \sum_{n\in\mathbb{N}} \mathbb{r}(X^{(n)}, X^{(n+1)}),\ \text{therefore}\ \infty \overset{\text{as}}{>} \sum_{n\in\mathbb{N}} \mathbb{r}(X^{(n)}, X^{(n+1)}).$$

As $\mathbb{r}$ is a complete metric, there exists an $\mathbb{r}$-limit $Y(\omega)$ of the sequence $X^{(n)}(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, and then we immediately get $X(\omega) = Y(\omega)$ holds for $\mathbb{P}$-a.e. $\omega \in \Omega$ by the definition of X. □

Corollary A.9. *Let $X^{(n)} \overset{abs}{\rightsquigarrow} X$, then $\mathbb{r}(X^{(n)}, X) \to 0$ almost surely as $n \to \infty$.*

Proof. By Lemma A.8, there exists $Y \in \mathbb{C}(\Omega, \mathcal{A})$ such that $\mathbb{r}(X^{(n)}, Y) \to 0$ as $n \to \infty$ almost surely. Then we get by the Dominated Convergence Theorem, by the definition of $\mathbb{p}$ and by the triangle inequality that $\mathbb{p}(X, Y) \leq \mathbb{p}(X^{(n)}, X) + \mathbb{E}[\mathbb{r}(X^{(n)}, Y)] \to 0$ as $n \to \infty$, i.e., $\mathbb{p}(X, Y) = 0$ which means that $X \overset{\text{as}}{=} Y$. □

Proposition A.10. *Metric $\mathbb{p}$ defined by* (2.3) *on $\mathbb{C}(\Omega, \mathcal{A})$ is complete.*

Proof. By Proposition A.4, it is enough to show that every absolutely $\mathbb{p}$-Cauchy sequence, say $(X^{(n)})_{n=1}^\infty$, is convergent in $\mathbb{p}$. By Lemma A.8, there exists $X \in \mathbb{C}(\Omega, \mathcal{A})$ such that $\mathbb{r}(X^{(n)}, X) \to 0$ as $n \to \infty$ almost surely. Then we get by the Dominated Convergence Theorem that $\mathbb{p}(X^{(n)}, X) = \mathbb{E}[\mathbb{r}(X^{(n)}, X)] \to 0$ as $n \to \infty$. □

Lemma A.11. *Let $\mathscr{F}, \mathscr{G}$ be filtrations on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that $\mathscr{F}_t \subseteq \mathscr{G}_t, t \in [0, \infty)$. Let $X \in \mathbb{CA}(\mathscr{F})$,$Y \in \mathbb{CM}_{loc}(\mathscr{G})$ be such that $X \overset{\text{as}}{=} Y$, then $X \in \mathbb{CM}_{loc}(\mathscr{F})$.*

Proof. As $\mathscr{F}$ is a subfiltration of $\mathscr{G}$, we get that $X \in \mathbb{CA}(\mathscr{F}) \subseteq \mathbb{CA}(\mathscr{G})$ and as $X \overset{\text{as}}{=} Y \in \mathbb{CM}_{loc}(\mathscr{G})$, we have that $0 \overset{\text{as}}{=} D \overset{\text{def}}{=} X - Y \in \mathbb{CM}(\mathscr{G}) \subseteq \mathbb{CM}_{loc}(\mathscr{G})$. Then also $X = D + Y \in \mathbb{CM}_{loc}(\mathscr{G})$ and as $\mathbb{CM}_{loc}(\mathscr{G}) \cap \mathbb{CA}(\mathscr{F}) = \mathbb{CM}_{loc}(\mathscr{F})$ holds by Theorem 1.4.3 (a), we get that also $X \in \mathbb{CM}_{loc}(\mathscr{F})$. □

A.2. **Simple processes and elementary integration.**

Definition A.12. Let $(\Omega, \mathcal{A}, \mathbb{P}, \mathscr{F})$ be a filtered probability space. A process H is called *$\mathscr{F}$-simple* if there exists an increasing sequence $(s_n)_{n=0}^{\infty}$ starting from $s_0 = 0$ tending to $\lim_n s_n = \infty$ as $n \to \infty$ such that $H_{s_k} \in \mathbb{L}(\mathscr{F}_{s_{k-1}})$ holds for every $k \in \mathbb{N}$ and that

$$H_t = \sum_{k \in \mathbb{N}} H_{s_k} 1_{[s_{k-1} < t \le s_k]}, \quad t \in [0, \infty). \tag{A.3}$$

The set of $\mathscr{F}$-simple processes will be denoted as $\mathscr{S}(\mathscr{F})$. Further, let $X = (X_t)_{t \ge 0}$ be a real-valued random process. By

$$\oint H \, \mathrm{d}X \overset{\text{def}}{=} \left(\oint_0^t H \mathrm{d}X\right)_{t \ge 0} \quad \text{with} \quad \oint_0^t H \mathrm{d}X \overset{\text{def}}{=} \sum_{k \in \mathbb{N}} H_{s_k} (X_{t \wedge s_k} - X_{t \wedge s_{k-1}}) \tag{A.4}$$

we denote the corresponding *simple integral* of the process H w.r.t. X. If it is helpful, we are also allowed to emphasize the time variable in the notation as follows $\oint H \mathrm{d}X = \oint H_s \, \mathrm{d}X_s$.

Remark A.13. Note that $Y \overset{\text{def}}{=} \oint H \mathrm{d}X \in \mathbb{CA}(\mathscr{F})$ holds if $H \in \mathscr{S}(\mathscr{F})$ and $X \in \mathbb{CA}(\mathscr{F})$, and that $\int H \mathrm{d}X \overset{\text{as}}{=} Y \in \mathbb{CM}_{loc}(\mathscr{F})$ holds if additionally $X \in \mathbb{CM}_{loc}(\mathscr{F})$ and that the same holds with $\mathbb{CM}_{loc}$ replaced by $\mathbb{CS}$.

A.3. **Enriched filtration.** Here A^{c} will stand for the complement of a set A and $A \triangle B$ will stand for the symmetric difference of sets A and B.

Lemma A.14. *Let $(\Omega, \mathcal{A})$ be a measurable space, $\mathcal{N} \subseteq \mathcal{A}$ be a family of sets closed under countable unions and containing all $\mathcal{A}$-measurable subsets of its elements and let $\mathscr{D}$ be a sub σ-algebra of $\mathcal{A}$. (i) Then $\mathscr{D} \vee \sigma(\mathcal{N}) = \mathcal{D} \overset{\text{def}}{=} \{D \triangle N; D \in \mathscr{D}, N \in \mathcal{N}\}$.*
(ii) If $X \in \mathbb{L}(\mathcal{D})$, then there exists $Y \in \mathbb{L}(\mathscr{D})$ such that $[X \neq Y] \in \mathcal{N}$.
(iii) Let $(\mathsf{S}, \mathfrak{S})$ be a measurable space and $\mathsf{F} \in \mathfrak{S} \otimes \mathcal{D}$, then

$$\exists \mathbb{F} \in \mathfrak{S} \otimes \mathscr{D} \; \exists N \in \mathcal{N} \quad \mathsf{F} \triangle \mathbb{F} \subseteq \mathsf{S} \times N. \tag{A.5}$$

Proof. (i) Since any σ-algebra is closed under the symmetric differences, we immediately get that $\mathcal{D} \subseteq \mathscr{D} \vee \sigma(\mathcal{N})$. Since our assumptions ensure that $\varnothing \in \mathcal{N} \cap \mathscr{D}$, we get that $\mathscr{D} \cup \mathcal{N} \subseteq \mathcal{D}$, and hence the only thing we have to show in order to verify (i) is that $\mathcal{D}$ is a σ-algebra. As already stated $\varnothing \in \mathscr{D} \subseteq \mathcal{D}$ and $\mathcal{D}$ is easily shown to be closed under complements. Indeed, if $D \in \mathscr{D}$ and $N \in \mathcal{N}$, then $(D \triangle N)^{\mathrm{c}} = D^{\mathrm{c}} \triangle N \in \mathcal{D}$ as $D^{\mathrm{c}} \in \mathscr{D}$. Hence, it is enough to verify that $\mathcal{D}$ is closed under countable unions. Let $D_n \in \mathscr{D}, N_n \in \mathcal{N}$ whenever $n \in \mathbb{N}$. We will show that also

$$\mathsf{D} \overset{\text{def}}{=} \cup_{n=1}^{\infty} \mathsf{D}_n \in \mathcal{D}, \quad \text{where} \quad \mathsf{D}_n \overset{\text{def}}{=} D_n \triangle N_n, \quad n \in \mathbb{N}. \tag{A.6}$$

As $\mathscr{D}$ is a σ-algebra containing $D_n, n \in \mathbb{N}$, we have that also $D \overset{\text{def}}{=} \cup_{n=1}^{\infty} D_n \in \mathscr{D}$. Put $N \overset{\text{def}}{=} D \triangle \mathsf{D}$. As $N_n = D_n \triangle \mathsf{D}_n, n \in \mathbb{N}$, we get that $\mathcal{A} \ni N \subseteq \cup_{n=1}^{\infty} N_n$ and as $N_n \in \mathcal{N}$, we get from assumptions on $\mathcal{N}$ that also $N \in \mathcal{N}$. Thus, (A.6) is verified since we have that $\mathsf{D} = D \triangle N$, where $D \in \mathscr{D}$ and $N \in \mathcal{N}$.

(ii) Since $X \in \mathbb{L}(\mathcal{D})$, we have that $[X < c] \in \mathcal{D}$ holds whenever $c \in \mathbb{R}$, and then from (i) we get that there exist $D_c \in \mathscr{D}, N_c \in \mathcal{N}$ such that $[X < c] \triangle D_c = N_c, c \in \mathbb{Q}$. Then $N \overset{\text{def}}{=} \cup_{c \in \mathbb{Q}} N_c \in \mathcal{N}$ and

$$Y \overset{\text{def}}{=} \left(\mathrm{Inf}\{c \in \mathbb{Q}; \omega \in D_c\}\right)_{\omega \in \Omega} \in \mathbb{L}(\mathscr{D}), \quad \text{where} \quad \mathrm{Inf}\, A \overset{\text{def}}{=} \inf A \cdot 1_{[\inf A \in \mathbb{R}]} \quad \text{if} \quad A \subseteq \mathbb{R}.$$

Since also $X = \mathrm{Inf}\{c \in \mathbb{Q}; X < c\} \in \mathbb{L}(\mathcal{A})$, we get that $\mathcal{A} \ni [X \neq Y] \subseteq \cup_{c \in \mathbb{Q}} N_c = N \in \mathcal{N}$, which by assumptions on $\mathcal{N}$ of the lemma ensures that also $[X \neq Y] \in \mathcal{N}$.

(iii) We will just show that $\mathfrak{S} \otimes \mathcal{D} \subseteq \mathcal{F} \overset{\text{def}}{=} \{\mathsf{F} \in \mathfrak{S} \otimes \mathcal{D}; \text{(A.5) holds}\}$. If $S \in \mathfrak{S}$ and $D \in \mathcal{D}$, then we have from (i) that $D = \mathbb{D} \triangle N$ for some $\mathbb{D} \in \mathscr{D}$ and $N \in \mathcal{N}$, and then $\mathbb{F} \overset{\text{def}}{=} S \times \mathbb{D} \in \mathfrak{S} \otimes \mathscr{D}$ is such that $(S \times D) \triangle \mathbb{F} = S \times N \subseteq \mathsf{S} \times N$, which verifies that $\mathsf{F} \overset{\text{def}}{=} S \times D \in \mathcal{F}$. Since this kind of sets generates the σ-algebra $\mathfrak{S} \otimes \mathcal{D}$, it remains to show that $\mathcal{F}$ is also a σ-algebra. We already have that $\emptyset \in \mathcal{F}$. Further, we will show that $\mathcal{F}$ is closed under complements and finally under countable unions. Let $\mathsf{F} \in \mathcal{F}$, and let $\mathbb{F}, N$ be as in (A.5). Then $\mathsf{F}^c \in \mathfrak{S} \otimes \mathcal{D}, \mathbb{F}^c \in \mathfrak{S} \otimes \mathscr{D}$ and $\mathsf{F}^c \triangle \mathbb{F}^c = \mathsf{F} \triangle \mathbb{F} \subseteq \mathsf{S} \times N$, i.e., $\mathsf{F}^c \in \mathcal{F}$. Finally, let $\mathsf{F}_n \in \mathfrak{S} \otimes \mathcal{D}, \mathbb{F}_n \in \mathfrak{S} \otimes \mathcal{D}, N_n \in \mathcal{N}$ be s.t. $\mathsf{F}_n \triangle \mathbb{F}_n \subseteq \mathsf{S} \times N_n, n \in \mathbb{N}$, then

$$\mathsf{F} \overset{\text{def}}{=} \cup_{n=1}^{\infty} \mathsf{F}_n \in \mathfrak{S} \otimes \mathcal{D}, \quad \mathbb{F} \overset{\text{def}}{=} \cup_{n=1}^{\infty} \mathbb{F}_n \in \mathfrak{S} \otimes \mathscr{D}, \quad N \overset{\text{def}}{=} \cup_{n=1}^{\infty} N_n \in \mathcal{N}$$

and $\mathsf{F} \triangle \mathbb{F} \subseteq \cup_{n=1}^{\infty} (\mathsf{F}_n \triangle \mathbb{F}_n) \subseteq \cup_{n=1}^{\infty} \mathsf{S} \times N_n = \mathsf{S} \times N$. Hence, we have that $\mathsf{F} \in \mathcal{F}$. □

Lemma A.15. *In the context of Notation 2.7, whenever $t \in [0, \infty)$ we have that*

(1) $\mathcal{F}_t \overset{\text{def}}{=} \mathcal{F}_t^{\mathscr{F},\mathscr{G}}$ *is equal to*

$$\mathcal{G}_t \overset{\text{def}}{=} \{F \triangle N; F \in \mathscr{F}_t, N \in \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}\} = \{G; \exists F \in \mathscr{F}_t, F \triangle G \in \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}\},$$

(2) $\mathscr{N}_t^{\mathscr{G},\mathbb{P}} \subseteq \mathcal{N}_t \overset{\text{def}}{=} \{F \in \mathcal{F}_t; \mathbb{P}(F) = 0\} \subseteq \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}} = \mathscr{N}_\infty^{\mathcal{F},\mathbb{P}} \subseteq \mathcal{F}_0$.

(3) *Let $\mathcal{F}, \mathscr{G}$ be as in* (1) *and let the filtration $\mathscr{G}$ be enriched, i.e., let $\sigma(\mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}) \subseteq \mathscr{G}_0$. If $\mathbb{CA}(\mathscr{G}) \ni X \overset{\text{as}}{=} Y \in \mathbb{CA}(\mathcal{F})$, then also $X \in \mathbb{CA}(\mathcal{F})$.*

(4) *Let $\mathcal{X}$ be a filtration on $(\Omega, \mathcal{A}, \mathbb{P}), k \in \mathbb{N}$, $X \in \mathbb{CA}(\mathcal{X})^k$. Then for each $t \in [0, \infty)$*

$$\mathcal{F}_t^{X,\mathcal{X},\mathbb{P}} = \{[X \in C] \triangle N; C \in \mathscr{C}_t^k, N \in \mathscr{N}_\infty^{\mathcal{X},\mathbb{P}}\}, \quad \mathscr{N}_\infty^{X,\mathbb{P}} = \{[X \in D]; D \in \mathcal{C}_\infty^{\mathbb{P}_X}\}.$$

Proof. (1) Let $t \in [0, \infty)$ be fixed. The first equality follows from Remark 2.9 and Lemma A.14 (i) and the second equality from the following simple observation. If A, B, C are sets, then the following statements are equivalent

$$A = B \triangle C, \quad B = A \triangle C, \quad C = A \triangle B. \tag{A.7}$$

(2) Since $\emptyset \in \mathscr{F}_0$, we immediately get from the point (1) that $\mathscr{N}_\infty^{\mathscr{G},\mathbb{P}} \subseteq \mathcal{F}_0$. As stated in Remark 2.9, all elements of $\mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}$ are $\mathbb{P}$-null sets. Hence, we get that $\mathscr{N}_t^{\mathscr{G},\mathbb{P}} \subseteq \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}} \subseteq \mathcal{N}_0 \subseteq \mathcal{N}_t$ holds whenever $t \in [0, \infty)$.

Now, let $G \in \mathcal{N}_t$, i.e., $G \in \mathcal{F}_t$ is a $\mathbb{P}$-null set. By (1) there exists $F \in \mathscr{F}_t \subseteq \mathscr{G}_t \subseteq \mathscr{G}_\infty$ such that $N \overset{\text{def}}{=} G \triangle F \in \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}} \subseteq \mathscr{G}_\infty$. Then $F = G \triangle N$ is similarly as G and N a $\mathbb{P}$-null set and as it is also $\mathscr{F}_t \subseteq \mathscr{G}_t$-measurable, we get that $F \in \mathscr{N}_t^{\mathscr{G},\mathbb{P}} \subseteq \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}$. Since $N \in \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}$ and $\mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}$ is closed under (countable) unions, we get that also $F \cup N \in \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}$, and as the system $\mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}$ contains also $\mathscr{G}_\infty$-measurable subsets of its elements and $F \cup N \supseteq F \triangle N = G \in \mathscr{G}_\infty$, we get that also $G \in \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}$.

Since the last relation in (2) has already been shown, it remains to verify that $\mathscr{N}_\infty^{\mathscr{G},\mathbb{P}} = \mathscr{N}_\infty^{\mathcal{F},\mathbb{P}}$. Since $\mathscr{N}_\infty^{\mathscr{G},\mathbb{P}} \subseteq \sigma(\mathscr{F}_0 \cup \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}) = \mathcal{F}_0$, we get that each $N \in \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}$ is a $\mathbb{P}$-null set from $\mathcal{F}_0$, which by definition ensures that $N \in \mathscr{N}_0^{\mathcal{F},\mathbb{P}} \subseteq \mathscr{N}_\infty^{\mathcal{F},\mathbb{P}}$, and we have that $\mathscr{N}_\infty^{\mathscr{G},\mathbb{P}} \subseteq \mathscr{N}_\infty^{\mathcal{F},\mathbb{P}}$.

On the other hand, let $M \in \mathscr{N}_\infty^{\mathcal{F},\mathbb{P}}$. Then there are $M_n \in \mathscr{N}_n^{\mathcal{F},\mathbb{P}}$ such that $M = \cup_{n=0}^{\infty} M_n$, and by definition there are $\mathbb{P}$-null sets $G_n \in \mathcal{F}_n$ such that $M_n \subseteq G_n$. By Lemma A.15 point (1) there are $F_n \in \mathscr{F}_n$ and $N_n \in \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}$ such that $G_n = F_n \triangle N_n, n \in \mathbb{N}_0$. Then $\mathscr{G}_n \supseteq \mathscr{F}_n \ni F_n = G_n \triangle N_n$ is also a $\mathbb{P}$-null set (similarly as G_n and N_n), and we get that $F_n \in \mathscr{N}_n^{\mathscr{G},\mathbb{P}} \subseteq \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}$. Since also $N_n \in \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}$, we have that

$$\mathscr{G}_\infty \supseteq \mathcal{F}_\infty \ni M \subseteq \cup_{n=1}^{\infty} (F_n \cup N_n) \in \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}},$$

which by Remark 2.9 ensures that also $M \in \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}$. Hence $\mathscr{N}_\infty^{\mathscr{G},\mathbb{P}} \subseteq \mathscr{N}_\infty^{\mathcal{F},\mathbb{P}} \subseteq \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}$.

(3) Since $\mathscr{F}$ is a subfiltration of $\mathscr{G}$ which is already enriched, we get from (2) that also

$$\mathcal{F}_t = \mathcal{F}_t^{\mathscr{F},\mathscr{G}} = \mathscr{F}_t \vee \sigma(\mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}) \subseteq \mathscr{G}_t \vee \sigma(\mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}) = \mathscr{G}_t, \quad t \in [0, \infty). \tag{A.8}$$

We just have to show that $X \in \mathbb{A}(\mathcal{F})$, i.e., that $[X_t < c] \in \mathcal{F}_t$ holds whenever $t \in [0,\infty)$ and $c \in \mathbb{R}$. Since $\mathbb{CA}(\mathscr{G}) \ni X \overset{\text{as}}{=} Y \in \mathbb{CA}(\mathcal{F}) \subseteq \mathbb{CA}(\mathscr{G})$ by (A.8), we have by (2) that

$$N_t^{(c)} \overset{\text{def}}{=} [X_t < c] \vartriangle [Y_t < c] \in \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}} \subseteq \mathcal{F}_t, \quad \text{i.e.,} \quad [X_t < c] = [Y_t < c] \vartriangle N_t^{(c)} \in \mathcal{F}_t.$$

(4) See point (1) of the lemma and (2.9) in order to agree that it is enough to show that

$$\mathscr{N}_\infty^{X,\mathbb{P}} = \{[X \in D]; D \in \mathcal{C}_\infty^{\mathbb{P}_X}\}. \tag{A.9}$$

If $t \in [0,\infty)$, we get immediately from the definition that

$$\begin{aligned} \mathscr{N}_t^{X,\mathbb{P}} &= \{N = [X \in D]; N \subseteq [X \in C], \mathbb{P}_X(C) = 0, C \in \mathscr{C}_t^k, D \in \mathscr{C}_\infty^k\} \\ &\supseteq \{[X \in D]; \mathscr{C}_\infty^k \ni D \subseteq C \in \mathscr{C}_t^k, \mathbb{P}_X(C) = 0\} = \{[X \in D]; D \in \mathcal{C}_t^{\mathbb{P}_X}\} \end{aligned}$$

and then the "$\supseteq$" relation in (A.9) follows immediately from the definition of $\mathcal{C}_\infty^{\mathbb{P}_X}$ and $\mathscr{N}_\infty^{X,\mathbb{P}}$. On the other hand, let $N \in \mathscr{N}_\infty^{X,\mathbb{P}} \subseteq \mathscr{F}_\infty^X$, then it is of the form

$$N = [X \in D] = \cup_{n=0}^\infty N_n, \quad \text{where} \quad D \in \mathscr{C}_\infty^k, \quad N_n \in \mathscr{N}_n^{X,\mathbb{P}}.$$

Then there exist $\mathbb{P}$-null sets $\mathscr{F}_n^X \ni F_n \supseteq N_n, n \in \mathbb{N}_0$, of the form $F_n = [X \in C_n], C_n \in \mathscr{C}_n^k$, and as $0 = \mathbb{P}(F_n) = \mathbb{P}_X(C_n)$, we get that $C_n \in \mathcal{C}_n^{\mathbb{P}_X}$ and that $C \overset{\text{def}}{=} \cup_{n=0}^\infty C_n \in \mathcal{C}_\infty^{\mathbb{P}_X} \subseteq \mathscr{C}_\infty^k$. Finally, we get that $N = [X \in C \cap D]$, where $C \cap D \in \mathcal{C}_\infty^{\mathbb{P}_X}$, as $[X \in C] = \cup_n F_n \supseteq \cup_n N_n = N$. Hence, also the "$\subseteq$" relation in (A.9) is verified. □

In the next lemma, we will see that the enrichment of a filtration is independent of a locally equivalent change of measure, and that the measures remain locally equivalent also after enrichment of the filtration.

Lemma A.16. *In the context of Notation 2.7, let $\mathbb{P}, Q$ be locally $\mathscr{G}$-equivalent probability measures.*

(i) Then $\mathscr{N} \overset{\text{def}}{=} \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}} = \mathscr{N}_\infty^{\mathscr{G},Q}, \mathcal{F} \overset{\text{def}}{=} \mathcal{F}^{\mathscr{F},\mathscr{G},\mathbb{P}} = \mathcal{F}^{\mathscr{F},\mathscr{G},Q}$.

(ii) The measures $\mathbb{P}, Q$ are also locally $\mathcal{F}$-equivalent.

(iii) If $X, Y \in \mathbb{CA}(\mathcal{F})$ and $X = Y$ holds $\mathbb{P}$-a.s., then also $X = Y$ holds Q-almost surely.

(iv) Whenever $\{X^{(n)}\}_{n=1}^\infty \subseteq \mathbb{CA}(\mathcal{F})$ is a sequence convergent in the metric $\mathbb{p}$, there exists a process $X \in \mathbb{CA}(\mathcal{F})$ such that $\mathbb{p}(X^{(n)}, X) \to 0$ as $n \to \infty$.

Proof. (i) If $n \in \mathbb{N}$, then $\mathbb{P}|\mathscr{G}_n \sim Q|\mathscr{G}_n$ holds by assumption, and therefore $\mathscr{N}_n^{\mathscr{G},\mathbb{P}} = \mathscr{N}_n^{\mathscr{G},Q}$. Then we get that also

$$\mathscr{N}_\infty^{\mathscr{G},\mathbb{P}} = \mathscr{N}_\infty^{\mathscr{G},Q} \quad \text{and} \quad \mathcal{F}_t^{\mathscr{F},\mathscr{G},\mathbb{P}} = \mathscr{F}_t \vee \sigma(\mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}) = \mathscr{F}_t \vee \sigma(\mathscr{N}_\infty^{\mathscr{G},Q}) = \mathcal{F}_t^{\mathscr{F},\mathscr{G},Q}, \quad t \geq 0.$$

(ii) Let $G \in \mathcal{F}_t$ be such that $\mathbb{P}(G) = 0$, where $t \in [0,\infty)$. We will show that also $Q(G) = 0$. By Lemma A.15 point (1), we get that there exist $F \in \mathscr{F}_t, N \in \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}}$ such that $G = F \vartriangle N$, i.e., $G \vartriangle F = N$. As N is a $\mathbb{P}$-null set, we get that $\mathbb{P}(F) = \mathbb{P}(G) = 0$, and as $F \in \mathscr{F}_t$ and $\mathbb{P}|\mathscr{F}_t \sim Q|\mathscr{F}_t$, we get that also $Q(F) = 0$. By the point (i) $\mathscr{N}_\infty^{\mathscr{G},Q} = \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}} \ni N$, and we get that $Q(N) = 0$ and then also that $Q(G) = Q(F) = 0$. From the symmetry between $\mathbb{P}$ and Q, we obtain that these measures are really locally $\mathcal{F}$-equivalent.

(iii) Let $t \in [0,\infty)$. By assumption $N_t \overset{\text{def}}{=} [X_t \neq Y_t] \in \mathcal{F}_t$ is a $\mathbb{P}$-null set and as $\mathbb{P}|\mathcal{F}_t \sim Q|\mathcal{F}_t$ holds by the point (ii), we get that $Q(N_t) = 0$. Then we get from the continuity of $X - Y$ that also $Q(X \neq Y) = 0$.

(iv) First, we may assume that $(X^{(n)})_{n=1}^\infty$ is an absolutely $\mathbb{p}$-Cauchy sequence as otherwise we replace it by a suitable subsequence and use triangle inequality in the end. Then we have that

$$\infty > \textstyle\sum_n \mathbb{p}(X^{(n)}, X^{(n+1)}) = \mathbb{E}_\mathbb{P} \sum_n r(X^{(n)}, X^{(n+1)}) \geq \mathbb{E}_\mathbb{P} \sum_n 2^{-k} \wedge |X^{(n)} - X^{(n+1)}|_k^*, \quad k \in \mathbb{N}.$$

Hence, $N_k \overset{\text{def}}{=} [\sum_n |X^{(n)} - X^{(n+1)}|_k^* = \infty] \in \mathcal{F}_k$ is a $\mathbb{P}$-null set and then $N_k \in \mathcal{N}_k \subseteq \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}} \subseteq \mathcal{F}_0$ holds by Lemma A.15 point (2) whenever $k \in \mathbb{N}$. Then

$$X \overset{\text{def}}{=} \lim_{n\to\infty} X^{(n)} 1_{\Omega \setminus N} \in \mathbb{CA}(\mathcal{F}), \qquad \text{where} \qquad N \overset{\text{def}}{=} \cup_{k=1}^\infty N_k \in \mathscr{N}_\infty^{\mathscr{G},\mathbb{P}} \subseteq \mathcal{F}_0.$$

Obviously, $\mathbb{P}(N) = 0$ and it ensures that $r(X^{(n)}, X) \to 0$ a.s. and hence also $\wp(X^{(n)}, X) \to 0$ as $n \to \infty$ by the Dominated Convergence Theorem. □

Corollary A.17. *Let $\mathcal{F}$ be an enriched filtration. (i) If $X \in \mathbb{CS}(\mathcal{F})$, then there exists a non-decreasing process $\mathscr{K} \in \mathbb{CA}(\mathcal{F}^X) \subseteq \mathbb{CA}(\mathcal{F})$ starting from $\mathscr{K}_0 = 0$ s.t. $\mathscr{K} \overset{\text{as}}{=} \langle X \rangle$.*
(ii) Let $X \in \mathbb{CS}(\mathcal{F})$ and $H \in \mathbb{L}(\mathcal{M}(\mathcal{F}))$ be such that the following Itô stochastic integral is well defined $Y \overset{\text{def}}{=} \int H \mathrm{d}X$. If $\mathcal{G}$ is an enriched subfiltration of $\mathcal{F}$ such that $X \in \mathbb{CA}(\mathcal{G})$ and $H \in \mathbb{L}(\mathcal{M}(\mathcal{G}))$, then there exists $Z \in \mathbb{CA}(\mathcal{G})$ such that $Z \overset{\text{as}}{=} Y$.

Remark A.18. Note that if X, Y, H are as in (ii) of the corollary, then $\int H^{(n)} \mathrm{d}X \rightsquigarrow Y$ as $n \to \infty$ holds with $H^{(n)} \overset{\text{def}}{=} H 1_{[|H| \le n]}$ in general, and in case when $H, H^{(n)} \in \mathbb{L}(\mathcal{M}(\mathcal{F})), n \in \mathbb{N}$, are equally bounded, it is enough that $H_t^{(n)} \to H_t$ as $n \to \infty$ holds for almost every $t \in [0, \infty)$.

Proof of Corollary A.17. (i) Put $S^{(n)} \overset{\text{def}}{=} \{s_k^{(n)}\}_{k=0}^\infty$, where $s_k^{(n)} \overset{\text{def}}{=} k2^{-n}$. By Theorem (1.8) in [24, Chapter IV]

$$\text{(A.10)} \qquad |\langle X \rangle - \mathcal{V}^{(2)}(X; S^{(n)})|_t^* \overset{\mathbb{P}}{\to} 0, \qquad \text{where} \qquad \mathcal{V}_t^{(2)}(X; S^{(n)}) = \sum_{k=1}^\infty (X_{t\wedge s_k^{(n)}} - X_{t\wedge s_{k-1:}^{(n)}})^2,$$

as $n \to \infty, t \in [0, \infty)$. From (A.10) and Remark 2.4, we get that

$$\lim_{n\to\infty} \wp\big(\langle X \rangle, \mathcal{V}^{(2)}(X; S^{(n)})\big) = 0$$

and since $\mathcal{V}^{(2)}(X; S^{(n)}) \in \mathbb{CA}(\mathscr{F}^X) \subseteq \mathbb{CA}(\mathcal{F}^X)$ holds whenever $n \in \mathbb{N}$, we get from the point (iv) of the previous lemma that there exists $\mathcal{K} \in \mathbb{CA}(\mathcal{F}^X)$ starting from $\mathcal{K}_0 \overset{\text{as}}{=} 0$ s.t. $\mathcal{K}_t \overset{\text{as}}{=} \langle X \rangle_t$ if $t \in [0, \infty)$. Further, we get from the last equality in Lemma A.15 (2) that

$$\mathscr{N}_t^{\mathcal{K}} \subseteq \mathscr{N}_\infty^{\mathcal{F}^X} = \mathscr{N}_\infty^X \subseteq \mathcal{F}_0^X, \quad t \in [0, \infty),$$

as $\mathcal{K} \in \mathbb{CA}(\mathcal{F}^X)$, and whenever $0 \le s \le t < \infty$ we get that

$$N_0 \overset{\text{def}}{=} [\mathcal{K}_0 \ne 0] \in \mathscr{N}_0^{\mathcal{K}} \subseteq \mathscr{N}_\infty^X \subseteq \mathcal{F}_0^X, \quad N_{s,t} \overset{\text{def}}{=} [\mathcal{K}_s > \mathcal{K}_t] \in \mathscr{N}_t^{\mathcal{K}} \subseteq \mathscr{N}_\infty^X \subseteq \mathcal{F}_0^X.$$

Then $\langle X \rangle \overset{\text{as}}{=} \mathscr{K} \overset{\text{def}}{=} \mathcal{K} 1_{\Omega \setminus N} \in \mathbb{CA}(\mathcal{F}^X)$ is a non-decreasing process starting from $\mathscr{K}_0 = 0$, where

$$N \overset{\text{def}}{=} [\mathcal{K}_0 \ne 0, \exists\ 0 \le s \le t < \infty\ \mathcal{K}_s > \mathcal{K}_t] = N_0 \cup \cup_{\mathbb{Q}^+ \ni s < t \in \mathbb{Q}} N_{s,t} \in \mathscr{N}_\infty^X \subseteq \mathcal{F}_0^X.$$

(ii) First, we define certain maps

$$\kappa^{[n]}(h) \overset{\text{def}}{=} h \cdot 1_{[|h| \le n]}, \quad \kappa^{(n)}(h) \overset{\text{def}}{=} \big(n \textstyle\int_{(t-1/n)^+}^t h_s \mathrm{d}s\big)_{t \ge 0}, \quad \kappa^{\langle n \rangle}(h) \overset{\text{def}}{=} \sum_{k=0}^\infty h_{\frac{k}{n}} 1_{[\frac{k}{n} < t \le \frac{k+1}{n}]}$$

and their composition $\kappa^{(n,k,j)} \overset{\text{def}}{=} \kappa^{\langle j \rangle} \circ \kappa^{(k)} \circ \kappa^{[n]}$ which is obviously well defined on $\mathbb{L}(\mathscr{B}_\infty)$. Since

$$\int \kappa^{[n]}(H) \,\mathrm{d}X \to Y, \quad \int \kappa^{(n)}(M) \,\mathrm{d}X \to \int M \,\mathrm{d}X, \quad \oint \kappa^{\langle n \rangle}(N) \,\mathrm{d}X \to \int N \,\mathrm{d}X$$

in $\wp$ as $n \to \infty$ holds by Remark A.18 whenever $N \in \mathbb{CA}(\mathcal{F}), M \in \mathbb{L}(\mathcal{M}(\mathcal{F}))$ are bounded processes, we get that there exists a sequence $(n_i, k_i, j_i)_{i=1}^\infty \in \mathbb{N}^{\mathbb{N}\times 3}$ such that

$$Y^{(i)} \overset{\text{def}}{=} \oint \kappa^{(n_i,k_i,j_i)}(H) \,\mathrm{d}X \to Y \quad \text{in } \wp \text{ as } i \to \infty.$$

Note that $\kappa^{(n_i,k_i,j_i)}(H) \in \mathscr{S}(\mathcal{G})$ holds as $H \in \mathbb{L}(\mathcal{M}(\mathcal{G}))$ and that $X \in \mathbb{CA}(\mathcal{G})$ holds by assumption. Then we get from Remark A.13 that also $Y^{(i)} \in \mathbb{CA}(\mathcal{G})$ and then from Lemma A.16 (iv) that there exists $\mathbb{CA}(\mathcal{G}) \ni Z \overset{\text{as}}{=} \wp \lim_i Y^{(i)} \overset{\text{as}}{=} Y \overset{\text{as}}{=} \int H \mathrm{d}X$. □

Further, $\langle X\rangle$ will stand for the $\mathcal{F}^X$-adapted version of $\langle X\rangle$ as mentioned in Notation 2.11.

Lemma A.19. *Let $\mathscr{D}$ be a filtration on a set D and let $f\in\mathbb{A}(\mathscr{D})$. Then (i)*

(A.11)
$$\mathbb{f}^{(n)} \overset{\text{def}}{=} \Big(\textstyle\sum_{k=0}^{\infty} f_{k2^{-n}} 1_{[k\le t2^n<k+1]}\Big)_{t\ge0}\in\mathbb{L}(\mathcal{M}(\mathscr{D})),\ n\in\mathbb{N},\qquad \mathbb{y}\overset{\text{def}}{=}\exists\lim_{n\to\infty}\mathbb{f}^{(n)}\in\mathbb{L}(\mathcal{M}(\mathscr{D})),$$

(ii) $\mathbb{z}\in\mathbb{L}(\mathcal{M}(\mathscr{D}))$, where

$$\mathbb{z}_t(x)\overset{\text{def}}{=}\exists\lim_{n\to\infty} n[\mathbb{y}_t(x)-\mathbb{y}_{(t-1/n)^+}(x)],\quad x\in\mathsf{D},\ t\in[0,\infty). \tag{A.12}$$

Proof. (i) By Proposition A.7, it is enough to verify the first relation in (A.11) as follows

$$\{(u,x)\in[0,t]\times\mathsf{D};\mathbb{f}^{(n)}_u(x)<c\}=\bigcup_{k\in[0,2^nt]\cap\mathbb{Z}} U^{(n)}_{k,t}\times\{x\in\mathsf{D};f_{k2^{-n}}(x)<c\}\in\mathscr{B}_t\otimes\mathscr{D}_t,$$

$c\in\mathbb{R}$, where $U^{(n)}_{k,t}\overset{\text{def}}{=}\{u\in[0,t];k\le u2^n<k+1\}=[0,t]\cap[\frac{k}{2^n},\frac{k+1}{2^n})$.
(ii) If we use Proposition A.7 again, we get that the only thing we have to verify is that

$$\mathbb{y}^{[s]}\overset{\text{def}}{=}(\mathbb{y}_{(u-s)^+})_{u\ge0}\in\mathbb{L}(\mathcal{M}(\mathscr{D})),\quad s\in[0,\infty), \tag{A.13}$$

i.e., to show that

$$\mathbb{B}\overset{\text{def}}{=}\{(u,x)\in[0,t]\times\mathsf{D};\mathbb{y}^{[s]}_u(x)<c\}\in\mathscr{B}_t\otimes\mathscr{D}_t,\quad s,t\in[0,\infty),\ c\in\mathbb{R}. \tag{A.14}$$

If $s\ge t$, then $\mathbb{B}=[0,t]\times\{x\in\mathsf{D};\mathbb{y}_0(x)<c\}\in\mathscr{B}_t\otimes\mathscr{D}_0\subseteq\mathscr{B}_t\otimes\mathscr{D}_t$. Let s,t,c from (A.14) be such that $s<t$. If $\mathcal{D}$ is a σ-algebra on D, it is easy to verify that the sets $\mathbb{D}\in\mathscr{B}_{t-s}\otimes\mathcal{D}$ satisfying

$$\mathbb{D}\in\mathscr{B}_{t-s}\otimes\mathcal{D}\ \Rightarrow\ \mathbb{D}^{[s]}\overset{\text{def}}{=}\{(u,x)\in[0,\infty)\times\mathsf{D};((u-s)^+,x)\in\mathbb{D}\}\in\mathscr{B}_t\otimes\mathcal{D} \tag{A.15}$$

form a σ-algebra which contains a generator of $\mathscr{B}_{t-s}\otimes\mathcal{D}$. Indeed, if $\mathbb{D}=[0,r]\times D$, where $r\in[0,t-s]$ and $D\in\mathcal{D}$, then $\mathbb{D}^{[s]}=[0,r+s]\times D\in\mathscr{B}_{r+s}\otimes\mathcal{D}\subseteq\mathscr{B}_t\otimes\mathcal{D}$. Hence, we get that (A.15) holds whenever $\mathcal{D}$ is a σ-algebra on D. By (i), we get that

$$\mathbb{D}\overset{\text{def}}{=}\{(u,x)\in[0,t-s]\times\mathsf{D};\mathbb{y}_u(x)<c\}\in\mathscr{B}_{t-s}\otimes\mathscr{D}_{t-s}$$

and then by (A.14, A.15) that $\mathbb{B}=\{(u,x)\in[0,t]\times\mathsf{D};\mathbb{y}_{(u-s)^+}(x)<c\}=\mathbb{D}^{[s]}\in\mathscr{B}_t\otimes\mathscr{D}_{t-s}\subseteq\mathscr{B}_t\otimes\mathscr{D}_t$. □

A.4. **Additional parameter.**

Lemma A.20. *Let $\mathscr{F}$ be a filtration, $\mathcal{D}$ be a σ-algebra, then $\mathbb{CA}(\mathscr{F}\otimes\mathcal{D})\subseteq\mathbb{L}(\mathcal{M}(\mathscr{F})\otimes\mathcal{D})$.*

Proof. 1. First, we will show that $\mathscr{S}(\mathscr{F}\otimes\mathcal{D})\subseteq\mathbb{L}(\mathcal{M}(\mathscr{F})\otimes\mathcal{D})$. Let $s\in[0,\infty)$ be fixed.

(a) We will show that

$$\mathfrak{F}_s\overset{\text{def}}{=}\{\Phi\in\mathscr{F}_s\otimes\mathcal{D};(s,\infty)\times\Phi\in\mathcal{M}(\mathscr{F})\otimes\mathcal{D}\}=\mathscr{F}_s\otimes\mathcal{D}. \tag{A.16}$$

Obviously, $\mathfrak{F}_s$ is a σ-algebra, and therefore in order to verify that (A.16) holds, it is sufficient to show that it contains the sets of type $F\times D$, where $F\in\mathscr{F}_s$ and $D\in\mathcal{D}$. If $F\in\mathscr{F}_s$, then $\mathbb{F}\overset{\text{def}}{=}(s,\infty)\times F\in\mathcal{M}(\mathscr{F})$ holds as

$$\mathbb{F}\cap\Omega_t=((s,\infty)\cap[0,t])\times F=\begin{Bmatrix}\varnothing & \text{if } t\le s\\ (s,t]\times F & \text{if } s<t\end{Bmatrix}\in\mathscr{B}_t\otimes\mathscr{F}_t,\quad t\in[0,\infty),$$

and hence we get that $(s,\infty)\times F\times D=\mathbb{F}\times D\in\mathcal{M}(\mathscr{F})\otimes\mathcal{D}$, which verifies that $F\times D\in\mathfrak{F}_s$ holds whenever $D\in\mathcal{D}$. As mentioned above, this gives the equality in (A.16).

(b) Now, we will show that

$$\mathcal{X}_s\overset{\text{def}}{=}\{X\in\mathbb{L}(\mathscr{F}_s\otimes\mathcal{D});(X\,1_{[s<t]})_{t\ge0}\in\mathbb{L}(\mathcal{M}(\mathscr{F})\otimes\mathcal{D})\}=\mathbb{L}(\mathscr{F}_s\otimes\mathcal{D}). \tag{A.17}$$

Obviously, $\mathcal{X}_s$ is a linear set closed under pointwise convergence, and hence in order to verify the equality in (A.17), it is sufficient to show that

$$1_\Phi \in \mathcal{X}_s \quad \text{holds whenever} \quad \Phi \in \mathscr{F}_s \otimes \mathcal{D}. \tag{A.18}$$

Let $X = 1_\Phi$ and $\Phi \in \mathscr{F}_s \otimes \mathcal{D}$. Then we get by the first step of the proof that $\Phi \in \mathscr{F}_s \otimes \mathcal{D} = \mathfrak{F}_s$ and then we get from the definition of $\mathfrak{F}_s$ that

$$(X\,1_{[s<t]})_{t\geq 0} = 1_{(s,\infty)\times\Phi} \in \mathbb{L}(\mathcal{M}(\mathscr{F}) \otimes \mathcal{D}).$$

Hence, we have shown (A.18) and as mentioned above, we obtain (A.17).

(c) Let $H \in \mathscr{S}(\mathscr{F} \otimes \mathcal{D})$, then there exists an increasing sequence $(s_n)_{n=0}^\infty \in [0,\infty)^{\mathbb{N}}$ starting from $s_0 = 0$ and tending to infinity, and a sequence $Y_n \in \mathbb{L}(\mathscr{F}_{s_n} \otimes \mathcal{D}), n \in \mathbb{N}_0$, s.t.

$$H = \textstyle\sum_{n=1}^\infty X^{(n)}, \quad \text{where} \quad X^{(n)} = (Y_n 1_{[s_n<t]})_{t\geq 0}. \tag{A.19}$$

By point (b) of this proof, $Y_n \in \mathbb{L}(\mathscr{F}_{s_n} \otimes \mathcal{D}) = \mathcal{X}_{s_n}$ which means that $X^{(n)} \in \mathbb{L}(\mathcal{M}(\mathscr{F}) \otimes \mathcal{D})$ holds whenever $n \in \mathbb{N}$. As the sum of measurable random variables is again measurable, we get that also $H \in \mathbb{L}(\mathcal{M}(\mathscr{F}) \otimes \mathcal{D})$. Hence, we have that $\mathscr{S}(\mathscr{F} \otimes \mathcal{D}) \subseteq \mathbb{L}(\mathcal{M}(\mathscr{F}) \otimes \mathcal{D})$ holds.

2. Finally, if $H \in \mathbb{CA}(\mathscr{F} \otimes \mathcal{D})$, then there are $H^{(n)} \in \mathscr{S}(\mathscr{F} \otimes \mathcal{D}) \subseteq \mathbb{L}(\mathcal{M}(\mathscr{F}) \otimes \mathcal{D}), n \in \mathbb{N}$, tending to $H - H_0$ pointwise, and we get that also $H - H_0 \in \mathbb{L}(\mathcal{M}(\mathscr{F}) \otimes \mathcal{D})$. For a suitable choice of $(H^{(n)})_{n=1}^\infty$ see for example (4.24) with X replaced by $H - H_0$.

Hence, it remains to show that $\mathbb{H}_0 : (t,\omega,y) \mapsto H_0(\omega,y)$ is in $\mathbb{L}(\mathcal{M}(\mathscr{F}) \otimes \mathcal{D})$, i.e., that $\mathbb{H}_0^{-1}(-\infty,c) = [0,\infty) \times H_0^{-1}(-\infty,c) \in \mathcal{M}(\mathscr{F}) \otimes \mathcal{D}$ holds whenever $c \in \mathbb{R}$. Since $H \in \mathbb{CA}(\mathscr{F} \otimes \mathcal{D})$, we have that $H_0 \in \mathbb{L}(\mathscr{F}_0 \otimes \mathcal{D})$ which gives that

$$\{0\} \times H_0^{-1}(-\infty,c) \subseteq \mathscr{B}_0 \otimes \mathscr{F}_0 \otimes \mathcal{D} \subseteq \mathcal{M}(\mathscr{F}) \otimes \mathcal{D}, \quad c \in \mathbb{R}.$$

Then it is enough to verify that $(0,\infty) \times H_0^{-1}(-\infty,c) \in \mathcal{M}(\mathscr{F}) \otimes \mathcal{D}$, but this follows from $H_0^{-1}(-\infty,c) \in \mathscr{F}_0 \otimes \mathcal{D} = \mathfrak{F}_0$, see (A.16). □

Lemma A.21. *Let $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$ be a filtration on $(\Omega,\mathcal{A},\mathbb{P})$ and let $(\mathsf{D},\mathcal{D})$ be a measurable space. (i) Then $\mathcal{M}(\mathscr{F}) \otimes \mathcal{D} \subseteq \mathcal{M}(\mathscr{F} \otimes \mathcal{D})$.*
(ii) If $H \in \mathbb{L}(\mathcal{M}(\mathscr{F} \otimes \mathcal{D}))$, then there exists $\tilde{H} \in \mathbb{L}(\mathcal{M}(\mathscr{F}) \otimes \mathcal{D})$ such that $\int_0^\infty 1_{[H_t \neq \tilde{H}_t]} \mathrm{d}t = 0$ and

$$\forall\, \omega \in \Omega, y \in \mathsf{D} \quad H(\omega,y) \in \mathbb{C} \;\Rightarrow\; \tilde{H}(\omega,y) \in \mathbb{C}. \tag{A.20}$$

(iii) Let $X : \Omega_\infty \times \mathsf{D} \to \mathbb{R}$ and τ be an $\mathscr{F}$-stopping time such that $H \overset{\text{def}}{=} (X_t 1_{[t<\tau]})_{t\geq 0} \in \mathbb{L}(\mathcal{M}(\mathscr{F} \otimes \mathcal{D}))$, then there exists $Z : \Omega \times \mathsf{D} \to \mathbb{R}$ such that

$$\tilde{X} \overset{\text{def}}{=} (X_t 1_{[t<\tau]} + Z 1_{[\tau\leq t]})_{t\geq 0} \in \mathbb{L}(\mathcal{M}(\mathscr{F} \otimes \mathcal{D})) \tag{A.21}$$

and that (A.20) holds with $(H,\tilde{H})$ replaced by $(X,\tilde{X})$.

Proof. (i) It is enough to show that $\mathbb{F} \times D \in \mathcal{M}(\mathscr{F} \otimes \mathcal{D})$ holds whenever $\mathbb{F} \in \mathcal{M}(\mathscr{F})$ and $D \in \mathcal{D}$. Let $\mathbb{F}, D$ be as above. Then $\mathbb{F} \cap \Omega_t \in \mathscr{B}_t \otimes \mathscr{F}_t$ holds if $t \in [0,\infty)$ and we get that

$$\{(s,\omega,y) \in \mathbb{F} \times D; s \leq t\} = (\mathbb{F} \cap \Omega_t) \times D \in \mathscr{B}_t \otimes \mathscr{F}_t \otimes \mathcal{D}, \quad t \in [0,\infty),$$

which verifies that $\mathbb{F} \times D \in \mathcal{M}(\mathscr{F} \otimes \mathcal{D})$.

(ii) Note that $\arctan(H) \in \mathbb{L}(\mathcal{M}(\mathscr{F} \otimes \mathcal{D}))$ is bounded. We get from Lemma A.20 that

$$\mathcal{H} \overset{\text{def}}{=} \int \arctan(H_t)\,\mathrm{d}t \in \mathbb{CA}(\mathscr{F} \otimes \mathcal{D}) \subseteq \mathbb{L}(\mathcal{M}(\mathscr{F}) \otimes \mathcal{D}). \tag{A.22}$$

Note that $|\mathcal{H}_t - \mathcal{H}_s| < \frac{\pi}{2}|t-s|$ holds if $s \neq t$ are from $[0,\infty)$. Then we get from Proposition A.7 (ii) that

$$\tilde{H} \overset{\text{def}}{=} \exists \lim_{n\to\infty} \tan\big(n[\mathcal{H}_s - \mathcal{H}_{(s-1/n)^+}]\big) \in \mathbb{L}(\mathcal{M}(\mathscr{F}) \otimes \mathcal{D}) \tag{A.23}$$

and we have that $\tilde{H}_t = \tan(\mathcal{H}'_t) = H_t$ for almost every $t \geq 0$ and (A.20) clearly holds.

(iii) Let $\mathcal{H}$ be as in (A.22) and $\tilde{H}$ as in (A.23). Then we use (i) in order to conclude that

$$Z \overset{\text{def}}{=} \tilde{H}_{\mathbb{T}} \in \mathbb{L}(\mathcal{G}_{\mathbb{T}}), \quad \text{where} \quad \mathcal{G} \overset{\text{def}}{=} \mathscr{F} \otimes \mathcal{D}, \quad \text{and where } \mathbb{T} \overset{\text{def}}{=} (\tau(\omega))_{(\omega,y)\in\Omega\times\mathsf{D}}.$$

It follows from the definition of Z that (A.20) holds with $(H, \tilde{H})$ replaced by $(X, \tilde{X})$, and hence it remains to verify (A.21). As $Z \in \mathbb{L}(\mathcal{G}_{\mathbb{T}})$, we get that

$$U_t^{(c)} \overset{\text{def}}{=} \{(\omega, y) \in \Omega \times \mathsf{D}; Z(\omega, y) < c, \tau(\omega) \le t\} \in \mathcal{G}_t = \mathscr{F}_t \otimes \mathcal{D}, \quad t \in [0,\infty), c \in \mathbb{R},$$

i.e., $\mathbb{U}_t^{(c)} \overset{\text{def}}{=} [0,t] \times U_t^{(c)} \in \mathscr{B}_t \otimes \mathcal{G}_t$. As $\mathbb{T}$ is a $\mathcal{G}$-stopping time, $1_{\mathbb{T}} = (1_{[\mathbb{T}>t]})_{t\ge0} \in \mathbb{A}(\mathcal{G})$, where

$$\mathbb{T} \overset{\text{def}}{=} \{(t,\omega,y) \in \Omega_\infty \times \mathsf{D}; t < \mathbb{T}(\omega, y)\} = \{(t,\omega) \in \Omega_\infty; t < \tau(\omega)\} \times \mathsf{D},$$

and since the process $1_{\mathbb{T}}$ is also right-continuous, it is $\mathcal{G}$-progressive, i.e., $\mathbb{T} \in \mathcal{M}(\mathcal{G}) = \mathcal{M}(\mathscr{F} \otimes \mathcal{D})$. Then we obtain (A.21) from the definition of $\tilde{X}$ as follows

$$\{(\omega, y) \in \Omega_t \times \mathsf{D}; \tilde{X}(\omega, y) < c\} = \big(\{(\omega, y) \in \Omega_t \times \mathsf{D}; H(\omega, y) < c\} \cap \mathbb{T}\big) \cup \big(\mathbb{U}_t^{(c)} \setminus \mathbb{T}\big) \in \mathscr{B}_t \otimes \mathcal{G}_t,$$

$t \in [0,\infty), c \in \mathbb{R}$, since $H = (X_t 1_{[t<\tau]})_{t\ge0} \in \mathbb{L}(\mathcal{M}(\mathscr{F} \otimes \mathcal{D})) = \mathbb{L}(\mathcal{M}(\mathcal{G}))$ by assumption. □

Remark A.22. In the proof of the subsequent lemma, we will need the following simple observation. Let $(\mathsf{D}, \mathcal{D})$ be a measurable space, $X : \Omega \to \mathsf{D}$ and let $(C_q)_{q\in\mathbb{Q}} \in \mathcal{D}^{\mathbb{Q}}$ be a family of $\mathcal{D}$-measurable sets. Put $\mathfrak{R}(x) \overset{\text{def}}{=} x 1_{[x\in\mathbb{R}]}$ if $x \in [-\infty, \infty]$, then

$$\mathfrak{R} \circ f \in \mathbb{L}(\mathcal{D}), \quad \text{where} \quad f(y) \overset{\text{def}}{=} \inf\{q \in \mathbb{Q}; y \in C_q\} \in [-\infty,\infty], \quad y \in \mathsf{D},$$

and $[q < f(X)] \subseteq [X \notin C_q]$ and $[f(X) < q] \subseteq \cup_{c\in\mathbb{Q}\cap(-\infty,q)}[X \in C_c]$ whenever $q \in \mathbb{Q}$.

Lemma A.23. *Let $\mathcal{X}$ be a filtration on $(\Omega, \mathcal{A}, \mathbb{P})$ and $X \in \mathbb{CA}(\mathcal{X})^k, k \in \mathbb{N}$. (i) If $Z \in \mathbb{L}(\mathcal{M}(\mathcal{F}^{X,\mathcal{X},\mathbb{P}}))$, then there exist $\mathbb{z} \in \mathbb{L}(\mathcal{M}(\mathscr{C}^k))$ and $N \in \mathscr{N}_\infty^{\mathcal{X},\mathbb{P}}$ such that*

$$0 = 1_{\Omega\setminus N} \int_0^\infty 1_{[Z_t \ne \mathbb{z}_t(X)]} \mathrm{d}t. \tag{A.24}$$

(ii) If $(\mathsf{S}, \mathfrak{S})$ is a measurable space and $Z \in \mathbb{L}(\mathcal{M}(\mathcal{F}^{X,\mathcal{X},\mathbb{P}}) \otimes \mathfrak{S})$, then there exist $N \in \mathscr{N}_\infty^{\mathcal{X},\mathbb{P}}$ and $\mathbb{z} \in \mathbb{L}(\mathcal{M}(\mathscr{C}^k) \otimes \mathfrak{S})$ such that (A.24) *holds with $\mathbb{z}_t(X) \overset{\text{def}}{=} \mathbb{z}_t(X, \mathbb{s})_{\mathbb{s}\in\mathsf{S}}$.*

Proof. (i,a) Let $Z \in \mathbb{L}(\mathcal{M}(\mathcal{F}^{X,\mathcal{X},\mathbb{P}}))$ first attain values within $[-n, n]$, where $n \in \mathbb{N}$ is fixed. Then

$$W \overset{\text{def}}{=} \int Z_t \mathrm{d}t \in \mathbb{CA}(\mathcal{F}^{X,\mathcal{X},\mathbb{P}}) \quad \text{and} \quad 0 = \int_0^\infty |W_t' - Z_t| \mathrm{d}t. \tag{A.25}$$

Then we use Lemma A.15 (4) in order to get that for each $t \in [0,\infty)$ and $c \in \mathbb{R}$, there exist $C_t^{(c)} \in \mathscr{C}_t^k$ and $N_t^{(c)} \in \mathscr{N}_\infty^{\mathcal{X},\mathbb{P}}$ such that $[W_t < c] = [X \in C_t^{(c)}] \vartriangle N_t^{(c)}$. By Remark A.22

$$F_t \overset{\text{def}}{=} \mathfrak{R} \circ f_t \in \mathbb{L}(\mathscr{C}_t^k), \quad \text{where} \quad f_t \overset{\text{def}}{=} \big(\inf\{q \in \mathbb{Q}; y \in C_t^{(c)}\}\big)_{y\in\mathbb{C}^k}, \tag{A.26}$$

$$[W_t < f_t(X)] = \bigcup_{c\in\mathbb{Q}} [W_t < c < f_t(X)] \subseteq \bigcup_{c\in\mathbb{Q}} [W_t < c, X \notin C_t^{(c)}] \subseteq N_t \overset{\text{def}}{=} \bigcup_{c\in\mathbb{Q}} N_t^{(c)}, \tag{A.27}$$

$$[W_t > f_t(X)] = \bigcup_{q\in\mathbb{Q}} [W_t > q > f_t(X)] \subseteq \bigcup_{c\in\mathbb{Q}} [W_t \ge c, X \in C_t^{(c)}] \subseteq N_t \in \mathscr{N}_\infty^{\mathcal{X},\mathbb{P}}. \tag{A.28}$$

Since $F \overset{\text{def}}{=} (F_t)_{t\ge0} \in \mathbb{A}(\mathscr{C}^k)$, we get from Lemma A.19 that there exists $\mathbb{y} \in \mathbb{L}(\mathcal{M}(\mathscr{C}^k)) \subseteq \mathbb{A}(\mathscr{C}^k)$ such that (A.11) holds with f replaced by F. As W has continuous trajectories, we get that

$$[W \ne \mathbb{y}(X)] \subseteq \bigcup_{q\in\mathbb{Q}^+} [W_q \ne F_q(X)] \subseteq \bigcup_{q\in\mathbb{Q}^+} [W_q \ne f_q(X)] \subseteq N \overset{\text{def}}{=} \bigcup_{q\in\mathbb{Q}^+} N_q \in \mathscr{N}_\infty^{\mathcal{X},\mathbb{P}}. \tag{A.29}$$

Further, consider $\mathbb{z} \in \mathbb{L}(\mathcal{M}(\mathscr{C}^k))$ from (A.12) in Lemma A.19 (ii) with $\mathsf{D} \overset{\text{def}}{=} \mathbb{C}^k$. Then

$$[W = \mathbb{y}(X), W_t' = Z_t] \subseteq [\mathbb{z}_t(X) = \mathbb{y}_t'(X) = Z_t], \quad t \in (0,\infty), \tag{A.30}$$

and we get (A.24) from (A.25,A.29,A.30).

(i,b) For $n \in \mathbb{N}$, put $Z^{(n)} \overset{\text{def}}{=} Z1_{[|Z|\le n]} \in \mathbb{L}(\mathcal{M}(\mathcal{F}^{X,\mathcal{X},\mathbb{P}}))$. By step (i,a) there exist $\mathbb{z}^{(n)} \in \mathbb{L}(\mathcal{M}(\mathscr{C}^k))$ and $N^{(n)} \in \mathscr{N}_\infty^{\mathcal{X},\mathbb{P}}$ such that (A.24) holds with $(\mathbb{z}, N)$ replaced by $(\mathbb{z}^{(n)}, N^{(n)})$. Then

$$N \overset{\text{def}}{=} \cup_{n=1}^\infty N^{(n)} \in \mathscr{N}_\infty^{\mathcal{X},\mathbb{P}} \quad \text{and} \quad \mathbb{z} \overset{\text{def}}{=} \exists \lim_{n\to\infty} \mathbb{z}^{(n)} \in \mathbb{L}(\mathcal{M}(\mathscr{C}^k)),$$

(see Proposition A.7) satisfy (A.24) as

$$1_{\Omega\setminus N}\int_0^\infty 1_{[Z_t \ne \mathbb{z}_t(X)]}\mathrm{d}t \le \sum_n 1_{\Omega\setminus N^{(n)}}\int_0^\infty 1_{[Z_t^{(n)} \ne \mathbb{z}_t^{(n)}(X)]}\mathrm{d}t = 0.$$

(ii) The exact proof of (ii) would be very similar to the proof of (i) and for this reason, we only suggest what should be changed hoping that the reader will be able to modify the proof of (i) carefully in order to obtain the correct proof of (ii). First, replace $\mathcal{F}^{X,\mathcal{X},\mathbb{P}}$ by $\mathcal{F}^{X,\mathcal{X},\mathbb{P}} \otimes \mathfrak{S}$ at the beginning of the proof until (A.25). Then instead of using Lemma A.15 (4) in case (i), we recommend to use Lemma A.14 (iii) in order to get that

$$\text{(A.31)} \qquad \forall\ t \in [0,\infty)\ \forall\, c \in \mathbb{R}\ \exists\, \mathbb{F}_t^{(c)} \in \mathscr{F}_t^X \otimes \mathfrak{S}\ \exists\, N_t^{(c)} \in \mathscr{N}_\infty^{\mathcal{X},\mathbb{P}}$$

$$\text{(A.32)} \qquad \{(\omega, \mathbb{s}) \in \Omega \times \mathsf{S}; W_t(\omega,\mathbb{s}) < c\} \vartriangle \mathbb{F}_t^{(c)} \subseteq N_t^{(c)} \times \mathsf{S}.$$

Then we obtain from Lemma A.1 that there are $C_t^{(c)} \in \mathscr{C}_t^k \otimes \mathfrak{S}$ s.t. $\mathbb{F}_t^{(c)} = (X \odot \mathbb{1}_{\mathsf{S}})^{-1}C_t^{(c)}$, $c \in \mathbb{R}, t \in [0,\infty)$. Further, as in (A.26) we obtain f, F, but this time with $F \in \mathbb{A}(\mathscr{C}^k \otimes \mathfrak{S})$, and similarly as in (A.27,A.28), we obtain that

$$\text{(A.33)} \quad \{(\omega, \mathbb{s}) \in \Omega \times \mathsf{S}; W_t(\omega,\mathbb{s}) \ne f_t(X(\omega),\mathbb{s})\} \subseteq N_t \times \mathsf{S}, \quad N_t \overset{\text{def}}{=} \cup_{c\in\mathbb{Q}} N_t^{(c)} \in \mathscr{N}_\infty^{\mathcal{X},\mathbb{P}}.$$

Note that the relation in (A.33) on the left can be also read as follows $[W_t \ne f_t(X)] \subseteq N_t$. Then by Lemma A.19 there exist

$$\mathbb{y} \in \mathbb{L}(\mathcal{M}(\mathscr{C}^k \otimes \mathfrak{S})) \subseteq \mathbb{A}(\mathscr{C}^k \otimes \mathfrak{S}) \quad \text{and} \quad \mathbb{z} \in \mathbb{L}(\mathcal{M}(\mathscr{C}^k \otimes \mathfrak{S}))$$

such that (A.29) holds in the sense of the notation mentioned just below (A.33). Then the part of (a) from (A.30) can remain as it is in (i). If $\mathbb{z} \notin \mathbb{L}(\mathcal{M}(\mathscr{C}^k) \otimes \mathfrak{S})$, use Lemma A.21 in order to obtain $\tilde{\mathbb{z}} \in \mathbb{L}(\mathcal{M}(\mathscr{C}^k) \otimes \mathfrak{S})$ such that $\int_0^\infty 1_{[\mathbb{z}_t \ne \tilde{\mathbb{z}}_t]}\,\mathrm{d}t = 0$. The process $\tilde{\mathbb{z}}$ can obviously play the role of $\mathbb{z}$ from the statement of the lemma in part (ii). (b) We leave modifying of the point (i,b) in order to get the point (ii,b) of the proof to the reader as we believe that it is straightforward. ☐

Lemma A.24. *Let $(\mathsf{D}, \mathcal{D})$ be a measurable space. (i) Let g, h be $\mathscr{C}^k$ and $\mathscr{C}^k \otimes \mathcal{D}$-progressive processes, respectively, i.e., $g \in \mathbb{L}(\mathcal{M}(\mathscr{C}^k)), h \in \mathbb{L}(\mathcal{M}(\mathscr{C}^k \otimes \mathcal{D}))$, and let $X \in \mathbb{CA}(\mathscr{F})^k, k \in \mathbb{N}$, then*

$$\text{(A.34)} \quad G \overset{\text{def}}{=} g(X) \overset{\text{def}}{=} \big(g_t(X)\big)_{t\ge0} \in \mathbb{L}(\mathcal{M}(\mathscr{F})), \quad H \overset{\text{def}}{=} \big(h(X,y)\big)_{y\in\mathsf{D}} \in \mathbb{L}(\mathcal{M}(\mathscr{F} \otimes \mathcal{D})).$$

In particular, whenever $p \in [1,\infty)$, we have that $H \in \mathcal{M}^p(\mathscr{F} \otimes \mathcal{D})$ if $h \in \mathcal{M}^p(\mathscr{C}^k \otimes \mathcal{D})$ and that $G \in \mathcal{M}^p(\mathscr{F})$ if $g \in \mathcal{M}^p(\mathscr{C}^k)$. (ii) If even $h \in \mathbb{L}(\mathcal{M}(\mathscr{C}^k) \otimes \mathcal{D})$, then $H \in \mathbb{L}(\mathcal{M}(\mathscr{F}) \otimes \mathcal{D})$. (iii) The points (i,ii) of the lemma hold also with $\mathscr{C}$ replaced by $\tilde{\mathscr{C}}$ if X attains values in $(0,\infty)^k$. (iv) If $h \in \mathbb{L}(\mathcal{M}(\mathscr{A}) \otimes \mathcal{D})$, and $X \in \mathbb{A}(\mathscr{F}, \mathscr{A})$, where $\mathscr{A}$ is a filtration, then

$$H \overset{\text{def}}{=} \big(h(X,y)\big)_{y\in\mathsf{D}} \in \mathbb{L}(\mathcal{M}(\mathscr{F}) \otimes \mathcal{D}).$$

Proof. (i) Let $h \in \mathbb{L}(\mathcal{M}(\mathscr{C}^k \otimes \mathcal{D}))$. Then $h|_t \overset{\text{def}}{=} (h_s)_{s\le t} \in \mathbb{L}(\mathscr{B}_t \otimes \mathscr{C}_t^k \otimes \mathcal{D})$ holds if $t \in [0,\infty)$. Since $\mathbb{1}_{[0,t]} \in \mathbb{L}(\mathscr{B}_t, \mathscr{B}_t), \mathbb{1}_{\mathsf{D}} \in \mathbb{L}(\mathcal{D},\mathcal{D})$ and $X \in \mathbb{L}(\mathscr{F}_t, \mathscr{C}_t^k)$ if $t \in [0,\infty)$, we get from Lemma A.1 that

$$\mathfrak{X}|_t \overset{\text{def}}{=} \mathbb{1}_{[0,t]} \odot X \odot \mathbb{1}_{\mathsf{D}} \in \mathbb{L}(\mathscr{B}_t \otimes \mathscr{F}_t \otimes \mathcal{D}, \mathscr{B}_t \otimes \mathscr{C}_t^k \otimes \mathcal{D}), \quad t \in [0,\infty).$$

Hence, we finally get that $H|_t \overset{\text{def}}{=} (H_s)_{s\le t} = h|_t \circ \mathfrak{X}|_t \in \mathbb{L}(\mathscr{B}_t \otimes \mathscr{F}_t \otimes \mathcal{D})$ whenever $t \in [0,\infty)$, which verifies that $H \in \mathbb{L}(\mathcal{M}(\mathscr{F} \otimes \mathcal{D}))$. The property in (A.34) on the left can be obtained similarly, and then the next part can be verified easily from the definition of $\mathcal{M}^p(\mathscr{C}^k \otimes \mathcal{D}), \mathcal{M}^p(\mathscr{C}^k)$.

(ii) This is a special case of (iv) with $\mathscr{A} \overset{\text{def}}{=} \mathscr{C}^k$, since $\mathbb{CA}(\mathscr{F})^k = \mathbb{A}(\mathscr{F},\mathscr{C}^k)$.

(iv) 1. First, we will show that $f(X) \overset{\text{def}}{=} (f_s(X))_{s\geq 0} \in \mathbb{L}(\mathcal{M}(\mathscr{F}))$ holds whenever $f \in \mathbb{L}(\mathcal{M}(\mathscr{A}))$ attains values in $\{0,1\}$. Put $C \overset{\text{def}}{=} f^{-1}\{1\} \in \mathbb{L}(\mathcal{M}(\mathscr{A}))$, then

$$C|_t \overset{\text{def}}{=} \{(s,x) \in C; s \leq t\} \in \mathscr{B}_t \otimes \mathscr{A}_t, \quad t \in [0,\infty).$$

Since $X \in \mathbb{A}(\mathscr{F},\mathscr{A}) \subseteq \mathbb{L}(\mathscr{F}_t,\mathscr{A}_t)$, we get from Lemma A.1 that $\mathbb{1}_{[0,t]} \odot X \in \mathbb{L}(\mathscr{B}_t \otimes \mathscr{F}_t, \mathscr{B}_t \otimes \mathscr{A}_t)$, and then $\{(s,\omega) \in \Omega_t; f_s(X(\omega)) = 1\} = (\mathbb{1}_{[0,t]} \odot X)^{-1}(C|_t) \in \mathbb{L}(\mathscr{B}_t \otimes \mathscr{F}_t), t \in [0,\infty)$, which verifies that $f(X) \in \mathbb{L}(\mathcal{M}(\mathscr{F}))$.

2. Again from Lemma A.1, we obtain that $H = f(X) \odot 1_D \in \mathbb{L}(\mathcal{M}(\mathscr{F}) \otimes \mathcal{D}), D \in \mathcal{D}$.

3. By point 2, $\mathscr{D} \overset{\text{def}}{=} \{C \times D; C \in \mathcal{M}(\mathscr{A}), D \in \mathcal{D}\}$ is a subset of the following set

$$\mathscr{X} \overset{\text{def}}{=} \{\mathbb{D} \in \mathcal{M}(\mathscr{A}) \otimes \mathcal{D}; (1_{\mathbb{D}}(X,y))_{y\in \mathsf{D}} \in \mathbb{L}(\mathcal{M}(\mathscr{F}) \otimes \mathcal{D})\},$$

which is obviously a σ-algebra. Hence, also the σ-hull of $\mathscr{D}$ is a subset of $\mathscr{X}$, which means that $\mathscr{X} = \mathcal{M}(\mathscr{A}) \otimes \mathcal{D}$. Hence, we have shown (iv) for h attaining values in $\{0,1\}$ and the extension of this result to all $h \in \mathbb{L}(\mathcal{M}(\mathscr{A}) \otimes \mathcal{D})$ is left to the reader. (iii) Just replace $\mathscr{C}$ by $\tilde{\mathscr{C}}$ in the proof. □

A.5. **Enriched filtration continued.**

Lemma A.25. *Let $\mathcal{F}$ be an enriched filtration, $L, X \in \mathbb{CS}(\mathcal{F})$ and $Y \in \mathbb{L}(\mathcal{F}_0)$ be s.t.*

$$X_t \overset{\text{as}}{=} Y + \int_0^t X \mathrm{d}L, \quad t \in [0,\infty). \tag{A.35}$$

Then $X_t \overset{\text{as}}{=} Y \exp\{L_t - L_0 - \frac{1}{2}\langle L\rangle_t\}, t \in [0,\infty)$.

Proof. Put $R \overset{\text{def}}{=} X \exp\{-L + L_0 + \frac{1}{2}\langle L\rangle\}$ and use the Itô rule in order to obtain that its paths are constant almost surely. Then we get that $R_t \overset{\text{as}}{=} R_0 \overset{\text{as}}{=} Y, t \in [0,\infty)$. □

Proposition A.26 (Lévy Characterization Theorem)**.** *Let $L \in \mathbb{CM}_{loc}(\mathscr{F})^n, n \in \mathbb{N}$. If $L_0 = 0$ and $\langle\langle L\rangle\rangle_t \overset{\text{as}}{=} t \cdot \mathbb{1}_n, t \in [0,\infty)$, then L is an n-dimensional standard $\mathscr{F}$-Brownian motion.*

Proof. Let $(\Omega,\mathcal{A},\mathbb{P})$ be the underlying probability space. Then we get that

$$L \in \mathbb{CM}_{loc}(\mathscr{F})^n \subseteq \mathbb{CM}_{loc}(\mathcal{F})^n$$

by Theorem 1.4.3 (c) in [10, part III], where $\mathcal{F} \overset{\text{def}}{=} (\mathscr{F}_t \vee \{A \in \mathcal{A}; \mathbb{P}(A) \in \{0,1\}\})_{t\geq 0}$. Then we are allowed to use the Lévy Characterization Theorem (3.6) in [24, Chapter IV] in order to conclude that L is a standard n-dimensional $\mathcal{F}$-Brownian motion, but as $L \in \mathbb{CM}_{loc}(\mathscr{F})^n \subseteq \mathbb{CA}(\mathscr{F})^n$, we get from the definition of a standard Brownian motion, for example, that the same holds with $\mathcal{F}$ replaced by a weaker filtration $\mathscr{F}$. □

Lemma A.27. *Let $\mathscr{F}$ be a filtration and let $L \in \mathbb{CM}_{loc}(\mathscr{F})$ start from $L_0 \overset{\text{as}}{=} 0$. If $\langle L\rangle$ is an integrable process, then $L \in \mathbb{CM}(\mathscr{F})$.*

Proof. If $\mathscr{F}_0$ contains all null sets from $\mathscr{F}_\infty$, then it follows from Corollary (1.25) [24, Chapter IV]. If this is not the case, we consider another filtration, say $\mathcal{F}$, as in the proof of Proposition A.26 in order to get that $L \in \mathbb{CM}_{loc}(\mathscr{F}) \subseteq \mathbb{CM}_{loc}(\mathcal{F})$ ensuring by the first step of the proof and by assumption that $L \in \mathbb{CM}(\mathcal{F}) \cap \mathbb{CA}(\mathscr{F}) = \mathbb{CM}(\mathscr{F})$. Note that the last "⊆" and "=" here follow from Theorems 1.2.9 and 1.4.3 in [10, part III] which give us the conditions for the stability of (local) martingale property. □

Lemma A.28. *Let $\mathcal{F}$ be an enriched filtration, let $X \in \mathbb{CM}_{loc}(\mathcal{F})$ and let $H \in \mathcal{M}^2_K(\mathcal{F})$ hold with $K \overset{\text{def}}{=} \langle X\rangle \in \mathbb{CI}_0(\mathcal{F})$, then there exists $Y \in \mathbb{CM}_{loc}(\mathcal{F})$ such that $Y \overset{\text{as}}{=} \int H \mathrm{d}X$.*

Proof. If the filtration $\mathcal{F}$ satisfies the usual conditions, then the statement follows from Proposition 3.2.24 in [19]. Otherwise, let $\mathcal{G}$ be the smallest super-filtration of $\mathcal{F}$ satisfying the usual conditions, i.e., we put $\mathcal{G}_t \overset{\text{def}}{=} \mathcal{F}_{t+} \vee \{A \in \mathcal{A}; \mathbb{P}(A) \in \{0,1\}\}, t \in [0,\infty)$. Then we get that $X \in \mathbb{CM}_{loc}(\mathcal{F}) \subseteq \mathbb{CM}_{loc}(\mathcal{G})$ holds by Theorem 1.4.3 (b,c) in [10, part III] and obviously also $H \in \mathcal{M}^2_K(\mathcal{F}) \subseteq \mathcal{M}^2_K(\mathcal{G})$. Then as stated at the beginning of the proof, there exists $Z \in \mathbb{CM}_{loc}(\mathcal{G})$ such that $Z \overset{\text{as}}{=} \int H \mathrm{d}X$. Since $X \in \mathbb{CA}(\mathcal{F})$ and $H \in \mathcal{M}^2(\mathcal{F}) \subseteq \mathbb{L}(\mathcal{M}(\mathcal{F}))$ hold by assumption, we get from Corollary A.17 that there exists $Y \in \mathbb{CA}(\mathcal{F})$ such that $Y \overset{\text{as}}{=} Z \overset{\text{as}}{=} \int H \mathrm{d}X$. Then we get from Lemma A.11 that $Y \in \mathbb{CM}_{loc}(\mathcal{F})$. □

Remark A.29. Later on, we will use the so called Continuity Theorem on continuous local martingales. It can be found as Theorem 1.7.7 in [10, part III], where the complete filtration is assumed and it can be reformulated in the view of Remark 2.4 in terms of convergence in metric $\wp$ denoted as $\rightsquigarrow$ as follows. If $M^{(n)}, M \in \mathbb{CM}_{loc}(\mathcal{F}), n \in \mathbb{N}$, start from $M^{(n)}_0 \overset{\text{as}}{=} M_0 \overset{\text{as}}{=} 0$, then

$$M^{(n)} \rightsquigarrow M \quad \text{if and only if} \quad \langle M^{(n)} - M \rangle \rightsquigarrow 0, \quad n \to \infty. \tag{A.36}$$

Note that the assumption that the underlying probability space is complete is not essential here, and therefore it can be simply omitted as we will do. We will also use the following simple consequence of (A.36). Under the same assumptions, we have that

$$M^{(n)} \rightsquigarrow M \quad \Rightarrow \quad [\langle M^{(n)} \rangle \rightsquigarrow \langle M \rangle, \quad \langle M^{(n)}, N \rangle \rightsquigarrow \langle M, N \rangle, \quad N \in \mathbb{CM}_{loc}(\mathcal{F})], \tag{A.37}$$

since $|\langle M^{(n)} - M, N \rangle|^*_t \overset{\text{as}}{\leq} (\langle M^{(n)} - M \rangle_t \cdot \langle N \rangle_t)^{1/2}$ and

$$|\langle M^{(n)} \rangle - \langle M \rangle|^*_t \overset{\text{as}}{\leq} f(\langle M^{(n)} - M \rangle_t, \langle M \rangle_t), \qquad \text{where} \qquad f(x,y) \overset{\text{def}}{=} x + 2\sqrt{xy}.$$

Finally note that (A.36) obviously holds also with $M \equiv 0$ and with n replaced by (n,i) giving that

$$M^{(n,i)} \overset{\text{def}}{=} M^{(n)} - M^{(i)} \rightsquigarrow 0 \quad \text{iff} \quad \langle M^{(n)} - M^{(i)} \rangle \overset{\text{as}}{=} \langle M^{(n,i)} \rangle \rightsquigarrow 0, \quad n,i \to \infty, \tag{A.38}$$

which help us characterize a $\wp$-Cauchy sequence of continuous local $\mathcal{F}$-martingales starting from zero in terms of quadratic variation.

Remark A.30 (On convergence of local martingales). Note that a process $M \in \mathbb{CA}(\mathcal{F})$ is in $\mathbb{CM}_{loc}(\mathcal{F})$ if and only if $\aleph_c(M) \in \mathbb{CM}(\mathcal{F})$ holds for every $c \in (0,\infty)$, where

$$\aleph_c(x) \overset{\text{def}}{=} (x_{t\wedge\tau^x_c} - x_0)_{t\geq 0}, \qquad \text{where} \qquad \tau^x_c \overset{\text{def}}{=} \inf\{t \in [0,\infty); |x_t - x_0| \geq c\}.$$

Then it easily follows from Theorem 1 in [8] that

$$\mathbb{CM}_{loc}(\mathcal{F}) \ni M^{(n)} \overset{\text{as}}{\to} M \in \mathbb{CA}(\mathcal{F}) \quad \Rightarrow \quad M \in \mathbb{CM}_{loc}(\mathcal{F}). \tag{A.39}$$

In order to clarify the convergence of local martingales completely, it is helpful to mention Theorem 3 from the same paper which states that $M \in \mathfrak{C}(\aleph_c)$ almost surely if M is a local martingale and $c \in (0,\infty)$, where $\mathfrak{C}(\aleph_c)$ stands for the points of continuity of $\aleph_c$. In particular and with the use of (A.39), we have that

$$\mathbb{CM}(\mathcal{F}) \ni \aleph_c(M^{(n)}) \overset{\text{as}}{\to} \aleph_c(M), \qquad n \to \infty, \quad c \in (0,\infty),$$

if the assumption of (A.39) is satisfied. Then we easily get from the Dominated Convergence Theorem and the definition of a martingale that $\aleph_c(M) \in \mathbb{CM}(\mathcal{F}), c \in (0,\infty)$, i.e., that $M \in \mathbb{CM}_{loc}(\mathcal{F})$ again, but this time the conclusion is perhaps clearer.

Lemma A.31. *Let $\mathcal{F}$ be an enriched filtration and let $M \in \mathbb{CM}_{loc}(\mathcal{F})^k$ and $H \in \mathbb{L}(\mathcal{M}(\mathcal{F}))^k$, $k \in \mathbb{N}$, satisfy the condition in* (2.12) *on the right. Then the integral in* (2.12) *on the left is well defined and there exists $L \in \mathbb{CM}_{loc}(\mathcal{F})$ s.t. $L \overset{\text{as}}{=} \int H^{\mathsf{T}} \mathrm{d}M$. Further, $\langle L, N \rangle \overset{\text{as}}{=} \int H^{\mathsf{T}} \mathrm{d}\langle M, N \rangle$ if $N \in \mathbb{CM}_{loc}(\mathcal{F})$. Moreover, if $G \in \mathbb{L}(\mathcal{M}(\mathcal{F}))^k$, then*

$$\int_0^t tr\{GG^{\mathsf{T}} \mathrm{d}\langle\!\langle M \rangle\!\rangle\} \overset{\text{as}}{<} \infty, \ t \in [0,\infty) \ \Rightarrow \ \langle \int H^{\mathsf{T}} \mathrm{d}M, \int G^{\mathsf{T}} \mathrm{d}M \rangle \overset{\text{as}}{=} \int tr\{GH^{\mathsf{T}} \mathrm{d}\langle\!\langle M \rangle\!\rangle\}. \tag{A.40}$$

Proof. We get from Lemma A.28 that there are $L^{(n)} \in \mathbb{CM}_{loc}(\mathcal{F})$ such that $L^{(n)} \overset{\text{as}}{=} \int 1_{[H^{\mathsf{T}}H \le n]} H^{\mathsf{T}} \mathrm{d}M$, $n \in \mathbb{N}$. Note that if the condition in (2.12) on the right holds, then $\langle L^{(n)} - L^{(i)} \rangle \rightsquigarrow 0$ as $i, n \to \infty$, which by (A.38) means that $(L^{(n)})_{n=1}^{\infty}$ is a $\mathbb{p}$-Cauchy sequence and since $\mathbb{p}$ is a complete metric by Proposition A.10, we get that the integral $\int H^{\mathsf{T}} \mathrm{d}M$ is well defined by (2.12). Further, we get from Lemma A.16 (iv) that there exists $L \in \mathbb{CA}(\mathcal{F})$ such that

$$L \overset{\text{as}}{=} \mathbb{p}\text{-}\lim_{n\to\infty} L^{(n)} \overset{\text{as}}{=} \int H^{\mathsf{T}} \mathrm{d}M. \tag{A.41}$$

We can then find a subsequence of $(L^{(n)})_{n=1}^{\infty}$ converging to L in $\mathbb{p}$ absolutely, which means by Corollary A.9 that we have convergence almost surely in (A.41) for such a subsequence, and since $L \in \mathbb{CA}(\mathcal{F})$ is a limit of local martingales almost surely, we get that from Remark A.30 that $L \in \mathbb{CM}_{loc}(\mathcal{F})$. Let $N \in \mathbb{CM}_{loc}(\mathcal{F})$. It follows from Remark A.29 that

$$\langle L, N \rangle \overset{\text{as}}{=} \mathbb{p}\text{-}\lim_{n\to\infty} \langle L^{(n)}, N \rangle \overset{\text{as}}{=} \mathbb{r}\text{-}\lim_{n\to\infty} \int 1_{[H^{\mathsf{T}}H \le n]} H^{\mathsf{T}} \mathrm{d}\langle M, N \rangle \overset{\text{as}}{=} \int H^{\mathsf{T}} \mathrm{d}\langle M, N \rangle. \tag{A.42}$$

Let $G \in \mathbb{L}(\mathcal{M}(\mathcal{F}))^k$ satisfy the property in (A.40) on the left. By the already proved part of the statement, there exists $N \in \mathbb{CM}_{loc}(\mathcal{F})$ s.t. $N \overset{\text{as}}{=} \int G^{\mathsf{T}} \mathrm{d}M$ and $\langle N, M^{(i)} \rangle \overset{\text{as}}{=} \int G^{\mathsf{T}} \mathrm{d}\langle M, M^{(i)} \rangle$, $i \le k$, hold together with (A.42). Then

$$\langle L, N \rangle \overset{\text{as}}{=} \int H^{\mathsf{T}} \mathrm{d}\langle M, N \rangle \overset{\text{as}}{=} \int H^{\mathsf{T}} \{ \mathrm{d}\langle\!\langle M \rangle\!\rangle\, G \} \overset{\text{as}}{=} \int \mathrm{tr}\{ G H^{\mathsf{T}} \mathrm{d}\langle\!\langle M \rangle\!\rangle \}.$$

□

Corollary A.32. *Let $\mathcal{F}$ be an enriched filtration, $k \in \mathbb{N}$, $M \in \mathbb{CM}_{loc}(\mathcal{F})^k$ and $H \in \mathbb{L}(\mathcal{M}(\mathcal{F}))^{k\times k}$. If $\int_0^t tr\{H^{\mathsf{T}} H \mathrm{d}\langle\!\langle M \rangle\!\rangle\} \overset{\text{as}}{<} \infty, t \in [0,\infty)$, then there exists $L \in \mathbb{CM}_{loc}(\mathcal{F})^k$ s.t.*

$$L \overset{\text{as}}{=} \int H \mathrm{d}M \overset{\text{def}}{=} \left(\int 1_{\{i\}}^{\mathsf{T}} H \mathrm{d}M\right)_{i=1}^{k}, \qquad \text{and} \qquad \langle\!\langle \int H \mathrm{d}M \rangle\!\rangle \overset{\text{as}}{=} \int \{ H \mathrm{d}\langle\!\langle M \rangle\!\rangle H^{\mathsf{T}} \}.$$

Moreover, if $G \in \mathbb{L}(\mathcal{M}(\mathcal{F}))^{1\times k}$ is such that $\int_0^t tr\{G^{\mathsf{T}} G \, \mathrm{d}\langle\!\langle L \rangle\!\rangle\} \overset{\text{as}}{<} \infty, t \in [0,\infty)$, then

$$\int G \, \mathrm{d}L \overset{\text{as}}{=} \int G H \mathrm{d}M. \tag{A.43}$$

Proof. The first part of the statement follows from Lemma A.31. Further, we get that

$$\int_0^t \mathrm{tr}\{(GH)^{\mathsf{T}} G H \mathrm{d}\langle\!\langle M \rangle\!\rangle\} \overset{\text{as}}{=} \int_0^t \mathrm{tr}\{G^{\mathsf{T}} G H \mathrm{d}\langle\!\langle M \rangle\!\rangle H^{\mathsf{T}}\} \overset{\text{as}}{=} \int_0^t \mathrm{tr}\{G^{\mathsf{T}} G \mathrm{d}\langle\!\langle L \rangle\!\rangle\} \overset{\text{as}}{<} \infty, \quad t \in [0,\infty),$$

which verifies that also the integral in (A.43) on the right is well defined. The one on the left is well defined by the assumption that $\int_0^t \mathrm{tr}\{G^{\mathsf{T}} G \, \mathrm{d}\langle\!\langle L \rangle\!\rangle\} \overset{\text{as}}{<} \infty, t \in [0,\infty)$. By Lemma A.31 there are $N, R \in \mathbb{CM}_{loc}(\mathcal{F})$ s.t. $N \overset{\text{as}}{=} \int G \, \mathrm{d}L$ and $R \overset{\text{as}}{=} \int G H \mathrm{d}M$ and that

$$\langle N, R \rangle \overset{\text{as}}{=} \int G \, \mathrm{d}\langle L, R \rangle \overset{\text{as}}{=} \int G H \, \mathrm{d}\langle M, R \rangle \overset{\text{as}}{=} \int \mathrm{tr}\{(GH)^{\mathsf{T}} G H \mathrm{d}\langle\!\langle M \rangle\!\rangle\} \overset{\text{as}}{=} \langle R \rangle \overset{\text{as}}{=} \langle N \rangle. \tag{A.44}$$

From (A.44) we get that the difference $D \overset{\text{def}}{=} N - R \in \mathbb{CM}_{loc}(\mathcal{F})$ has $\langle D \rangle \overset{\text{as}}{=} 0$. As $D_0 \overset{\text{as}}{=} 0$ holds by the definition of D, R, N, we get from Remark A.29 that $D \overset{\text{as}}{=} 0$, i.e., that $N \overset{\text{as}}{=} R$ which verifies (A.43). □

References

1. P.H. Algoet and T.M. Cover, *Asymptotic optimality and asymptotic equipartition properties of log-optimum investment*, Ann. Probab. **16** (1988), no. 2, 876–898.
2. A. Bain and D. Crisan, *Fundamentals of stochastic filtering*, Stochastic Modelling and Applied Probability, 60, Springer, New York, 2009.
3. R. Bell and T.M. Cover, *Game-theoretic optimal portfolios*, Manage. Sci. **34** (1998), no. 6, 724–733.
4. L. Breiman, *Optimal gambling system for flavorable games*, Proc. Fourth Berkeley Symp. on Math. Statist. and Prob. **1** (1961) (J. Neyman, ed.), Univ. of Calif. Press, Berkeley, 65–78.
5. S. Browne and W. Whitt, *Portfolio choice and the Bayesian Kelly criterion*, Adv. Appl. Probab. **28** (1996), no. 4, 1145–1176.

6. D. Crisan and M. Kouritzin, J. Xiong, *Nonlinear filtering with signal dependent observation noise*, Electron. J. Probab. **14** (2009), no. 63, 1863–1883.
7. D. Crisan and S. Ortiz-Latorre, *A high order time discretization of the solution of the non-linear filtering problem*, Stoch. PDE: Anal. Comp. (2019), https://doi.org/10.1007/s40072-019-00157-3
8. P. Dostál, *A note on weak convergence on martingale measures*, Acta Univ. Carol., Math. Phys. **43** (2002), no. 1, 3-12.
9. P. Dostál and J. Klůjová, *Log-Optimal Investment in the Long Run with Proportional Transaction Costs When Using Shadow Price*, Kybernetika **51** (2015), no. 4, 588–628.
10. J. Dupačová, J. Hurt J and J. Štěpán, *Stochastic Modelling in Economics and Finance* Kluwer Academic Publishers, New York, 2003.
11. M. Fujisaki, G. Kallianpur and H. Kunita, *Stochastic differential equations for the non linear filtering problem*, Osaka J. Math. **9** (1972), no. 1, 19–40.
12. J. Jacod, A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer-Verlag, Berlin, Heidelberg, 2003.
13. O. Kallenberg, *Foundations of Modern Probability*, Springer-Verlag, New York, 1997.
14. G. Kallianpur, *Stochastic Filtering Theory*, Springer-Verlag, New York, 1980.
15. G.Kallianpur and C. Striebel, *Estimation of Stochastic Systems: Arbitrary System Process with Additive White Noise Observation Errors*, The Annals of Mathematical Statistics **39** (1968), no. 3, 785–801.
16. R.E. Kalman, *A New Approach to Linear Filtering and Prediction Problems*, Journal of Basic Engineering **82** (1960), no. 1, 35–45.
17. R.E. Kalman and R.S. Bucy, *New Results in Linear Filtering and Prediction Theory*, Journal of Basic Engineering **83** (1961), no. 1, 95–108.
18. I. Karatzas, K. Kardaras, *The numéraire portfolio in semimartingale financial models*, Finance and Stochastics **11** (2007), no. 4, 447–493.
19. I. Karatzas and S.E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, Berlin, New York, Heidelberg, 1988.
20. J.L. Kelly, *A new interpretation of information rate*, Bell Sys. Tech. J. **35** (1956), no. 4, 917–926.
21. H. Kushner, *Dynamical equations for optimal nonlinear filtering*, J. Differ. Equations **3** (1967), 179–190.
22. O. Mostovyi, P. Siorpaes, *Differentiation of measures on a non-separable space, and the Radon-Nikodym theorem*, arXiv:1909.03505[math.CA], 2019.
23. R. Liptser and A.N. Shiryaev, *Statistics of Random Processes I (General Theory)*, Springer-Verlag, New York, 1977.
24. D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer-Verlag, Berlin, 1999.
25. L.C.G. Rogers and D. Williams, *Diffusions, Markov Processes and Martingales*, Second edition, Cambridge University Press, Cambridge, 2000.

Institute of Information Theory and Automation, Czech Academy of Science, Prague, Czech Republic, and Faculty of Mathematics and Physics, Charles University in Prague, Czech Republic

Current address: Institute of Information Theory and Automation, Czech Academy of Science, Prague, Czech Republic

E-mail address: dostal@utia.cas.cz, dostal@karlin.mff.cuni.cz

Institute of Information Theory and Automation, Czech Academy of Science, Prague, Czech Republic

Current address: Institute of Information Theory and Automation, Czech Academy of Science, Prague, Czech Republic

E-mail address: machtibor@gmail.com